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STABLE SEMIGROUPS ON HOMOGENEOUS TREES AND HYPERBOLIC SPACES

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ABSTRACT. We prove the kernel estimates for subordinated semigroups on homogeneous trees. We study the long time propagation problem. We exploit this to show exit time estimates for large balls in an abstract setting of metric measure spaces. Finally, we give estimates for the Poisson kernel of a ball.

1. Introduction

In 1961 Getoor [12] proposed subordinated semigroups in the context of the real hyperbolic spaces. It is only recently when the corresponding kernel estimates were found ([1], [14]).

The aim of this paper is to give a corresponding result in the context of homogeneous trees. Our motivations come from the fact that such structures make a discrete counterpart for hyperbolic spaces. Large scale analogy holds not only in geometry but also in analysis, see e.g. [8], [10], [11].

Our starting point is a diffusion semigroup considered in [8]. By subordination we obtain a new semigroup, which is referred to as to the stable one. We show estimates for the corresponding kernel (Theorem 3.1 below). Our present theorem sheds some light on a natural interpretation for the analogous result from [14] (see remarks after the proof).

We consider the long time propagation problem (Theorem 3.2). It turns out that for large time t the mass of our kernel is distributed at distances comparable with $t^{2/\alpha}$. We give two different proofs. First of them is of general nature and exploits properties of the underlying diffusion semigroup. The proof works for hyperbolic spaces or Riemannian manifolds as well. The other proof shows that in this context our Theorem 3.1 is useful as well.

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Getoor [12] raised the question of "stability" properties for semigroups of this type. Obviously, here we have neither classical scaling, nor its *weak* form which is typical for e.g. fractals [6]. However, one may interpret Theorem 3.2 as an *asymptotic* scaling property.

Finally, we give some applications of Theorem 3.1. We study exit time from balls for the *stable* process corresponding to our semigroup. For related results we refer the reader to [13] or [19]. In general, we were inspired by the approach from [4], for stable case see [6]. The results in Section 4 have their analogues in these papers. Observe, however, that the argument of [4] and [6] hinges on the *Ahlfors-regularity* of the measure, i.e. polynomial volume growth. Clearly, this excludes the homogeneous trees and hyperbolic spaces. Our contribution is to find a convenient setting so that the the argument can be adapted for stable processes in spaces with exponential volume growth (see (26) and (27) below). We use a more abstract framework of metric measure spaces (cf. [16]). In this way, we can obtain some results for homogeneous trees and hyperbolic spaces at the same time. For example, we get estimates for the Poisson kernel for balls. The interplay between (26) and (27) may be of independent interest.

The paper is organized as follows. The necessary notations are gathered in Preliminaries. Section 3 is devoted to the heat kernel estimates and the long time propagation problem. In Section 4, we introduce our abstract setting and prove the exit time estimates. As a consequence, in Section 5 the Poisson kernel estimates are obtained.

2. Preliminaries

In what follows, X denotes a homogeneous tree of degree $q+1$, i.e. a connected graph without loops, in which every vertex has $q+1$ neighbors. Fix an arbitrary reference point $o \in X$. For any vertex $x \in X$, the graph distance from x to o will be denoted by $|x|$. Consider the nearest-neighbor Laplacian Δ and the related heat semigroup $\mathcal{H}_t$ with continuous time on X of degree $q+1$ with $q \geq 2$, that is,

$$\Delta f(x) = f(x) - \frac{1}{q+1} \sum_{y \sim x} f(y), \quad x \in X \quad \text{and} \quad \mathcal{H}_t = e^{-t\Delta}, \quad t > 0.$$

See [8] for detailed exposition. We adopt the general setting from that paper. For the reader's convenience, we recall definitions needed in what follows. In particular, let h_t denote the corresponding heat kernel and $h_t^{\mathbb{Z}}$ the heat kernel in the case of $q = 1$, when the tree can be identified with the set of integers. We have

$$h_t^{\mathbb{Z}}(j) = e^{-t} I_{|j|}(t), \quad t > 0, j \in \mathbb{Z},$$

where $I_\nu(t)$ stands for the modified Bessel function of the first kind. Furthermore, set

$$\gamma = \frac{2\sqrt{q}}{q+1}$$

so that $b_2 = 1-\gamma$ is the bottom of the spectrum of the Laplacian acting on $L^2(X)$.

We adopt the convention that c (without subscripts) denotes a generic constant whose value may change from one place to another. To avoid some curiosities occasionally, we write $\tilde{c}$, $c', \dots$ with the same properties. Numbered constants (with subscripts) always keep their particular value throughout the current theorem or proof. We often write $f \asymp g$ to indicate that there exists $c>0$ such that $c^{-1} < f/g < c$. Similarly, $f(x) \asymp g(x)$, $x \to \infty$, means $c^{-1} < f/g < c$ for x large enough.

The kernel h_t is known to satisfy the following estimates [8]:

$$h_t(x) \asymp \frac{e^{-b_2 t}}{t} \phi_0(x) h^{\mathbb{Z}}_{t\gamma}\big(|x|+1\big),$$

where

$$\phi_0(x) = \left(1 + \frac{q-1}{q+1}|x|\right) q^{-\frac{|x|}{2}}, \quad x \in X \tag{1}$$

is the *spherical function.* Using the definition of $h^{\mathbb{Z}}_t$, we obtain

$$h_t(x) \asymp \frac{e^{-t}}{t} \phi_0(x) I_{1+|x|}(t\gamma), \quad t>0, x \in X. \tag{2}$$

In what follows, we fix $\alpha \in (0,2)$ and consider the *subordinate semigroup* $(T^{(\alpha)}_t)_{t\geq 0}$,

$$T^{(\alpha)}_t = \int_0^\infty e^{-u\Delta} \eta_t(u)\, du,$$

where the *subordinator* $\eta_t(\cdot)$ is a (defined on $\mathbb{R}^+$) continuous density function of a probability measure, determined by its Laplace transform,

$$\mathcal{L}\big[\eta_t(\cdot)\big](\lambda) = e^{-t\lambda^{\alpha/2}}.$$

By an analogy with the classical situation (subordination of the gaussian semigroup on $\mathbb{R}^n$), the generator of the new semigroup is denoted by $\Delta^{\alpha/2}$ and called the fractional Laplacian.

For the corresponding kernels, we have

$$p_t(x) = \int_0^\infty h_u(x) \eta_t(u)\, du. \tag{3}$$

Sometimes we refer to $p_t(x)$ as to the α-stable kernel. For more details concerning this construction, we refer the reader for example, to [5].

3. α-stable kernel

Our main result may be stated as follows.

THEOREM 3.1. *For any constants $K, M > 0$*

$$p_t(x) \asymp \begin{cases} \phi_0(x) t^{-3/2} \exp(-t(1-\gamma)^{\alpha/2}), & |x| < Kt^{1/2}, t \geq 1, \\ \phi_0(x) t |x|^{-2-\alpha/2} q^{-|x|/2}, & |x| > Mt^{2/\alpha}. \end{cases} \tag{4}$$

Proof. First, we collect some auxiliary estimates for Bessel function $I_\nu(z)$. Recall its integral representation (e.g., [15], (8.431.1))

$$\begin{aligned} I_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}} \int_{-1}^{1} \left(1-u^2\right)^{\nu-1/2} e^{-zu}\, du \\ &= \frac{(2\pi z)^{-1/2} e^z}{2^{\nu-1/2}\Gamma(\nu+1/2)} \int_0^{2z} \left[u(2-u/z)\right]^{\nu-1/2} e^{-u}\, du. \end{aligned}$$

We only need $\nu \geq 1$ here. Clearly, the last integral is bounded above by $2^{\nu-1/2}\Gamma(\nu+1/2)$ so that

$$I_\nu(z) \leq c z^{-1/2} e^z, \quad z > 0, \nu \geq 1. \tag{5}$$

Let us recall that ([8])

$$I_\nu(z) \asymp \frac{e^{\sqrt{\nu^2+z^2}}}{\sqrt{z+\nu}} \left(\frac{z}{\nu+\sqrt{\nu^2+z^2}}\right)^\nu, \quad z > 0, \nu \geq 1/2. \tag{6}$$

Assume that $z > \max(1, \nu^2/a)$ with some $a > 0$ and $\nu > 1$. Thus, $\sqrt{\nu^2+z^2} - z \leq a/2$ so that $\exp(\sqrt{\nu^2+z^2}) \asymp \exp(z)$ (in the upper bound there is a constant that depends on a, the lower bound holds with constant 1). Clearly, $\sqrt{z+\nu} \asymp \sqrt{z}$ and the quotient in the parentheses in (6) is bounded above by 1. Further,

$$\begin{aligned} \frac{z^\nu}{(\nu+\sqrt{\nu^2+z^2})^\nu} &\geq \frac{1}{(\sqrt{a}/\sqrt{z}+\sqrt{1+a/z})^{\sqrt{az}}} \\ &\geq \frac{1}{(1+2\sqrt{a}/\sqrt{z})^{\sqrt{z}/(2\sqrt{a})\times 2a}} \geq \frac{1}{e^{2a}}. \end{aligned}$$

Consequently, we obtain the desired simplification

$$I_\nu(z) \asymp z^{-1/2} e^z, \quad z > \max\bigl(1, \nu^2/a\bigr), \nu \geq 1. \tag{7}$$

We recall the exact estimates of the densities $\eta_t(\cdot)$ which will be fundamental in what follows (see, e.g., [17]). We have

$$\eta_t(u) \asymp t^{\frac{1}{2-\alpha}} u^{-\frac{4-\alpha}{4-2\alpha}} \exp\bigl(-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}\bigr), \quad t^{-2/\alpha} u \leq c, \tag{8}$$

where

$$c_1 = c_1(\alpha) = \frac{2-\alpha}{2}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2-\alpha}}$$

and

$$\eta_t(u) \asymp t u^{-1-\alpha/2}, \quad t^{-2/\alpha} u > c. \tag{9}$$

According to (8) and (9), it is convenient to split the integral (3) as follows

$$p_t(x) = \int_0^{c_0 t^{2/\alpha}} h_u(x)\eta_t(u)\,du + \int_{c_0 t^{2/\alpha}}^{\infty} h_u(x)\eta_t(u)\,du \tag{10}$$

$$\begin{aligned} &\asymp \phi_0(x) t^{\frac{1}{2-\alpha}} \int_0^{c_0 t^{2/\alpha}} e^{-u} I_{1+|x|}(\gamma u) \\ &\quad \times u^{-\frac{4-\alpha}{4-2\alpha}-1} \exp\bigl(-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}\bigr)\,du \\ &\quad + \phi_0(x) t \int_{c_0 t^{2/\alpha}}^{\infty} e^{-u} I_{1+|x|}(\gamma u) u^{-2-\alpha/2}\,du \\ &\stackrel{\text{def}}{=} \phi_0(x)(\mathbf{A}^{(x,t)} + \mathbf{B}^{(x,t)}). \end{aligned} \tag{11}$$

Fix $K > 0$. We assume that $c_0 = 1$ and $|x| \le K\sqrt{t}$ with x and t large enough. Note that neither x, nor t is fixed. It follows that $(1 + |x|)^2 \le (1 + K\sqrt{t})^2 \le \gamma t^{2/\alpha}$. Hence, by (7) with $a = 1$ we get

$$I_{1+|x|}(\gamma u) \le c u^{-1/2} e^{\gamma u}, \quad u > t^{2/\alpha}.$$

In consequence,

$$\begin{aligned} \mathbf{B}^{(x,t)} &\le ct \int_{t^{2/\alpha}}^{\infty} e^{-(1-\gamma)u} u^{-(5+\alpha)/2}\,du \\ &\le c t^{-5/\alpha} \int_{t^{2/\alpha}}^{\infty} e^{-(1-\gamma)u}\,du \\ &= c t^{-5/\alpha} e^{-(1-\gamma)t^{2/\alpha}}. \end{aligned}$$

To estimate $\mathbf{A}^{(x,t)}$ let us split it as follows

$$\begin{aligned} \mathbf{A}^{(x,t)} &= t^{\frac{1}{2-\alpha}} \left(\int_0^{\alpha t/2} + \int_{\alpha t/2}^{t^{2/\alpha}} \right) e^{-u} I_{1+|x|}(\gamma u) \\ &\quad \times u^{-\frac{4-\alpha}{4-2\alpha}-1} \exp\bigl(-c_1 t^{\frac{2}{2-\alpha}} u^{-\frac{\alpha}{2-\alpha}}\bigr)\,du \\ &= \mathbf{A}_1^{(x,t)} + \mathbf{A}_2^{(x,t)}. \end{aligned}$$

We apply (7) to the integral $\mathbf{A}_2^{(x,t)}$. After the change of variable $u \to tu$ we get

$$\mathbf{A}_2^{(x,t)} \asymp c t^{-1} \int_{\frac{\alpha}{2}}^{t^{\frac{2}{\alpha}-1}} u^{-\frac{4-\alpha}{4-2\alpha}-\frac{3}{2}} \exp\Bigl(-t\bigl((1-\gamma)u + c_1 u^{-\frac{\alpha}{2-\alpha}}\bigr)\Bigr)\,du. \tag{12}$$

Observe that the minimum of function $p(u) = (1-\gamma)u + c_1 u^{-\alpha/(2-\alpha)}$ is attained at $u_0 = (1-\gamma)^{-(1-\alpha/2)}(\frac{\alpha c_1}{2-\alpha})^{1-\alpha/2}$. Putting in c_1 we get

$$u_0 = \frac{\alpha/2}{(1-\gamma)^{1-\alpha/2}} \quad \text{and} \quad p(u_0) = (1-\gamma)^{\alpha/2}.$$

For t large enough u_0 is in the integration range. The integral (12) is bounded by integrals with the following limits independent of t

$$\int_{\frac{\alpha}{2}}^{u_0} \le \int_{\frac{\alpha}{2}}^{t^{\frac{2}{\alpha}-1}} \le \int_0^{\infty}.$$

The Laplace method [21] applied to the extreme members of this inequality gives the same asymptotic result

$$ct^{-1/2}e^{-tp(u_0)} \quad \text{as } t \to \infty.$$

Consequently,

$$\mathbf{A}_2^{(x,t)} \asymp t^{-3/2}\exp\bigl(-(1-\gamma)^{\alpha/2}t\bigr), \quad |x| < K\sqrt{t}, t \ge 1.$$

Similarly, using (5) we get

$$\mathbf{A}_1^{(x,t)} \le ct^{-1}\int_0^{\alpha/2} u^{-\frac{4-\alpha}{4-2\alpha}-\frac{3}{2}}\exp\Bigl(-t\bigl((1-\gamma)u + c_1 u^{-\frac{\alpha}{2-\alpha}}\bigr)\Bigr)\,du.$$

Since the minimum of $p(u)$ is *not* attained in $(0,\alpha/2)$ and p is nondecreasing in this interval, the Laplace method gives the following upper bound:

$$\mathbf{A}_1^{(x,t)} \le ct^{-2}\exp\bigl(-p(\alpha/2)t\bigr).$$

It follows that $p_t(x) \asymp \mathbf{A}_2^{(x,t)}$ and the first of the desired estimates follows.

Now, assume that $|x| > Mt^{2/\alpha}$. We put $c_0 = aM$ in the decomposition (11), where $a \in (0,1)$ is to be specified later. To simplify the notation, first we estimate the integral obtained by replacing $1+|x|$ by $|x|$ in the definition of $\mathbf{A}^{(x,t)}$. By (6) and the elementary inequalities $e^{\sqrt{|x|^2+\gamma^2u^2}} \le e^{|x|+\gamma u}$, $|x| + \sqrt{|x|^2+\gamma^2u^2} \ge 2|x|$, we get

$$\begin{aligned}
&t^{\frac{1}{2-\alpha}}\int_0^{aMt^{2/\alpha}} e^{-u}I_{|x|}(\gamma u)u^{-\frac{4-\alpha}{4-2\alpha}-1}e^{-c_1t^{\frac{2}{2-\alpha}}u^{-\frac{\alpha}{2-\alpha}}}\,du \\
&\quad\le c|x|^{\frac{\alpha}{4-2\alpha}}\int_0^{a|x|}\frac{e^{\sqrt{|x|^2+\gamma^2u^2}-u}(\gamma u)^{|x|}u^{-\frac{4-\alpha}{4-2\alpha}-1}e^{-c_1t^{\frac{2}{2-\alpha}}u^{-\frac{\alpha}{2-\alpha}}}}{\sqrt{|x|+\gamma u}(|x|+\sqrt{|x|^2+\gamma^2u^2})^{|x|}}\,du \\
&\quad\le c|x|^{\frac{\alpha}{4-2\alpha}-\frac{1}{2}}\left(\frac{ae\gamma}{2}\right)^{|x|}\int_0^{a|x|} e^{-(1-\gamma)u}u^{-\frac{4-\alpha}{4-2\alpha}-1}e^{-c_1t^{\frac{2}{2-\alpha}}u^{-\frac{\alpha}{2-\alpha}}}\,du.
\end{aligned}$$

Clearly, the last integral is convergent and bounded above by a constant independent of $|x|$. Therefore, for $t \geq 1$ we have

$$\mathbf{A}^{(x,t)} \leq c(|x|+1)^{\frac{\alpha}{4-2\alpha}-\frac{1}{2}}\left(\frac{ae\gamma}{2}\right)^{|x|+1} \leq c|x|^{\frac{\alpha}{4-2\alpha}-\frac{1}{2}}\left(\frac{ae\gamma}{2}\right)^{|x|}. \tag{13}$$

On the other hand, again by (6) and the change of variable $u \to u|x|$, we obtain

$$\begin{aligned}
\mathbf{B}^{(x,t)} &= t\int_{aMt^{2/\alpha}}^{\infty} I_{|x|}(\gamma u)u^{-2-\alpha/2}e^{-u}\,du \\
&\geq ct\int_{a|x|}^{\infty} \frac{e^{\sqrt{|x|^2+\gamma^2u^2}-u}}{\sqrt{|x|+\gamma u}}\frac{(\gamma u)^{|x|}u^{-2-\alpha/2}}{(|x|+\sqrt{|x|^2+\gamma^2u^2})^{|x|}}\,du \\
&\geq ct\gamma^{|x|}|x|^{-\frac{\alpha+3}{2}}\int_a^{\infty}\frac{e^{|x|(\sqrt{1+\gamma^2u^2}-u)}u^{-2-\alpha/2+|x|}}{\sqrt{1+\gamma u}(1+\sqrt{1+\gamma^2u^2})^{|x|}}\,du \\
&\asymp t\gamma^{|x|}|x|^{-\frac{\alpha+3}{2}} \\
&\quad\times\int_a^{\infty} e^{|x|(\sqrt{1+\gamma^2u^2}-u+\log(u)-\log(1+\sqrt{1+\gamma^2u^2}))}\frac{u^{-2-\alpha/2}}{\sqrt{1+\gamma u}}\,du.
\end{aligned}$$

Observe that a similar computation with the lower limit of integration equal to 0 gives

$$\begin{aligned}
\mathbf{B}^{(x,t)} &\leq t\gamma^{|x|}|x|^{-\frac{\alpha+3}{2}} \\
&\quad\times\int_0^{\infty} e^{|x|(\sqrt{1+\gamma^2u^2}-u+\log(u)-\log(1+\sqrt{1+\gamma^2u^2}))}\frac{u^{-2-\alpha/2}}{\sqrt{1+\gamma u}}\,du.
\end{aligned}$$

Let

$$p(u) = \sqrt{1+\gamma^2u^2} - u + \log(u) - \log\big(1+\sqrt{1+\gamma^2u^2}\big)$$

and $g = \sqrt{1+\gamma^2u^2}$. Then $p'(u) = -1 + g/u$ and, consequently, $p(u)$ attains the maximum at $u_1 = \frac{q+1}{q-1} > 1$. Hence, u_1 belongs to the integration range for integrals in both upper and lower bound for $\mathbf{B}^{(x,t)}$. Consequently, by the Laplace method, both of them have the same asymptotic as $|x| \to \infty$. Since

$$\begin{aligned}
p(u_1) &= \sqrt{1+\frac{4q}{(q-1)^2}} - \frac{q+1}{q-1} + \log\left(\frac{q+1}{(q-1)(1+\sqrt{1+\frac{4q}{(q-1)^2}})}\right) \\
&= -\log\left(\frac{2q}{q+1}\right) = -\log(\gamma\sqrt{q}),
\end{aligned}$$

it follows that

$$\mathbf{B}^{(x,t)} \asymp t|x|^{-2-\alpha/2}e^{|x|(\log\gamma-\log(\gamma\sqrt{q}))} = t|x|^{-2-\alpha/2}q^{-|x|/2},$$

if $|x| \geq Mt^{2/\alpha}$ and $|x|$ is large enough (and hence for $|x| > 1$). Moreover, if we take $a = 1/e$ then $ae\gamma/2 \leq q^{-1/2}$ so that $\mathbf{A}^{(x,t)} = o(\mathbf{B}^{(x,t)})$, $|x| \to \infty$ and $p_t(x) \asymp \mathbf{B}^{(x,t)}$. The assertion follows. □

REMARK 1. Our theorem can be compared with the following result of [14]. For reader's convenience we give it below, specialized to the real hyperbolic space $\mathbb{H}^n$. The corresponding α-stable kernel and spherical function are denoted with the tilde.

THEOREM ([14], Corollary 5.6). *Let* $|\rho| = (n-1)/2$. *If* $K, M > 0$ *and* $t + |x| > 1$ *then*

$$\tilde{p}_t(x) \asymp \begin{cases} \tilde{\phi}_0(x) t^{-3/2} e^{-|\rho|^\alpha t}, & |x| \leq Kt^{1/2} \\ \tilde{\phi}_0(x) t|x|^{-2-\alpha/2} e^{-|\rho||x|}, & |x| \geq Mt^{2/\alpha}. \end{cases} \tag{14}$$

In the context of hyperbolic space (or, more generally, symmetric space of noncompact type), the parameter $|\rho|$ plays a double role: it is the square root of the bottom of the spectrum of the Laplace–Beltrami operator; at the same time, the volume of the ball of the radius r is equivalent to $e^{2|\rho|r}$ as $r \to \infty$. One may ask, whether it is the spectral data or the geometry which appears in the above estimates. The comparison with Theorem 3.1 gives us a natural interpretation: in the first part (i.e., in the long time asymptotics) we deal with the spectral data, in the other case the volume growth intervenes.

REMARK 2. Note that for the remaining region $Kt^{1/2} < |x| < Mt^{2/\alpha}$, in the continuous setting there is no *simple homogeneous* estimate of $\tilde{p}_t(x)$ (see [14], Corollary 5.6).

Before we state our next result for stable processes on trees, we provide some motivations and classical background. The Brownian motion and α-stable processes in $\mathbb{R}^d$ share the same type of long time heat repartition. Namely, with the standard understanding that $\alpha = 2$ corresponds to the Brownian motion, for $A_1 < A_2$ we have

$$\int_{A_1 t^{1/\alpha} \leq |x| \leq A_2 t^{1/\alpha}} p_t(x)\, dx = c(A_1, A_2) \in (0,1).$$

This follows immediately from the scaling property

$$p_t(x) = t^{-d/\alpha} p_1\big(t^{-1/\alpha} x\big). \tag{15}$$

Moreover, $c(A_1, A_2) \to 1$ if $A_1 \to 0$ and $A_2 \to \infty$ so that

$$\int_{A_1 t^{\beta} \leq |x| \leq A_2 t^{\beta}} p_t(x)\, dx \to 0, \quad t \to \infty, \tag{16}$$

provided $\beta \neq 1/\alpha$ (cf. [2], p. 50).

On the other hand, for the Brownian motion in the real hyperbolic space $\mathbb{H}^n$, a nonclassical phenomenon of concentration was observed in [9]. Namely,

$$\int_{A_1 t\le|x|\le A_2 t} h_t(x)\,dx \to 1, \quad t\to\infty,$$

provided $A_1 < n-1 < A_2$. Notice the change of the "space-time scaling", that is, t instead of $t^{1/2}$ in the integration bounds, as compared to (16) with $\alpha=2$. This result was sharpened and generalized to symmetric space setting ([2], [3]). In the context of homogeneous trees an analogous result was shown in [20] and [23]:

$$\sum_{R_0 t-r(t)\le|x|\le R_0 t+r(t)} h_t(x)\to 1, \quad t\to\infty,$$

where $R_0=(q-1)/(q+1)$ and $r(t)$ is a positive function such that $r(t)t^{-1/2}\to\infty$, $t\to\infty$. This might suggest a hypothesis of the same kind for our kernel $p_t(x)$, for example, the asymptotic concentration of the mass of the heat kernel on the region $\{A_1 t^{2/\alpha}\le|x|\le A_2 t^{2/\alpha}\}$. The following theorem shows that the actual behavior of the kernel is different.

THEOREM 3.2. *For* $0<A_1<A_2$ *let* $R(t)=\{x\in X: A_1 t^{2/\alpha}\le|x|\le A_2 t^{2/\alpha}\}$. *Then there exist* c_1 *and* c_2 *such that*

$$0<c_1<\sum_{x\in R(t)} p_t(x)<c_2<1, \quad t\to\infty. \tag{17}$$

Conversely, for any given $0<c_1<c_2<1$ *there exist* A_1 *and* A_2 *such that* (17) *holds true.*

Proof. Set $R_0=(q-1)/(q+1)$ and let R_1, R_2 be such that $R_1<R_0<R_2$. Then, by Theorem 1 of [20], we have

$$\sum_{R_1 u\le|x|\le R_2 u} h_u(x)\to 1, \quad u\to\infty. \tag{18}$$

Moreover, let $c_3=A_1/R_1$ and $c_4=A_2/R_2$. We require additionally that R_1 and R_2 be close to R_0 so that $c_3<c_4$. Then $c_3 t^{2/\alpha}<u<c_4 t^{2/\alpha}$ yields

$$|x|\in(R_1 u, R_2 u) \quad\Longrightarrow\quad x\in R(t). \tag{19}$$

From the definition of $p_t(x)$, (18) and (19), we get

$$\begin{aligned}\sum_{x\in R(t)} p_t(x) &= \int_0^\infty\Big(\sum_{x\in R(t)} h_u(x)\Big)\eta_t(u)\,du\\ &\ge \int_{c_3 t^{2/\alpha}}^{c_4 t^{2/\alpha}}\Big(\sum_{R_1 u\le|x|\le R_2 u} h_u(x)\Big)\eta_t(u)\,du\\ &\to \int_{c_3 t^{2/\alpha}}^{c_4 t^{2/\alpha}} \eta_t(u)\,du, \quad t\to\infty.\end{aligned}$$

Formally, the last integral depends on t. By the scaling property (15), however, it equals

$$t^{-2/\alpha}\int_{c_3t^{2/\alpha}}^{c_4t^{2/\alpha}}\eta_1\big(t^{-2/\alpha}u\big)\,du=\int_{c_3}^{c_4}\eta_1(u)\,du=c_0. \tag{20}$$

This is an absolute constant which depends on c_3, c_4 and α only. The lower bound in (17) follows. Since the lower bound holds for *any* $A_1<A_2$, the mass of the annulus $R(t)$ (with A_1 and A_2 fixed) is strictly less than 1. In other words, $c_2<1$ in (17) and there is no mass concentration. The proof of (17) is complete.

By choosing R_1 (R_2, respectively) sufficiently close to R_0, we get c_3 (c_4, resp.) arbitrarily close to A_1/R_0 (A_2/R_0, resp.) so that

$$\lim_{t\to\infty}\sum_{x\in R(t)}p_t(x)\geq\int_{A_1/R_0}^{A_2/R_0}\eta_1(u)\,du. \tag{21}$$

Since (21) holds for any $A_1<A_2$ and $t>0$ we have $\sum_{x\in X}p_t(x)=1=\int_0^\infty\eta_1(u)\,du$, it follows that we have the equality in (21). The proof is complete. $\square$

The following corollary is an analogue of the classical counterpart (16).

COROLLARY 3.3. *For $0<\tilde{A}_1<\tilde{A}_2$ and some $\beta>0$ let $\bar{R}(t)=\{x\in X:\tilde{A}_1t^\beta\leq|x|\leq\tilde{A}_2t^\beta\}$. If $\beta\neq 2/\alpha$ then*

$$\sum_{x\in\bar{R}(t)}p_t(x)\longrightarrow 0,\quad t\to\infty. \tag{22}$$

Proof. For t large enough, $R(t)$ and $\bar{R}(t)$ are disjoint. $\square$

COROLLARY 3.4. *Let $\nu(x):=\lim_{t\to 0}p_t(x)/t$ be the the Lévy measure density for our semigroup. Then*

$$\nu(x)\asymp|x|^{-1-\alpha/2}q^{-|x|},\quad |x|\geq 1.$$

Proof. From Theorem 3.1 and (1) we get

$$\nu(x)\asymp\phi_0(x)|x|^{-2-\alpha/2}q^{-|x|/2}\asymp|x|^{-1-\alpha/2}q^{-|x|}. \qquad\square$$

REMARK 3. Evidently, the summation bounds (space-time scaling) in (17) is characteristic for the Brownian motion in hyperbolic spaces and homogeneous trees. However, the concentration phenomenon is not observed. From the probabilistic point of view this may be explained by the influence of the long jumps of the corresponding stable process. Indeed, the Lévy measure density is of the same exponential order as volume growth because it arises from the second estimate in (4).

Below we include an alternative approach that relies directly on the α-stable kernel estimates (4). It shows that the mass of the region in Theorem 3.1 is large enough to be useful in some applications.

Second proof of (17). For $x \in R(t)$ we have

$$p_t(x) \asymp t\phi_0(x)|x|^{-2-\alpha/2}q^{-|x|/2}. \tag{23}$$

By (1),

$$\phi_0(x) \asymp |x|q^{-|x|/2}, \quad |x| \to \infty.$$

Therefore,

$$\sum_{x\in R(t)} p_t(x) \asymp t \sum_{x\in R(t)} |x|^{-1-\alpha/2}q^{-|x|}, \quad \text{as } t \to \infty.$$

We use an analogue of polar coordinates. At each sphere $\{|x| = n\}$ we have exactly $(q+1)q^{n-1}$ vertices, so that

$$\sum_{x\in R(t)} p_t(x) \asymp t \sum_{A_1 t^{2/\alpha} \le n \le A_2 t^{2/\alpha}} n^{-1-\alpha/2} \asymp t \int_{A_1 t^{2/\alpha}}^{A_2 t^{2/\alpha}} y^{-1-\alpha/2}\, dy.$$

Clearly, the last integral behaves as

$$\bigl(A_1^{-\alpha/2} - A_2^{-\alpha/2}\bigr)(\alpha/2)^{-1}t^{-1}, \quad t \to \infty. \tag{24}$$

Thus, we get the lower bound in (17). Since this holds true for any any $A_1 < A_2$, as before, the upper bound by a constant $c_2 < 1$ follows. The proof of (17) is complete. □

Remark. This direct argument enables us to prove (22) for $\beta > 2/\alpha$ as well. Indeed, (23) holds also for $\bar{R}(t)$ with $\beta > 2/\alpha$. In this case, (24) implies that

$$\sum_{x\in \bar{R}(t)} p_t(x) \to 0, \quad t \to \infty.$$

However, the argument fails for $\beta < 2/\alpha$. (Actually, if (23) held for $|x| \ge At^\beta$ with some $\beta < 2/\alpha$, then we would obtain $t^{-\alpha\beta/2}$ in (24). Consequently, the mass of the annulus goes to infinity, which is impossible.)

The proof of Theorem 3.2 with only minor modifications can be applied in the context of the symmetric spaces of noncompact type with Theorem 1 of [2] instead of (18). We prefer, however, to take the opportunity given by Theorem 2 of that article to state our result in the a general setting of manifolds. For reader's convenience, we recall the framework. We assume that M is a complete, noncompact Riemannian manifold with the volume growth controlled by

$$\operatorname{vol}\bigl(B(x,r)\bigr) \asymp r^\kappa e^{2Kr}, \quad r \to \infty,$$

with some positive constants κ and K, and the spectral gap $E^2 = \inf \operatorname{spec}(-\Delta) > 0$. In general we have $E \le K$, while for the symmetric spaces of noncompact type $E = K = |\rho|$. Set $R_1 = 2(K - \sqrt{K^2 - E^2})$, $R_2 = 2(K + \sqrt{K^2 - E^2})$. Let $A(t)$ be a function such that

$$\begin{aligned} &A(t) - \frac{\kappa - 1}{2\sqrt{K^2 - E^2}} \log t \nearrow \infty && \text{if } K < E, \\ &A(t) = (2\kappa t \log t)^{1/2} && \text{if } K = E \text{ and } \kappa > 0, \\ &A(t) t^{-1/2} \nearrow \infty && \text{if } K = E \text{ and } \kappa = 0. \end{aligned}$$

Let $h(x,y)$ denote the heat kernel on M (see [2] for more details). By Theorem 2 from [2]

$$\int_{R_1 t - A(t) \le d(x,y) \le R_2 t + A(t)} h_t(x,y) \to 1, \quad t \to \infty.$$

Note that we may and do require $A(t) = o(t)$, which is essential for our proof to work (cf. (19)). Since the heat kernel h depends on two variables, we fix an arbitrary $y \in M$ and redefine slightly $R(t) = \{x \in X : A_1 t^{2/\alpha} \le d(x,y) \le A_2 t^{2/\alpha}\}$ for any $0 < A_1 < A_2$. We obtain the following corollary.

Corollary 3.5. *If* $0 < A_1 < A_2$, *then there exist* c_1 *and* c_2 *such that*

$$0 < c_1 < \int_{R(t)} p_t(x)\, dx < c_2 < 1, \quad t \to \infty. \tag{25}$$

Conversely, for any $0 < c_1 < c_2 < 1$ *there exist* $0 < A_1 < A_2$ *such that* (25) *holds true.*

4. Exit time

We conclude our work by giving an application of Theorem 3.1. Since the results below are very similar for both homogeneous trees and hyperbolic spaces, we will use the following notation of metric spaces.

Let (E, d) be a locally compact separable metric space and μ be a Radon measure with full support. Suppose that E admits a *fractional diffusion* (X_t, P_x) [3] with a *heat kernel* $p_t(x,y)$ in the sense of the axiomatic Definition 2.1 of [16] (see also [22]). For the reader's convenience, we recall it shortly. We assume that $p_t(\cdot,\cdot)$ is a $\mu \times \mu$ nonnegative measurable function and for μ-almost all $x, y \in E$ and all $s, t > 0$ we have $p_t(x,y) = p_t(y,x)$,

$$\int_E p_t(x,y)\, d\mu(y) = 1, \qquad p_{t+s}(x,y) = \int_E p_t(x,z) p_s(z,y)\, d\mu(z),$$

and for each $u \in L^2(E,\mu)$

$$\int_E p_t(x,y) u(y)\, d\mu(y) \xrightarrow{L^2} u(x), \quad t \to 0^+.$$

In the case of the hyperbolic spaces or homogeneous trees, we have $p_t(x,y) = p_t(d(x,y))$, where $d(x,y)$ is the distance.

This kernel is the transition density for the process, that is,

$$P_x[X_t \in B] = \int_B p_t(x,y)\,d\mu(y).$$

For simplicity, we suppose that the space is homogeneous, that is, there exists a function $V(r)$, called a volume growth, such that $V(r) = \mu(B(x,r))$, $x \in E$. It can be seen that for the proofs below this assumption is not essential and we could deal with nonhomogeneous version $V(x,r)$ as well.

Further, assume that there exist $A \geq 1$ and $c_1 < 1$ such that

$$V(r) \leq c_1 V(r+A) \quad \text{and} \quad V(r+1) \asymp V(r), \quad r \geq 1. \tag{26}$$

Note that (26) is satisfied in the case of trees and hyperbolic spaces (with e.g. $A = 1$). Suppose also that for any $M > 0$

$$p_t(x,y) \asymp t d(x,y)^{-1-\alpha/2} V\big(d(x,y)\big)^{-1}, \tag{27}$$

provided $d(x,y) > Mt^{2/\alpha}$ and $d(x,y) > 1$. The condition is satisfied in the context of trees and hyperbolic spaces as well (cf. Theorem 3.1 and (14) resp.).

Note that the first part of (26) implies that $\lim_{r\to\infty} V(r) = \infty$. In particular, our space is not bounded. Below, we use this fact without further mention.

PROPOSITION 4.1. *For any $M > 0$ and $r > 1$, we have*

$$P_x\big[X_t \notin B(x,r)\big] \asymp tr^{-\alpha/2}, \quad r > Mt^{2/\alpha}.$$

Proof. By (27), we get

$$\begin{aligned}
&P_x\big[X_t \notin B(x,r)\big] \\
&\quad\asymp \int_{d(x,y)>r} p_t(x,y)\,d\mu(y) \\
&\quad\asymp t\sum_{k=0}^{\infty}\int_{r+k<d(x,y)\leq r+k+1} d(x,y)^{-1-\alpha/2} V\big(d(x,y)\big)^{-1}\,d\mu(y) \\
&\quad\leq ct\sum_{k=0}^{\infty}(r+k)^{-1-\alpha/2} V(r+k)^{-1}\big(V(r+k+1) - V(r+k)\big).
\end{aligned}$$

Clearly, by (26) we get

$$V(r+k)^{-1}\big(V(r+k+1) - V(r+k)\big) \leq c.$$

Moreover, by a comparison of the series with the corresponding integral it can be easily seen that

$$\begin{aligned}
\sum_{k=0}^{\infty}(r+k)^{-1-\alpha/2} &= r^{-1-\alpha/2} + \sum_{k=1}^{\infty}(r+k)^{-1-\alpha/2} \\
&\leq r^{-\alpha/2} + \int_r^{\infty} z^{-1-\alpha/2}\,dz \leq cr^{-\alpha/2}
\end{aligned}$$

and the upper bound in the assertion follows.

On the other hand, we have similarly

$$\begin{aligned}
&P_x\big[X_t \notin B(x,r)\big] \\
&\quad \asymp t\sum_{k=0}^{\infty}\int_{r+kA<d(x,y)\le r+(k+1)A} d(x,y)^{-1-\alpha/2}V\big(d(x,y)\big)^{-1}\,d\mu(y) \\
&\quad \ge ct\sum_{k=0}^{\infty}(r+kA+A)^{-1-\alpha/2}\frac{V(r+kA+A)-V(r+kA)}{V(r+kA+A)}.
\end{aligned}$$

Again, by (26)

$$\frac{V(r+kA+A)-V(r+kA)}{V(r+kA+A)} = 1-\frac{V(r+kA)}{V(r+kA+A)} \ge 1-c_1>0.$$

Moreover,

$$\sum_{k=0}^{\infty}(r+kA+A)^{-1-\alpha/2} \ge \int_{r+A}^{\infty} z^{-1-\alpha/2}\,dz = c(r+A)^{-\alpha/2} \ge cr^{-\alpha/2},$$

since $r>1$. The proof is complete. □

For a measurable set D define the exit time $\tau_D = \inf\{t\ge 0; X_t\notin D\}$. Then we have the following proposition.

PROPOSITION 4.2. *For any $M>0$ and $r>1$, we have*

$$P_x[\tau_{B(x,r)}<t] \le ctr^{-\alpha/2}, \quad r>Mt^{2/\alpha}.$$

Proof. The proof follows the lines of [4] (or [6]). Since it is short, we sketch it for the reader's convenience. Denote $T=\tau_{B(x,2r)}$. Then

$$\begin{aligned}
P_x[T<t] &= P_x\big[X_t\notin B(x,r); T<t\big] + P_x\big[X_t\in B(x,r); T<t\big] \\
&\le P_x\big[X_t\notin B(x,r)\big] + P_x\big[X_t\in B(x,r); T<t\big] = A+B.
\end{aligned}$$

By Proposition 4.1 we obtain $A\le ctr^{-\alpha/2}$. By the strong Markov property, we have

$$\begin{aligned}
B &= E_x\big[P_{X(T)}\big[X_{t-u}\in B(x,r)\big]_{|u=T}; T<t\big] \\
&\le \sup_{u\le t}\sup_{z\in B(x,2r)^c} E_x\big[P_z\big[X_u\in B(x,r)\big]; T<t\big] \\
&\le \sup_{u\le t}\sup_{z\in B(x,r)^c} E_x\big[P_z\big[X_u\notin B(z,r)\big]; T<t\big] \\
&\le ctr^{-\alpha/2}.
\end{aligned}$$

The result follows. □

THEOREM 4.3. *For* $r > 1$,

$$E_y \tau_{B(x,r)} \le c r^{\alpha/2}, \quad y \in B(x,r)$$

and

$$E_x \tau_{B(x,r)} \asymp r^{\alpha/2}.$$

Proof. For any $y \in B(x,r)$ by Proposition 4.1, we have

$$P_y[\tau_{B(x,r)} > t] \le P_y\big[X_t \in B(x,r)\big] = 1 - P_y\big[X_t \notin B(x,r)\big] \le 1 - ctr^{-\alpha/2}$$

provided that $r > Mt^{2/\alpha}$ with some $M > 0$. Let $t_0 = r^{\alpha/2}$ so that for some c_0 we get

$$P_y[\tau_{B(x,r)} > t_0] \le 1 - c_0. \tag{28}$$

Then, by Markov property, for $k = 1, 2, \dots$ we have

$$\begin{aligned} P_y\big[\tau_{B(x,r)} > (k+1)t_0\big] &= P_y[\tau_{B(x,r)} \circ \theta_{t_0} > kt_0, \tau_{B(x,r)} > t_0] \\ &= E_y\big[P_{X(t_0)}[\tau_{B(x,r)} > kt_0]; \tau_{B(x,r)} > t_0\big] \\ &\le P_y[\tau_{B(x,r)} > t_0] \sup_{z \in B(x,r)} P_z[\tau_{B(x,r)} > kt_0] \end{aligned}$$

(here θ stands for the standard shift operator on the space of trajectories). By induction we get

$$P_y[\tau_{B(x,r)} > kt_0] \le (1 - c_0)^k, \quad y \in B(x,r), k = 0, 1, 2, \dots$$

Thus,

$$\begin{aligned} E_y \tau_{B(x,r)} &= \int_0^\infty P_y[\tau_{B(x,r)} > t]\, dt \\ &\le \sum_{k=0}^\infty t_0 P_y[\tau_{B(x,r)} > kt_0] \le r^{\alpha/2} \sum_{k=0}^\infty (1 - c_0)^k \end{aligned}$$

and the upper bound in the assertion follows.

On the other hand, let $t_1 = c_1 r^{\alpha/2}$ with c_1 to be specified below. From Proposition 4.2, we get

$$P_x[\tau_{B(x,r)} < t_1] \le c_1 c_2.$$

Observe that the constant c_2 above does not depend on c_1 provided $c_1 < 1$. Hence, we may and do choose c_1 small enough to get $c_1 c_2 < 1$. It follows that

$$E_x \tau_{B(x,r)} \ge t_1 P_x[\tau_{B(x,r)} > t_1] \ge (1 - c_1 c_2) t_1 \asymp r^{\alpha/2}$$

The proof is complete. □

5. Poisson kernel

In this section, we give estimates for the Poisson kernel for balls. Since in general our development follows ideas of [6], we give only a short sketch of the construction. For more detailed exposition, we refer the reader to Sections 5 and 6 of that article. Since the results in what follows are similar for both the homogeneous trees and hyperbolic spaces, we continue to use the notation introduced in the previous section.

In what follows, we assume that for $x,y\in X$ the following limit exists

$$N(x,y)=\lim_{t\to 0}\frac{p_t(x,y)}{t}>0.$$

This is verified whenever our α-stable kernel arises by a subordination of a reasonable diffusion with η_t described above. Clearly, the case of homogeneous trees and hyperbolic spaces is included. From (27) it follows that

$$N(x,y)\asymp d(x,y)^{-1-\alpha/2}V\bigl(d(x,y)\bigr)^{-1},\quad d(x,y)\geq 1. \tag{29}$$

Let

$$n(x,E)=\int_E N(x,y)\,d\mu(y). \tag{30}$$

For an open set D let (P_t^D) be the semigroup generated by the process killed on exiting D, i.e.

$$P_t^D f(x)=E_x\bigl[f(X_t);t<\tau_D\bigr].$$

This semigroup possesses transition densities denoted by $p_t^D(x,y)$ (see [7]; the argument applies here as well). Let $G_D(x,y)$ be the Green function for D, that is, the potential for (P_t^D):

$$G_D(x,y)=\int_0^\infty p_t^D(x,y)\,dt.$$

With these definitions, one verifies the assumptions of the following Ikeda–Watanabe formula (see [6] or [18]). For homogeneous trees and hyperbolic spaces, this is straightforward and we omit the details. We get

Proposition 5.1 (Ikeda–Watanabe formula). *Assume that $D\subset X$ is an open nonempty bounded set, $E\subset X$ is a Borel set and* $\operatorname{dist}(D,E)>0$. *Then*

$$P_x[X_{\tau_D}\in E]=\int_D G_D(x,y)n(x,E)\,d\mu(y).$$

In particular, by (30) we get that $P_x[X_{\tau_D}\in\cdot]$ is absolutely continuous w.r. to μ on $(\bar{D})^c$ (this is nontrivial only for the hyperbolic spaces). Let $P_D(x,\cdot)$ denote the density of the measure (i.e., Poisson kernel).

Proposition 5.2. *For any $x_0\in X$ and $r\geq 1$ let $D=B(x_0,r)$. Then*

$$P_D(x,z)\leq c\frac{r^{\alpha/2}V(2r)}{d(x,z)^{1+\alpha/2}V(d(x,z))},\quad z\in B(x_0,3r)^c, x\in D.$$

If $r \geq 2$ *then*

$$P_D(x,z) \geq c\frac{r^{\alpha/2}}{V(2r)d(x,z)^{1+\alpha/2}V(d(x,z))}, \quad z \in D^c, x \in B(x_0, r/2).$$

Proof. By (29), we have

$$P_D(x,z) \asymp \int_D \frac{G_D(x,y)}{d(y,z)^{1+\alpha/2}V(d(y,z))}\, d\mu(y). \tag{31}$$

Clearly, $d(y,z) \asymp d(x,z)$. Moreover, for the hyperbolic spaces and homogeneous trees we have $V(r) \asymp C_1^r$ where C_1 depends on the dimension or the degree, respectively. It follows that

$$V\big(d(y,z)\big) \geq V\big(d(x,z) - d(x,y)\big) \geq V\big(d(x,z) - 2r\big) \asymp V(2r)^{-1}V\big(d(x,y)\big).$$

Since $\int_D G_D(x,y)\, d\mu(y) = E_x\tau_D$ the upper bound in the assertion follows by Theorem 4.3.

On the other hand, fix $x \in B(x_0, r/2)$. Then $d(y,z) \leq cd(x,z)$, $y \in D$, $z \in D^c$. Similarly as before, $V(d(y,z)) \leq V(d(y,x) + d(x,z)) \asymp V(2r)V(d(x,z))$. Moreover, $E_x\tau_D \geq E_x\tau_{B(x,r/2)} \asymp r^{\alpha/2}$. By (31) the lower bound follows. ☐

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