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NON-COMMUTATIVE VARIETIES WITH CURVATURE HAVING BOUNDED SIGNATURE

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ABSTRACT. A natural notion for the signature $C_\pm(\mathcal{V}(p))$ of the curvature of the zero set $\mathcal{V}(p)$ of a non-commutative polynomial p is introduced. The main result of this paper is the bound

$$\deg p \le 2C_\pm\big(\mathcal{V}(p)\big) + 2.$$

It is obtained under some irreducibility and nonsingularity conditions, and shows that the signature of the curvature of the zero set of p dominates its degree.

The condition $C_+(\mathcal{V}(p)) = 0$ means that the non-commutative variety $\mathcal{V}(p)$ has positive curvature. In this case, the preceding inequality implies that the degree of p is at most two. Non-commutative varieties $\mathcal{V}(p)$ with positive curvature were introduced in (*Indiana Univ. Math. J.* **56** (2007) 1189–1231). There a slightly weaker irreducibility hypothesis plus a number of additional hypotheses yielded a weaker result on p. The approach here is quite different; it is cleaner, and allows for the treatment of arbitrary signatures.

In (*J. Anal. Math.* **108** (2009) 19–59), the degree of a non-commutative polynomial p was bounded by twice the signature of its Hessian plus two. In this paper, we introduce a modified version of this non-commutative Hessian of p which turns out to be very appropriate for analyzing the variety $\mathcal{V}(p)$.

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1. Introduction

In the classical setting of a surface defined by the zero set $\nu(p)$ of a polynomial $p = p(x) = p(x_1, \ldots, x_g)$ in g commuting variables, the second fundamental form at a smooth point x_0 of $\nu(p)$ is the quadratic form,

$$-\big\langle (\operatorname{Hess} p)(x_0)h, h\big\rangle, \tag{1.1}$$

where $\operatorname{Hess} p$ is the Hessian of p, and $h \in \mathbb{R}^g$ is in the tangent space to the surface $p(x) = 0$ at x_0; i.e., $\nabla p(x_0) \cdot h = 0$.[1]

In this paper, we show that in the non-commutative setting even a modicum of positive curvature of the zero set $\mathcal{V}(p)$ for a non-commutative polynomial p (subject to appropriate irreducibility constraints) implies that p is convex—and thus, p has degree at most two—and $\mathcal{V}(p)$ has positive curvature everywhere; see Theorem 1.4 and its corollary, Corollary 1.3, for the precise statements. In addition, we introduce a natural notion of the signature $C_\pm(\mathcal{V}(p))$ of a variety $\mathcal{V}(p)$ and obtain the bound

$$\deg p \le 2C_\pm\big(\mathcal{V}(p)\big) + 2$$

on the degree of p in terms of the signature $C_\pm(\mathcal{V}(p))$.

Throughout the paper, we shall adopt the convention that $C_+(\mathcal{V}(p)) = 0$ corresponds to positive curvature, since in our examples, defining functions p are typically concave or quasiconcave. The convention $C_-(\mathcal{V}(p)) = 0$ or even $C_\pm(\mathcal{V}(p)) = 0$ would be equally reasonable. Either way gives the same mathematical consequences.

Now that the main results have been described informally, the remainder of this introduction turns to the precise statements. The setting of this paper overlaps that of [DHM07b]. The principal definitions are reviewed briefly for the convenience of the reader in Sections 1.1, 1.2, 1.3.2, 1.3.4, 1.3.1, 1.3.5; the notion of positive curvature is introduced in Section 1.3.3; and the main new results are stated in Sections 1.4 and 1.6. The Introduction concludes with a guide to the rest of the paper in Section 1.7.

1.1. NC polynomials. Let $x = \{x_1, \ldots, x_g\}$ denote non-commuting indeterminates and let $\mathbb{R}\langle x\rangle$ denote the set of polynomials

$$p(x) = p(x_1, \ldots, x_g)$$

[1] The choice of the minus sign in (1.1) is somewhat arbitrary. Classically the sign of the second fundamental form is associated with the choice of a smoothly varying vector that is normal to $\nu(p)$. The zero set $\nu(p)$ has positive curvature at x_0 if the second fundamental form is either positive semidefinite or negative semi-definite at x_0. For example, if we define $\nu(p)$ using a concave function p, then the second fundamental form is negative semidefinite, while for the same set $\nu(-p)$ the second fundamental form is positive semidefinite.

in the indeterminates x with real coefficients; i.e., the set of finite linear combinations

$$p = \sum_{|w|\le d} c_w w \quad \text{with } c_w \in \mathbb{R} \tag{1.2}$$

of words w in x. The degree of such a polynomial p is defined as the maximum of the lengths $|w|$ of the words w appearing (nontrivially) in the linear combination (1.2). Thus, for example, if $g = 3$, then

$$p_1 = 3x_1x_2^3 + x_2 + x_3x_1x_2 \quad \text{and} \quad p_2 = 2x_1x_2^3 + x_2^3x_1 + x_3x_1x_2 + x_2x_1x_3$$

are polynomials of degree four in $\mathbb{R}\langle x\rangle$.

There is a natural *involution* w^T on words given by the rule

$$x_j^T = x_j \quad \text{and if} \quad w = x_{i_1}x_{i_2}\cdots x_{i_k}, \quad \text{then} \quad w^T = x_{i_k}\cdots x_{i_2}x_{i_1},$$

which extends to polynomials $p = \sum c_w w$ by linearity:

$$p^T = \sum_{|w|\le d} c_w w^T.$$

A polynomial $p \in \mathbb{R}\langle x\rangle$ is said to be *symmetric* if $p = p^T$. The second polynomial p_2 listed above is symmetric, the first is not. Because of the requirement $x_j^T = x_j$, the variables are said to be symmetric.

A polynomial $p(x) = p(x_1, \dots, x_g)$ in non-commuting variables $\{x_1, \dots, x_g\}$ will be referred to as an *nc polynomial* for short; and *nc* will be used as a short hand notation for non-commutative.

1.1.1. *Substituting matrices for indeterminates.* Let $(\mathbb{R}^{n\times n}_{\text{sym}})^g$ denote the set of g-tuples $(X_1, \dots, X_g)$ of real symmetric $n \times n$ matrices. We shall be interested in evaluating a polynomial $p(x) = p(x_1, \dots, x_g)$ that belongs to $\mathbb{R}\langle x\rangle$ at a tuple $X = (X_1, \dots, X_g) \in (\mathbb{R}^{n\times n}_{\text{sym}})^g$. In this case, $p(X)$ is also an $n \times n$ matrix and the involution on $\mathbb{R}\langle x\rangle$ that was introduced earlier is compatible with matrix transposition, i.e.,

$$p^T(X) = p(X)^T,$$

where $p(X)^T$ denotes the transpose of the matrix $p(X)$. When $X \in (\mathbb{R}^{n\times n}_{\text{sym}})^g$ is substituted into p the constant term $p(0)$ of $p(x)$ becomes $p(0)I_n$. For example, if $p(x) = 3 + x^2$, then

$$p(X) = 3I_n + X^2.$$

A symmetric polynomial $p \in \mathbb{R}\langle x\rangle$ is said to be *matrix positive* if $p(X)$ is a positive semi-definite matrix for each tuple $X = (X_1, \dots, X_g) \in (\mathbb{R}^{n\times n}_{\text{sym}})^g$. Similarly, p is said to be *matrix convex* if

$$p\big(tX + (1-t)Y\big) \preceq tp(X) + (1-t)p(Y) \tag{1.3}$$

for every pair of tuples $X, Y \in (\mathbb{R}^{n\times n}_{\text{sym}})^g$ and $0 \le t \le 1$.

1.1.2. *Derivatives.* We define the directional derivative of the word $w = x_{j_1}x_{j_2}\cdots x_{j_n}$ with coefficient $c \in \mathbb{R}$ as the linear form:

$$w'[h] = h_{j_1}x_{j_2}\cdots x_{j_n} + x_{j_1}h_{j_2}x_{j_3}\cdots x_{j_n} + \cdots + x_{j_1}\cdots x_{j_{n-1}}h_{j_n}$$

and extend the definition to polynomials $p = \sum c_w w$ by linearity; i.e.,

$$p'(x)[h] = \sum c_w w'[h].$$

Thus, $p'(x)[h] \in \mathbb{R}\langle x, h\rangle$ is the

coefficient of t in the expression $p(x+th) - p(x)$;

it is an nc polynomial in $2g$ (symmetric) variables $(x_1, \dots, x_g, h_1, \dots, h_g)$. For connections with von Neumann algebras and free probability see [Vo06]. Higher order derivatives are computed by the same recipe, i.e., as the coefficient of t in the expression $q(x+th)[h] - q(x)[h]$: If $q(x)[h] = h_{j_1}x_{j_2}\cdots x_{j_n}$, then

$$q'(x)[h] = h_{j_1}h_{j_2}x_{j_3}\cdots x_{j_n} + h_{j_1}x_{j_2}h_{j_3}\cdots x_{j_n} + \cdots + h_{j_1}x_{j_2}\cdots x_{j_{n-1}}h_{j_n}$$

and the definition is extended to finite linear combinations of such terms by linearity. If p is symmetric, then so is p'. For g-tuples of symmetric matrices of a fixed size X, H, the evaluation formula

$$p'(X)[H] = \lim_{t\to 0} \frac{p(X+tH) - p(X)}{t}$$

holds, and if $q(t) = p(X+tH)$, then

$$p'(X)[H] = q'(0) \quad \text{and} \quad p''(X)[H] = q''(0). \tag{1.4}$$

The second formula in (1.4) is the evaluation of the *Hessian*, $\mathrm{Hess}(p) = p''(x)[h]$ of a polynomial $p \in \mathbb{R}\langle x\rangle$; it can be thought of as the formal second directional derivative of p in the "direction" h.

If $p'' \neq 0$, then the degree of p is two or more, and the degree of $p''(x)[h]$ as a polynomial in the $2g$ variables $(x_1, \dots, x_g, h_1 \dots, h_g)$ is equal to the degree of $p(x)$ as a polynomial in $(x_1, \dots, x_g)$ and is homogeneous of degree two in h.

The same conclusion holds for kth derivatives if $k \le d$, the degree of p. The expositions in [HMV06] and in [HP07] give more detail on the derivatives and the Hessian of nc polynomials.

EXAMPLE 1.1. A few concrete examples are listed for practice with the definitions, if the reader is so inclined.

(1) If $p(x) = x^4$, then

$$\begin{aligned}
p'(x)[h] &= hxxx + xhxx + xxhx + xxxh,\\
p''(x)[h] &= 2hhxx + 2hxhx + 2hxxh + 2xhhx + 2xhxh + 2xxhh,\\
p^{(3)}(x)[h &= 6(hhhx + hhxh + hxhh + xhhh),\\
p^{(4)}(x)[h] &= 24hhhh \quad \text{and} \quad p^{(5)}(x)[h] = 0.
\end{aligned}$$

(2) If $p(x) = x_2 x_1 x_2$, then $p'(x)[h] = h_2 x_1 x_2 + x_2 h_1 x_2 + x_2 x_1 h_2$.
(3) If $p(x) = x_1^2 x_2$, then $p''(x)[h] = 2(h_1^2 x_2 + h_1 x_1 h_2 + x_1 h_1 h_2)$.

1.1.3. *The Signature of a non-commutative quadratic.* Consider a symmetric polynomial $q(x)[h]$ in the $2g$ variables $(x_1, \dots, x_g, h_1 \dots, h_g)$ which is homogeneous of degree two in h. It admits a representation of the form (a sum and difference of squares)

$$q(x)[h] = \sum_{j=1}^{\sigma_+} f_j^+(x)[h]^T f_j^+(x)[h] - \sum_{\ell=1}^{\sigma_-} f_\ell^-(x)[h]^T f_\ell^-(x)[h], \tag{SDS}$$

where $f_j^+(x)[h], f_\ell^-(x)[h]$ are non-commutative polynomials which are homogeneous of degree one in h; see e.g., Lemmas 4.7 and 4.8 of [DHM07a] for details and for the related notion of a nc positive kernel see [KVV06]. Such representations are highly non-unique. However, there is a unique smallest number of positive (resp., negative squares) $\sigma_\pm^{\min}(q)$ required in an SDS decomposition of q. These numbers are called *the signature of* q. Later $\sigma_\pm^{\min}(p'')$ will be identified with $\mu_\pm(\mathcal{Z})$, the number of positive (resp., negative) eigenvalues of an appropriately chosen symmetric matrix $\mathcal{Z}$ appearing in a Gram type representation of the Hessian $p''(x)[h]$ that is discussed in Section 5.

1.2. Previous results. The number (or rather dearth) of negative squares in an SDS decomposition of the Hessian places serious restrictions on the degree of an nc polynomial. A theorem of Helton and McCullough [HM04] states that a symmetric nc polynomial that is matrix convex has degree at most two. Since a symmetric nc polynomial p is matrix convex if $\sigma_-^{\min}(p'') = 0$, this is a special case of the following more general result in [DHM07a].

Theorem 1.2. *If $p(x)$ is a symmetric nc polynomial of degree d in symmetric variables, then*

$$d \le 2\sigma_\pm^{\min}\big(p''\big) + 2. \tag{1.5}$$

1.3. Some basic definitions. We next define a number of basic geometric objects associated to the nc variety determined by an nc polynomial p.

1.3.1. *Varieties, tangent planes, and the second fundamental form.* The *variety* (zero set) for p is

$$\mathcal{V}(p) := \bigcup_{n \ge 1} \mathcal{V}_n(p),$$

where

$$\mathcal{V}_n(p) := \big\{(X, v) \in \big(\mathbb{R}_{\text{sym}}^{n\times n}\big)^g \times \mathbb{R}^n : p(X)v = 0\big\}.$$

The *clamped tangent plane* to $\mathcal{V}(p)$ at $(X, v) \in \mathcal{V}_n(p)$ is

$$\mathcal{T}_p(X, v) := \big\{H \in \big(\mathbb{R}_{\text{sym}}^{n\times n}\big)^g : p'(X)[H]v = 0\big\}.$$

The *clamped second fundamental form* for $\mathcal{V}(p)$ at $(X,v)\in\mathcal{V}_n(p)$ is the quadratic function

$$\mathcal{T}_p(X,v)\ni H\mapsto -\big\langle p''(X)[H]v,v\big\rangle.$$

Note that

$$\begin{aligned}&\big\{X\in\big(\mathbb{R}^{n\times n}_{\mathrm{sym}}\big)^g : (X,v)\in\mathcal{V}(p) \text{ for some } v\neq 0\big\}\\ &=\big\{X\in\big(\mathbb{R}^{n\times n}_{\mathrm{sym}}\big)^g :\ \det\big(p(X)\big)=0\big\}\end{aligned}$$

is a variety in $(\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$ and typically has a *true* (commutative) tangent plane at many points X, which of course has codimension one, whereas the clamped tangent plane at a typical point $(X,v)\in\mathcal{V}_n(p)$ has codimension on the order of n and is contained inside the true tangent plane.

1.3.2. *Full rank points.* The point $(X,v)\in\mathcal{V}(p)$ is *a full rank point* for p if the mapping

$$\big(\mathbb{R}^{n\times n}_{\mathrm{sym}}\big)^g\ni H\mapsto p'(X)[H]v\in\mathbb{R}^n$$

is onto. The full rank condition is a non-singularity condition which amounts to a smoothness hypothesis. Such conditions play a major role in real algebraic geometry, see Section 3.3 [BCR91].

As an example, consider the classical real algebraic geometry case of $n=1$ (and thus $X\in\mathbb{R}^g$) with the commutative polynomial $\tilde{p}$ (which can be taken to be the *commutative collapse* of the polynomial p). In this case, a full rank point $(X,1)\in\mathbb{R}^g\times\mathbb{R}$ is a point at which the gradient of $\tilde{p}$ does not vanish. Thus, X is a non-singular point for the zero variety of $\tilde{p}$.

Some perspective for $n>1$ is obtained by counting dimensions. If $(X,v)\in(\mathbb{R}^{n\times n}_{\mathrm{sym}})^g\times\mathbb{R}^n$, then $H\mapsto p'(X)[H]v$ is a linear map from the $g(n^2+n)/2$ dimensional space $(\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$ into the n dimensional space $\mathbb{R}^n$. Therefore, the codimension of the kernel of this map is no bigger than n. This codimension is n if and only if (X,v) is a full rank point and in this case the clamped tangent plane has codimension n.

1.3.3. *Positive curvature.* As noted earlier, a notion of positive (really nonnegative) curvature can be defined in terms of the clamped second fundamental form.

The variety $\mathcal{V}(p)$ has *positive curvature* at $(X,v)\in\mathcal{V}(p)$ if the clamped second fundamental form is nonnegative at (X,v); i.e., if

$$-\big\langle p''(X)[H]v,v\big\rangle\geq 0\quad\text{for every } H\in\mathcal{T}_p(X,v).$$

1.3.4. *Algebraically open sets.* Define an *algebraically open set* $\mathcal{O}$ to be one of the form $\mathcal{D}_{\mathcal{Q}}:=\bigcap_{q\in\mathcal{Q}}\{X: q(X)\succ 0\}$ where $\mathcal{Q}$ is some finite set of symmetric nc polynomials. Abusing notation a little, we set

$$\mathcal{V}(p)\cap\mathcal{O}:=\big\{(X,v): p(X)v=0 \text{ and } X\in\mathcal{O}\big\}.$$

We are primarily motivated by the case where $\mathcal{V}(p)$ has positive curvature on an algebraically open set intersected with the full rank points of p although the techniques produce a more general result of independent interest.

1.3.5. *Irreducibility: The minimum degree defining polynomial condition.* While there is no tradition of what is an effective notion of irreducibility for nc polynomials, there is a notion of minimal degree nc polynomial which is appropriate for the present context.

In the commutative case, the polynomial p on $\mathbb{R}^g$ is a minimal degree defining polynomial for $\mathcal{V}(p)$ if there does not exist a polynomial q of lower degree such that $\mathcal{V}(p) = \mathcal{V}(q)$. This is a key feature of irreducible polynomials.

There are several ways of generalizing the notion of minimal degree defining polynomial to the nc setting. In the next couple of paragraphs, we describe the natural and effective generalization used here.

Our main result is stated for certain subsets $\mathcal{S}$ of $\mathcal{V}(p)$, but the result remains of interest for $\mathcal{S}$ equal all of $\mathcal{V}(p)$. Thus, as we now turn to axiomatizing the admissible subsets $\mathcal{S}$ of $\mathcal{V}(p)$, the reader who is so inclined can simply choose $\mathcal{S} = \mathcal{V}(p)$ or $\mathcal{S}$ equal to the full rank points of $\mathcal{V}(p)$.

A symmetric nc polynomial p is a *k-minimum degree defining polynomial for $\mathcal{S} \subseteq \mathcal{V}(p)$* if

(1) $\mathcal{S}$ is a nonempty subset of $\mathcal{V}(p)$;
(2) if $q \neq 0$ is another (*not necessarily symmetric*) nc polynomial such that $q(X)v = 0$ for each $(X, v) \in \mathcal{S}$, then
$$\deg q(x) \geq \deg p(x) - k;$$
(3) there exists a (*not necessarily symmetric*) nc polynomial $q(x)$ with
$$\deg q(x) = \deg p(x) - k$$
such that $q(X)v = 0$ for each pair $(X, v) \in \mathcal{S}$.

A 0-minimum degree defining nc polynomial will be called a *minimum degree defining polynomial*. Note this contrasts with [DHM07b], where minimal degree meant 1-minimal degree in terms of the present usage.

1.4. Results I: Positive curvature and the degree of p. To ease the exposition, we start with a corollary of our main theorem. The proof of the corollary and the theorem will be supplied in Section 7.

Corollary 1.3. *Let p be a symmetric nc polynomial in symmetric variables, let $\mathcal{O}$ be an algebraically open set and let $\mathcal{S}$ denote the full rank points of p in $\mathcal{V}(p) \cap \mathcal{O}$. If*

(i) *There exists $(X, v) \in (\mathbb{R}^g \times \mathbb{R}) \cap \mathcal{S}$;*
(ii) *$\mathcal{V}(p)$ has positive curvature at each point of $\mathcal{S}$; and*
(iii) *p is a 0-minimum degree defining polynomial for $\mathcal{S}$;*

then the degree of p is at most two and p is concave.

1.5. The signature of the variety. The fact that if $(X,v),(Y,w)\in\mathcal{V}(p)$, then so is the direct sum

$$\left(\operatorname{diag}\{X,Y\},\begin{bmatrix} v\\ w\end{bmatrix}\right)=\left(\left(\begin{bmatrix} X_1 & 0\\ 0 & Y_1\end{bmatrix},\dots,\begin{bmatrix} X_g & 0\\ 0 & Y_g\end{bmatrix}\right),\begin{bmatrix} v\\ w\end{bmatrix}\right)$$
$$=\left(\begin{bmatrix} X & 0\\ 0 & Y\end{bmatrix},\begin{bmatrix} v\\ w\end{bmatrix}\right)$$

is a key feature of non-commutative varieties. This section begins with a discussion of direct sums. It continues with the notion of the signature of a quadratic form and concludes with our definition of the signature of $\mathcal{V}(p)$.

1.5.1. *Direct sums.* Given a finite set $F=\{(X^1,v^1),\dots,(X^t,v^t)\}$ with $X^j\in(\mathbb{R}^{n_j\times n_j}_{\rm sym})^g$ and $v^j\in\mathbb{R}^{n_j}$ for $j=1,2,\dots,t$, set

$$X_F=\operatorname{diag}\{X^1,\dots,X^t\}\quad\text{and}\quad v_F=\operatorname{col}(v^1,\dots,v^t). \tag{1.6}$$

Thus, if q is an nc polynomial, then

$$q(X_F)v_F=\operatorname{col}(q(X^1)v^1,\dots,q(X^t)v^t). \tag{1.7}$$

Now let

$$\mathcal{S}=\bigcup_{n=1}^{\infty}\mathcal{S}_n,$$

where $\mathcal{S}_n\subseteq(\mathbb{R}^{n\times n}_{\rm sym})^g\times\mathbb{R}^n$ for $n=1,2,\dots$, be given. The set $\mathcal{S}$ *respects direct sums* if for each finite set

$$F=\{(X^1,v^1),\dots,(X^t,v^t)\}\quad\text{with } (X^j,v^j)\in\mathcal{S}_{n_j}\text{ and } n=\sum_{j=1}^{t}n_j,$$

with repetitions allowed, the pair (X_F,v_F) is in $\mathcal{S}_n$.

Examples of sets which respect direct sums include

(i) the zero set $\mathcal{V}(p)$ of an nc polynomial p (see equation (1.7));
(ii) the set of full rank points of p in $\mathcal{V}(p)$ (See Lemma 4.2); and
(iii) the intersection of sets which respect direct sums.

Indeed, as remarked before, the reader may choose to take $\mathcal{S}$ equal to either $\mathcal{V}(p)$ or to the full rank points in $\mathcal{V}(p)$.

1.5.2. *The signature of a quadratic form.* The Hessian $p''(x)[h]$ of an nc polynomial is a quadratic form in h. More generally, let $f(x)[h]$ be an nc symmetric polynomial in the $2g$ symmetric variables $x=(x_1,\dots,x_g)$ and $h=(h_1,\dots,h_g)$ that is of degree s in x and homogeneous of degree two in h. Given a subspace $\mathcal{H}$ of $(\mathbb{R}^{n\times n}_{\rm sym})^g$, let

(1.8) $e^n_\pm(X,v;f,\mathcal{H})$ denote the maximum dimension of a strictly positive/negative subspace of $\mathcal{H}$ with respect to the quadratic form $\mathcal{H}\ni H\mapsto\langle f(X)[H]v,v\rangle$.

Here strictly positive (resp., negative) subspace $\mathcal{H}$ means $\langle f(X)[H]v, v\rangle > 0$ (resp., < 0) for $H \in \mathcal{H}$, $H \neq 0$.

1.5.3. *The Signature of the curvature of $\mathcal{V}(p)$ relative to $\mathcal{S}$.* Given a symmetric nc polynomial p in symmetric variables and (X, v), let

$$\mathcal{T} = \left\{H \in \left(\mathbb{R}^{n\times n}_{\mathrm{sym}}\right)^g : p'(X)[H]v = 0\right\} \quad \text{and} \quad c^n_\pm(X, v; p) := e^n_\pm\left(X, v; p'', \mathcal{T}\right).$$

When $p(X)v = 0$, so that (X, v) is in the zero set of p, then the subspace $\mathcal{T}$ is the *clamped tangent space*.

The numbers $C_\pm(\mathcal{S})$, which are defined below in terms of $c^n_\pm(X, v; p)$ in (1.9), bound the signature of the second fundamental form of $\mathcal{V}(p)$ on $\mathcal{S}$. Because of the close connection between the second fundamental form and the curvature of $\mathcal{V}(p)$ at a smooth point, we shall call the numbers $\mathcal{C}_\pm(\mathcal{S})$ the *signature of the curvature of $\mathcal{V}(p)$ on $\mathcal{S}$.*

Note that if $\mathcal{S} = \cup_{n\geq 1}\mathcal{S}_n$ is closed with respect to direct sums, then

$$\mathcal{S}_n \neq \emptyset \quad \text{for every integer } n \geq 1 \quad \Longleftrightarrow \quad \mathcal{S}_1 \neq \emptyset.$$

1.6. Results II. We can now state the main result of this article.

THEOREM 1.4. *Let p be a symmetric nc polynomial in g symmetric variables, and let $\mathcal{S} = \bigcup_{n\geq 1}\mathcal{S}_n$ be a subset of $\mathcal{V}(p)$ which respects direct sums and for which $\mathcal{S}_1$ is nonempty.*

(A) (i) *The limit*

$$C_\pm(\mathcal{S}) := \lim_{n\uparrow\infty}\left(\sup\left\{\frac{c^n_\pm(X, v; p)}{n} : (X, v) \in \mathcal{S}_n\right\}\right) \tag{1.9}$$

exists.

(ii) *If p is a 0-minimum degree defining polynomial for $\mathcal{S}$, then*

$$\deg p \leq 2C_\pm(\mathcal{S}) + 2. \tag{1.10}$$

(B) *Moreover:*

$$\text{if } C_-(\mathcal{S}) = 0, \quad \text{then } p \text{ is a convex polynomial of degree } 2;$$
$$\text{if } C_+(\mathcal{S}) = 0, \quad \text{then } p \text{ is a concave polynomial of degree } 2.$$

(C) *Conversely, if p is convex (resp., concave), then p has degree at most two and $c^n_-(X, v; p) = 0$ (resp., $c^n_+(X, v; p) = 0$) for every $(X, v) \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g \times \mathbb{R}^n$.*

(D) *Finally, if $\mathcal{S}' = \bigcup \mathcal{S}'_n$ is any other subset of $\mathcal{V}(p)$ for which $\mathcal{S}'_1 \neq \emptyset$ and p is a 0-minimal degree defining polynomial, then $C_\pm(\mathcal{S}) = C_\pm(\mathcal{S}')$.*

In view of (D), the signature of $\mathcal{V}(p)$ is determined on any nonempty subset $\mathcal{S}$ of $\mathcal{V}(p)$ which respects direct sums and is large enough so that p is a minimal defining polynomial for $\mathcal{S}$. The next proposition shows how this phenomenon carries to surprising extremes, in that a single pair (X, v) of high enough dimension often determines $\mathcal{C}_\pm(\mathcal{V}(p))$. These principles are elaborated upon in Theorem 7.4. Here we offer the following consequence of Theorem 7.4.

PROPOSITION 1.5. *Let p be a symmetric nc polynomial in symmetric variables which is a minimal degree defining polynomial for $\mathcal{V}(p)$. Then there is an integer $\hat{n}_{g,d}$ not depending on p such that for every $n \geq \hat{n}_{g,d}$ there exists a pair $(X,v) \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g \times \mathbb{R}^n$ in $\mathcal{V}(p)$ such that*

$$\mathcal{C}_\pm\big(\mathcal{V}(p)\big) = \left\lceil \frac{c^n_\pm(X,v;p)}{n} \right\rceil.$$

Here $\lceil r \rceil$ is the ceiling function; i.e., the smallest integer at least as large as the real number r.

1.7. Reader's guide. The remainder of the paper is organized as follows. Section 2 contains examples which illustrate Theorem 1.4. In Section 3, the relaxed Hessian is introduced and the signature of the fundamental form (on the clamped tangent space) is shown to correspond closely to the signature of this relaxed Hessian. A key tool is a Gram like representation for nc quadratics, called the middle matrix-border vector representation, which is reviewed in Section 5. Section 4 shows that the minimal degree hypothesis translates into a linear independence condition on the border vector. This linear independence, via a "CHSY Lemma" (see Section 6) is enough to put the signature of the relaxed Hessian in close correspondence with that of the middle matrix. The signature of the middle matrix has been carefully analyzed and exploited in studying issues of matrix convexity [DHM07a], [DHM07b], [DGHM09], [HM04] and this produces our main inequality (1.10). The results outlined above are tied together in Section 7 to produce the proof of Theorem 1.4. The article concludes with Section 8 which discusses the non-commutative analog of the fact that the boundary of a convex sublevel set in $\mathbb{R}^g$ has nonnegative curvature.

The paper is in principle self contained except for two previous results. One is the Middle Matrix Congruence Theorem from [DHM07a]. This congruence is given in equation (5.3) and related necessary results are summarized early in Section 5. The other result is the CHSY Lemma [CHSY03], which is stated and elaborated upon in Section 6.

2. Examples

In this section, we compute some examples to illustrate the notation and objects from Theorem 1.4.

2.1. A very simple example. In the following example, the null space

$$\mathcal{T} = \mathcal{T}_p(X,v) = \left\{ H \in \left(\mathbb{R}^{n\times n}_{\mathrm{sym}}\right)^g : p'(X)[H]v = 0 \right\}$$

is computed for certain choices of p, X, and v. Recall that if $p(X)v = 0$, then the subspace $\mathcal{T}$ is the *clamped tangent plane* introduced in Section 1.3.1.

EXAMPLE 2.1. Let $X \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, $v \in \mathbb{R}^n$, $v \neq 0$, let $p(x) = x^k$ for some integer $k \geq 1$. Suppose that $(X,v) \in \mathcal{V}(p)$, that is, $X^k v = 0$. Then, since

$$X^k v = 0 \quad \Longleftrightarrow \quad Xv = 0 \quad \text{when } X \in \mathbb{R}^{n\times n}_{\mathrm{sym}},$$

it follows that p is a minimum degree defining polynomial for $\mathcal{V}(p)$ if and only if $k = 1$.

It is readily checked that

$$(X,v) \in \mathcal{V}(p) \quad \Longrightarrow \quad p'(X)[H]v = X^{k-1}Hv,$$

and hence that X is a full rank point for p if and only if X is invertible.

Now suppose $k \geq 2$. Then,

$$\langle p''(X)[H]v, v\rangle = 2\langle HX^{k-2}Hv, v\rangle.$$

Therefore, if $k > 2$

$$(X,v) \in \mathcal{V}(p) \quad \text{and} \quad p'(X)[H]v = 0 \quad \Longrightarrow \quad XHv = 0, \quad \text{and so} \\ \langle p''(X)[H]v, v\rangle = 0.$$

To count the dimension of $\mathcal{T}$, we can suppose without loss of generality that

$$X = \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $Y \in \mathbb{R}^{(n-1)\times(n-1)}_{\mathrm{sym}}$ is invertible. Then, for the simple case under consideration,

$$\mathcal{T} = \{H \in \mathbb{R}^{n\times n}_{\mathrm{sym}} : h_{21}, \ldots, h_{n1} = 0\},$$

where h_{ij} denotes the ij entry of H. Thus,

$$\dim \mathcal{T} = \frac{n^2+n}{2} - (n-1), \quad \text{i.e.,} \quad \operatorname{codim} \mathcal{T} = n-1.$$

REMARK 2.2. We remark that

$$X^k v = 0 \quad \text{and} \quad \langle p''(X)[H]v, v\rangle = 0 \quad \Longrightarrow \quad p'(X)[H]v = 0 \quad \text{if } k = 2t \geq 4,$$

as follows easily from the formula

$$\langle p''(X)[H]v, v\rangle = 2\langle X^{t-1}Hv, X^{t-1}Hv\rangle.$$

2.2. Computation of $c_\pm$ and direct sums. We now turn to computing examples of the quantities $c^n_\pm(X,p,v)$ with special attention paid to their behavior under direct sums demonstrating the inequality of Lemma 7.1.

EXAMPLE 2.3. Specializing the previous example, choose $p(x)=x^3$ and suppose $X\in\mathbb{R}^{n\times n}_{\rm sym}$ satisfies $X^2=I_n$. Soon we will make a concrete choice of X and consider various choices for the vector v. To compute $c_\pm(X,v,p)$, we must compute both the Hessian and the subspace $\mathcal{T}$.

In this case,

$$p'(X)[H]=HX^2+XHX+X^2H \quad \text{and}$$
$$p''(X)[H]=2\big(H^2X+HXH+XH^2\big).$$

Next, upon imposing the supplementary constraint $X^2=I_n$, it is readily seen that

$$p'(X)[H]v=0 \quad\Longleftrightarrow\quad HXv=-2XHv$$

and hence

$$p'(X)[H]v=0 \quad\Longrightarrow\quad \big\langle p''(X)[H]v,v\big\rangle=-6\langle XHv,Hv\rangle. \tag{2.1}$$

To illustrate more detail, let

$$X=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix},\qquad v=\begin{bmatrix}1\\ \beta\\ \gamma\end{bmatrix}\quad\text{and}\quad \mathcal{T}=\big\{H\in\mathbb{R}^{3\times3}_{\rm sym}:p'(X)[H]v=0\big\},$$

Then it is easily checked that

$$\mathcal{T}=\operatorname{span}\left\{\begin{bmatrix}\beta^2&-3\beta&0\\-3\beta&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}2\beta\gamma&-3\gamma&-3\beta\\-3\gamma&0&1\\-3\beta&1&0\end{bmatrix},\begin{bmatrix}\gamma^2&0&-3\gamma\\0&0&0\\-3\gamma&0&1\end{bmatrix}\right\},$$

and correspondingly,

$$\begin{aligned}\operatorname{span}\{Hv:H\in\mathcal{T}\}&=\operatorname{span}\left\{\begin{bmatrix}\beta^2\\ \beta\\ 0\end{bmatrix},\begin{bmatrix}2\beta\gamma\\ \gamma\\ \beta\end{bmatrix},\begin{bmatrix}\gamma^2\\ 0\\ \gamma\end{bmatrix}\right\}\\ &=\operatorname{span}\left\{\beta\begin{bmatrix}\beta\\1\\0\end{bmatrix},\beta\begin{bmatrix}\gamma\\0\\1\end{bmatrix}+\gamma\begin{bmatrix}\beta\\1\\0\end{bmatrix},\gamma\begin{bmatrix}\gamma\\0\\1\end{bmatrix}\right\}.\end{aligned}$$

Thus, if $0<\beta^2+\gamma^2$, then

$$\operatorname{span}\{Hv:H\in\mathcal{T}\}=\operatorname{span}\left\{\begin{bmatrix}\beta\\1\\0\end{bmatrix},\begin{bmatrix}\gamma\\0\\1\end{bmatrix}\right\},$$

and, if

$$Hv=\lambda\begin{bmatrix}\beta\\1\\0\end{bmatrix}+\mu\begin{bmatrix}\gamma\\0\\1\end{bmatrix},$$

it follows that

$$\begin{aligned}\langle p''(X)[H]v, v\rangle &= 6\{\lambda^2 + \mu^2 - (\lambda\beta + \mu\gamma)^2\} \\ &\geq 6(\lambda^2 + \mu^2)\{1 - (\beta^2 + \gamma^2)\} \geq 0,\end{aligned}$$

since $(\lambda\beta + \mu\gamma)^2 \leq (\lambda^2 + \mu^2)(\beta^2 + \gamma^2)$, by the Cauchy–Schwartz inequality. Consequently,

$$0 < \beta^2 + \gamma^2 < 1 \quad \Longrightarrow \quad c_-^n(X, v; p) = 0 \quad \text{and} \quad c_+^n(X, v; p) = 2 = \mu_-(X).$$

We now turn to the behavior of $c_\pm$ under direct sums. Supposing again only that $X^2 = I$ (and allowing for general n) let

$$Y = \operatorname{diag}\{X, \ldots, X\} \quad \text{and} \quad w = \operatorname{col}(v, \ldots, v) \quad k \text{ times},$$

then $Y^2 = I_{kn}$ and hence

$$\begin{aligned} p'(Y)[H]w = 0 \quad &\Longleftrightarrow \quad HYw = -2YHw, \\ p'(Y)[H]w = 0 \quad &\Longrightarrow \quad \langle p''(Y)[H]w, w\rangle = -6\langle YHw, Hw\rangle, \end{aligned}$$

and

$$\mathcal{T} = \{H \in \mathbb{R}^{kn\times kn}_{\text{sym}} : HYw + 2YHw = 0\},$$

just as before.

Let

$$\mathcal{D}_k = \{H \in \mathbb{R}^{kn\times kn}_{\text{sym}} : H = \operatorname{diag}\{H^1, \ldots, H^k\} \text{ with } H^j \in \mathbb{R}^{n\times n}_{\text{sym}}\} \tag{2.2}$$

and

$$\mathcal{T}_{\mathcal{D}_k} = \mathcal{T} \cap \mathcal{D}_k. \tag{2.3}$$

Then clearly $\mathcal{T} \supseteq \mathcal{T}_{\mathcal{D}_k}$ and $\{Hv : H \in \mathcal{T}\} \supseteq \{Hv : H \in \mathcal{T}_{\mathcal{D}_k}\}$. Therefore,

$$c_\pm^{kn}(Y, w; p) \geq k c_\pm^n(X, v; p), \tag{2.4}$$

an inequality which holds generally, see Lemma 7.1.

A finer analysis of a specialization of the previous example shows that the inequality in equation (2.4) (and Lemma 7.1) can be strict.

Example 2.4. Let $p(x) = x^3$ and $X = \operatorname{diag}\{I_r, -I_q\}$ with $r \geq 1$, $q \geq 1$ and $r + q = n$ (so that $X^2 = I_n$ as in the previous example) and correspondingly partition $H \in \mathbb{R}^{n\times n}_{\text{sym}}$ and $v \in \mathbb{R}^n$ as

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

with $H_{11} \in \mathbb{R}^{r\times r}_{\text{sym}}$, $H_{22} \in \mathbb{R}^{q\times q}_{\text{sym}}$, $v_1 \in \mathbb{R}^r$ and $v_2 \in \mathbb{R}^q$. Then

$$\mathcal{T} = \{H \in \mathbb{R}^{n\times n}_{\text{sym}} : 3H_{11}v_1 = -H_{12}v_2 \text{ and } H_{21}v_1 = -3H_{22}v_2\}.$$

Thus, if $H \in \mathcal{T}$, then

$$Hv = \begin{bmatrix} H_{11}v_1 + H_{12}v_2 \\ H_{21}v_1 + H_{22}v_2 \end{bmatrix} = -2\begin{bmatrix} H_{11}v_1 \\ H_{22}v_2 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} H_{12}v_2 \\ H_{21}v_1 \end{bmatrix}$$

and

$$3v_1^T H_{11} v_1 = -v_1^T H_{12} v_2 = -v_2^T H_{21} v_1 = 3v_2^T H_{22} v_2.$$

Therefore, since H_{12} is an arbitrary $r \times q$ matrix and $H_{21} = H_{12}^T$,

$$\{Hv : H \in \mathcal{T}\} = \begin{cases} \left\{ \left[\begin{smallmatrix} Av_2 \\ A^T v_1 \end{smallmatrix} \right] : A \in \mathbb{R}^{r \times q} \right\}, & \text{if } v_1 \neq 0 \text{ and } v_2 \neq 0, \\ 0, & \text{if } v_1 = 0 \text{ or } v_2 = 0. \end{cases}$$

If, say, $r = 2$, $q = 3$, $v_1^T = [a \;\; b] \neq 0$ and $v_2^T = [c \;\; d \;\; e] \neq 0$, then

$$\{Hv : H \in \mathcal{T}\} = \operatorname{span} \left\{ \begin{bmatrix} c \\ 0 \\ a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ 0 \\ 0 \\ a \\ 0 \end{bmatrix}, \begin{bmatrix} e \\ 0 \\ 0 \\ 0 \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ c \\ b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d \\ 0 \\ b \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e \\ 0 \\ 0 \\ b \end{bmatrix} \right\}.$$

If $b = c = d = 0$, then

$$\{Hv : H \in \mathcal{T}\} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \\ 0 \end{bmatrix}, \begin{bmatrix} e \\ 0 \\ 0 \\ 0 \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ e \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

If also $a > e > 0$, then this span splits into the orthogonal sum of a negative space of dimension one and a positive space of dimension three with respect to the bi-linear form $\langle p''(X)[H]v, v\rangle$. Correspondingly (in view of (2.1)),

$$c_-^5(X, v; p) = 1 \quad \text{and} \quad c_+^5(X, v; p) = \mu_-(X).$$

Turn now to direct sums. If

$$Y = \operatorname{diag}\{X, \dots, X\} \quad \text{and} \quad w = \operatorname{col}(v, \dots, v) \quad k \text{ times},$$

then

$$c_\pm^{kn}(Y, w; p) = c_\pm^{kn}(\widetilde{Y}, \widetilde{w}; p),$$

where

$$\widetilde{Y} = \operatorname{diag}\{I_{kr}, -I_{kq}\}, \qquad \widetilde{w} = \operatorname{col}(w_1, w_2), \quad w_j = \operatorname{col}(v_j, \dots, v_j) \quad \text{for } j = 1, 2.$$

Thus,

$$\{H\widetilde{w} : H \in \mathcal{T}\} = \begin{cases} \left\{ \left[\begin{smallmatrix} Aw_2 \\ A^T w_1 \end{smallmatrix} \right] : A \in \mathbb{R}^{kr \times kq} \right\}, & \text{if } w_1 \neq 0 \text{ and } w_2 \neq 0, \\ 0, & \text{if } w_1 = 0 \text{ or } w_2 = 0. \end{cases}$$

If, $r = 2$, $q = 3$, $k = 2$, $v_1^T = [a \;\; b] \neq 0$ and $v_2^T = [c \;\; d \;\; e] \neq 0$, then

$$c_-^{10}(Y, w; p) = 3 \quad \text{and} \quad c_+^{10}(Y, w; p) = \mu_-(Y) = 2\mu_-(X).$$

In particular, with $k = 5$ and $n = 2$,

$$c_-^{kn}(Y, w; p) > kc_-^n(X, v; p).$$

3. Curvature: The Hessian on a tangent plane vs. the relaxed Hessian

Our main tool for analyzing the curvature of non-commutative real varieties is a variant of the Hessian for symmetric nc polynomials p of degree d in g non-commuting variables. The curvature of $\mathcal{V}(p)$ is defined in terms of $\operatorname{Hess}(p)$ compressed to tangent planes, for each dimension n. This compression of the Hessian is awkward to work with directly, and so we associate to it a quadratic polynomial $q(x)[h]$ (defined for all $H \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$, not just $H \in \mathcal{T}_p(X,v)$) called the relaxed Hessian.

Let $V_k(x)[h]$ denote the vector of polynomials with entries $h_j w(x)$, where $w(x)$ runs through the set of g^k words of length k, $j=1,\dots,g$. Although the order of the entries is fixed in some of our earlier applications (see, e.g., formula (2.3) in [DHM07a]) it is irrelevant for the moment. Thus, $V_k = V_k(x)[h]$ is a vector of height g^{k+1}, and the vectors

$$V(x)[h] = \operatorname{col}(V_0,\dots,V_{d-2}) \quad \text{and} \quad \widetilde{V}(x)[h] = \operatorname{col}(V_0,\dots,V_{d-1}) \tag{3.1}$$

are vectors of height $g\alpha_{d-2}$ and $g\alpha_{d-1}$, respectively, where

$$\alpha_t = 1 + g + \cdots + g^t. \tag{3.2}$$

Note that

$$\widetilde{V}(x)[h]^T \widetilde{V}(x)[h] = \sum_{j=1}^{g} \sum_{|w|\le d-1} w(x)^T h_j^2 w(x),$$

where $|w|$ denotes the degree (length) of the word w.

The *relaxed Hessian* of the symmetric nc polynomial p of degree d is defined to be the polynomial

$$p''_{\lambda,\delta} := p''(x)[h] + \delta \widetilde{V}(x)[h]^T \widetilde{V}(x)[h] + \lambda p'(x)[h]^T p'(x)[h]. \tag{3.3}$$

Suppose $X \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$ and $v \in \mathbb{R}^n$. We say that the *relaxed Hessian is positive* at (X,v) if for each $\delta > 0$ there is a $\lambda_\delta > 0$ so that for all $\lambda > \lambda_\delta$

$$0 \le \langle p''_{\lambda,\delta}(X)[H]v, v\rangle$$

for all $H \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$. Correspondingly, we say that the *relaxed Hessian is negative* at (X,v) if for each $\delta < 0$ there is a $\lambda_\delta < 0$ so that for all $\lambda \le \lambda_\delta$,

$$0 \le -\langle p''_{\lambda,\delta}(X)[H]v, v\rangle$$

for all $H \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$. Given a subset $\mathcal{S} = \bigcup_{n=1}^\infty \mathcal{S}_n$, with $\mathcal{S}_n \subseteq ((\mathbb{R}^{n\times n}_{\mathrm{sym}})^g \times \mathbb{R}^n)$, we say that the *relaxed Hessian is positive (resp., negative) on* $\mathcal{S}$ if it is positive (resp., negative) at each $(X,v) \in \mathcal{S}$.

EXAMPLE 3.1. Consider the classical $n=1$ case. Suppose that p is strictly smoothly quasi-concave, meaning that all superlevel sets of p are strictly convex with strictly positively curved smooth boundary ν. Suppose that the gradient ∇p (written as a row vector) never vanishes on $\mathbb{R}^g$. Then $G = \nabla p(\nabla p)^T$

is strictly positive, and at each point X in $\mathbb{R}^g$ the relaxed Hessian can be decomposed as a block matrix subordinate to the tangent plane to the level set at X and to its orthogonal complement (the gradient direction). In this decomposition, the relaxed Hessian with $\delta = 0$ has the form

$$R = \begin{bmatrix} A & B \\ B^T & D + \lambda G \end{bmatrix},$$

where, by convention the second fundamental form is A or $-A$, depending on the rather arbitrary choice of inward or outward normal to ν. If we select our normal direction to be ∇p, then $-A$ is the classical second fundamental form as is consistent with the choice of sign in our definition in Section 1.3.3. (All this concern with the sign is irrelevant to the content of this paper and can be ignored by the reader.)

Next, in view of the presumed strict positive curvature of ν, the matrix A at each point of ν is negative definite. Thus, by standard Schur complement arguments, R will be negative definite on any compact region of $\mathbb{R}^g$ if

$$D + \lambda G - B^T A^{-1} B \prec 0$$

on this region. Thus, strict convexity assumptions on the sublevel sets of p make the relaxed Hessian negative definite even if $\delta = 0$.

In the non-commutative case, Remark 6.8 (below) implies that if n is large enough, then the second fundamental form will have a nonzero null space. Consequently, in this paper we shall be forced to consider the case where A is negative semi-definite and has a nonzero null vector η. Then, to obtain negative definite R, we must add another negative term, say δI, with arbitrarily small $\delta < 0$. After this, the argument based on choosing $-\lambda$ large succeeds as before. This δ term plus the λ term produces the relaxed Hessian, and proper selection of these terms make it negative definite.

Some additional detail on the connection between convexity of a sublevel set and non-negativity of the relaxed Hessian (positive curvature) is provided in Section 8.

The following theorem provides a link between the signature of the clamped second fundamental form with that of the Hessian.

THEOREM 3.2. *Suppose p is a symmetric nc polynomial of degree d in g symmetric variables and $(X, v) \in (\mathbb{R}^{n\times n}_{\text{sym}})^g \times \mathbb{R}^n$.*

(1) *There exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0]$ there exists a $\lambda_\delta > 0$ so that for every $\lambda \geq \lambda_\delta$,*

$$e^n_-\left(X, v; p''_{\lambda,\delta}, \left(\mathbb{R}^{n\times n}_{\text{sym}}\right)^g\right) = c^n_-(X, v; p).$$

(2) *There exists a $\delta_0 < 0$ such that for each $\delta \in [\delta_0, 0)$ there exists a $\lambda_\delta < 0$ so that for every $\lambda \leq \lambda_\delta$,*

$$e^n_+\left(X, v; p''_{\lambda,\delta}, \left(\mathbb{R}^{n\times n}_{\text{sym}}\right)^g\right) = c^n_+(X, v; p).$$

(3) *If $\mathcal{V}(p)$ has positive curvature at $(X,v)\in\mathcal{V}_n(p)$, i.e., if*

$$\langle p''(X)[H]v,v\rangle \le 0 \quad \textit{for every } H\in\mathcal{T}_p(X,v),$$

then $c_+^n(X,v;p)=0$ and for every $\delta<0$ there exists a $\lambda_\delta<0$ such that for all $\lambda\le\lambda_\delta$,

$$\langle p''_{\lambda,\delta}(X)[H]v,v\rangle \le 0 \quad \textit{for every } H\in\left(\mathbb{R}^{n\times n}_{\mathrm{sym}}\right)^g;$$

i.e., the relaxed Hessian of p is negative at (X,v).

Note: (1) and (2) do not require (X,v) to be in $\mathcal{V}_n(p)$.

The proof employs a variant of the Hessian which we now introduce. Let $p''(X)[H][K]$ denote the matrix obtained by differentiating $p'(X)[H]$ in the direction $K\in(\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$; i.e.,

$$p''(X)[H][K]=\lim_{t\to 0}\frac{1}{t}\big(p'(X+tK)[H]-p'(X)[H]\big).$$

In particular,

$$p''(X)[H]=p''(X)[H][H] \quad\text{and}\quad p''(X)[H][K]=p''(X)[K][H].$$

Proof of Theorem 3.2. Let d denote the degree of p and g the number of non-commutative symmetric variables. To verify (1), let $\mathcal{H}=(\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$ endowed with the Hilbert Schmidt norm:

$$\langle H,K\rangle_{\mathcal{H}}=\operatorname{trace}K^TH=\sum_{j=1}^{g}\operatorname{trace}K_jH_j. \tag{3.4}$$

The mapping

$$\mathcal{H}\times\mathcal{H}\ni(H,K)\mapsto\langle p''(X)[H][K]v,v\rangle_{\mathbb{R}^n}$$

is bilinear and, because $\mathcal{H}$ is finite dimensional, bounded. Thus, there is a bounded linear operator A on $\mathcal{H}$ so that

$$\langle AH,K\rangle_{\mathcal{H}}=\langle p''(X)[H][K]v,v\rangle_{\mathbb{R}^n}. \tag{3.5}$$

The operator A is selfadjoint with respect to this inner product, because

$$p''(X)[H][K]=p''(X)[K][H].$$

Similarly, there are bounded linear selfadjoint operators Q and E on $\mathcal{H}$ so that

$$\langle QH,K\rangle_{\mathcal{H}}=\langle p'(X)[H]v,p'(X)[K]v\rangle_{\mathbb{R}^n},$$
$$\langle EH,K\rangle_{\mathcal{H}}=\langle \widetilde{V}(X)[H]v,\widetilde{V}(X)[K]v\rangle_{\mathbb{R}^n}.$$

Thus,

$$\langle p''_{\lambda,\delta}(X)[H]v,v\rangle_{\mathbb{R}^n}=\langle(A+\lambda Q+\delta E)H,H\rangle_{\mathcal{H}}$$

for all $H\in(\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$. Let

$$\mathcal{N}=\{H\in\mathcal{H}:\widetilde{V}(X)[H]v=0\}.$$

Note that $H \in \mathcal{N}$ if and only if $H_j w(X)v = 0$ for $j = 1, \dots, g$ and all words w of degree at most $d-1$. In particular, $A\mathcal{N} = \{0\}$, $Q\mathcal{N} = \{0\}$ and $E\mathcal{N} = \{0\}$, and thus it suffices to focus on $\mathcal{N}^\perp$.

Let

$$\mathcal{M} = \left\{H \in \mathcal{N}^\perp : p'(X)[H]v = 0\right\}$$

and

$$\mathcal{L} = \mathcal{M}^\perp \cap \mathcal{N}^\perp.$$

Then

$$\mathcal{N}^\perp = \mathcal{M} \oplus \mathcal{L}.$$

Now let $\mathcal{M}_-$ denote the span of the eigenspaces corresponding to the negative eigenvalues of the compression of A to $\mathcal{M}$ (i.e., of $P_\mathcal{M} A|_\mathcal{M}$), let $\mathcal{M}_+$ denote the orthogonal complement of $\mathcal{M}_-$ in $\mathcal{M}$ and observe that:

(a) $\dim \mathcal{M}_- = c_-^n(X, v; p)$;
(b) Q is strictly positive definite on $\mathcal{L}$ and is 0 on $\mathcal{M}$;
(c) E is strictly positive definite on $\mathcal{N}^\perp = \mathcal{M} \oplus \mathcal{L}$;
(d) $P_{\mathcal{M}_+} A|_{\mathcal{M}_+} \succeq 0$ and $P_{\mathcal{M}_-} A|_{\mathcal{M}_-} \prec 0$.

Hence, there is a $\delta_0 > 0$ so that if $0 < \delta \leq \delta_0$, then the compression of $A + \delta E$ to $\mathcal{M}_-$ is negative definite and the compression of $A + \delta E$ to $\mathcal{M}_+$ is positive definite. Therefore, if $\lambda > 0$ is sufficiently large, the compression of $A + \delta E + \lambda Q$ to $\mathcal{M}_+ \oplus \mathcal{L}$ is positive definite; whereas its compression to $\mathcal{M}_-$ is equal to the compression of $A + \delta E$ to $\mathcal{M}_-$, which is negative definite. Thus, as

$$\mathcal{N}^\perp = \mathcal{M}_- \oplus (\mathcal{M}_+ \oplus \mathcal{L}),$$

it now follows that $A + \delta E + \lambda Q$ has $c_-(X, v; p)$ negative eigenvalues (counting with multiplicity).

The proof of (2) is similar to the proof of (1). To prove (3), fix $\delta_0 < 0$ so that (2) holds and choose δ so that $\delta_0 \leq \delta < 0$. Then there is a λ_δ satisfying the conclusion of (2); i.e., $e_+^n(X, v; p''_{\lambda,\delta}, (\mathbb{R}^{n\times n}_{\text{sym}})^g) = 0$ and thus (3) holds. Moreover, if $\delta_* < \delta$, then it is still the case that $e_+^n(X, v; p''_{\lambda,\delta_*}, (\mathbb{R}^{n\times n}_{\text{sym}})^g) = 0$ and hence (3) holds for all $\delta < 0$. $\square$

REMARK 3.3. Let $(\mathcal{H}_A)_-$ (resp., $(\mathcal{H}_A)_0$, $(\mathcal{H}_A)_+$) denote the span of the eigenvectors of A corresponding to negative (resp., zero, positive) eigenvalues of the operator A that was introduced in the proof of Theorem 3.2. Then

$$(\mathbb{R}^{n\times n}_{\text{sym}})^g = (\mathcal{H}_A)_- \oplus (\mathcal{H}_A)_0 \oplus (\mathcal{H}_A)_+. \tag{3.6}$$

In particular, $(\mathcal{H}_A)_-$ is a maximal strictly negative subspace of $(\mathbb{R}^{n\times n}_{\text{sym}})^g$ with respect to the indefinite inner product (3.5), and $(\mathcal{H}_A)_0 \oplus (\mathcal{H}_A)_+$ is complementary to it.

3.1. Example illustrating Theorem 3.2. In this subsection, we continue with Example 2.4 to illustrate the ingredients in the proof of Theorem 3.2.

EXAMPLE 3.4. Let $p(x) = x^3$, let $X = \operatorname{diag}\{I_2, -I_3\}$, and let

$$v^T = \begin{bmatrix} a & 0 & 0 & 0 & e \end{bmatrix}$$

with $a \geq e > 0$.

Let u_j denote the jth standard basis vector for $\mathbb{R}^5$ with $j = 1, \dots, 5$ and let S_{ij} denote the normalized symmetrized elementary matrices

$$S_{ij} = \begin{cases} \left(u_i u_j^T + u_j u_i^T\right)/\sqrt{2}, & \text{for } i \neq j, \\ u_i u_i^T, & \text{for } i = j. \end{cases}$$

Then the set of 15 matrices

$$\{S_{ij} : i, j = 1, \dots, 5 \text{ and } i \leq j\}$$

is an orthonormal basis for $\mathbb{R}^{5\times 5}_{\text{sym}}$ with respect to the trace inner product (3.4). In terms of the notation of Theorem 3.2,

$$\mathcal{N} = \left\{H \in \mathbb{R}^{5\times 5}_{\text{sym}} : \widetilde{V}(X)[H]v = 0\right\} = \operatorname{span}\{S_{22}, S_{23}, S_{24}, S_{33}, S_{34}, S_{44}\},$$
$$\mathcal{N}^\perp = \operatorname{span}\{S_{11}, S_{21}, S_{31}, S_{41}, S_{51}, S_{25}, S_{35}, S_{45}, S_{55}\}$$

and

$$\mathcal{M} = \left\{H \in \mathcal{N}^\perp : p'(X)[H]v = 0\right\} = \operatorname{span}\{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4\},$$

where

$$\begin{aligned}
\mathfrak{h}_1 &= -eS_{11} + 3a\sqrt{2}S_{15} - \frac{a^2}{e}S_{55}, \\
\mathfrak{h}_2 &= eS_{21} - 3aS_{25}, \\
\mathfrak{h}_3 &= 3eS_{31} - aS_{35}, \\
\mathfrak{h}_4 &= 3eS_{41} - aS_{45}.
\end{aligned}$$

Moreover,

$$\mathcal{L} := \mathcal{M}^\perp \cap \mathcal{N}^\perp = \operatorname{span}\{\mathfrak{h}_5, \mathfrak{h}_6, \mathfrak{h}_7, \mathfrak{h}_8, \mathfrak{h}_9\},$$

where

$$\begin{aligned}
\mathfrak{h}_5 &= 3aS_{21} + eS_{25}, \\
\mathfrak{h}_6 &= aS_{31} + 3eS_{35}, \\
\mathfrak{h}_7 &= aS_{41} + 3eS_{45}, \\
\mathfrak{h}_8 &= 6aS_{11} + \sqrt{2}eS_{15}, \\
\mathfrak{h}_9 &= -a^2 S_{11} + \frac{3\sqrt{2}a^3}{e}S_{15} + \left(e^2 + 18a^2\right)S_{55}
\end{aligned}$$

and

$$\operatorname{trace} \mathfrak{h}_j^T \mathfrak{h}_i = 0 \quad \text{for } i, j = 1, \dots, 9 \text{ if } i \neq j.$$

Thus, as

$$\begin{aligned}
&\mathfrak{h}_1 v = 2aeu_1 + 2a^2u_5, \qquad \mathfrak{h}_1 Xv = -4aeu_1 + 4a^2u_5,\\
&\mathfrak{h}_2 v = -\sqrt{2}aeu_2, \qquad \mathfrak{h}_2 Xv = 2\sqrt{2}aeu_2,\\
&\mathfrak{h}_3 v = \sqrt{2}aeu_3, \qquad \mathfrak{h}_3 Xv = 2\sqrt{2}aeu_3,\\
&\mathfrak{h}_4 v = \sqrt{2}aeu_4, \qquad \mathfrak{h}_4 Xv = 2\sqrt{2}aeu_4,
\end{aligned}$$

it is readily confirmed that if $H = \sum_{j=1}^4 \alpha_j \mathfrak{h}_j$, then

$$p'(X)[H]v = (2H + XHX)v = 0$$

and

$$\begin{aligned}
\big\langle p''(X)[H]v, v\big\rangle &= -6\langle XHv, Hv\rangle\\
&= 12\big\{2\alpha_1^2a^2\big(a^2 - e^2\big) - \alpha_2^2a^2e^2 + \alpha_3^2a^2e^2 + \alpha_4^2a^2e^2\big\}.
\end{aligned}$$

Consequently,

$$\mathcal{M}_- = \operatorname{span}\{\mathfrak{h}_2\} \quad \text{and} \quad \mathcal{M}_+ = \mathcal{M} \ominus \mathcal{M}_- = \operatorname{span}\{\mathfrak{h}_1, \mathfrak{h}_3, \mathfrak{h}_4\}.$$

Note that if $a > e$, then $\mathcal{M}_+$ contributes three positive squares, whereas, if $a = e$, then only two.

Next we look at the λ and δ terms used in the relaxed Hessian. Since

$$\begin{aligned}
&\mathfrak{h}_5 v = \frac{1}{\sqrt{2}}\big(3a^2 + e^2\big)u_2, \qquad \mathfrak{h}_5 Xv = \frac{1}{\sqrt{2}}\big(3a^2 - e^2\big)u_2,\\
&\mathfrak{h}_6 v = \frac{1}{\sqrt{2}}\big(a^2 + 3e^2\big)u_3, \qquad \mathfrak{h}_6 Xv = \frac{1}{\sqrt{2}}\big(a^2 - 3e^2\big)u_3,\\
&\mathfrak{h}_7 v = \frac{1}{\sqrt{2}}\big(a^2 + 3e^2\big)u_4, \qquad \mathfrak{h}_7 Xv = \frac{1}{\sqrt{2}}\big(a^2 - 3e^2\big)u_4,\\
&\mathfrak{h}_8 v = \big(6a^2 + e^2\big)u_1 + aeu_5, \qquad \mathfrak{h}_8 Xv = \big(6a^2 - e^2\big)u_1 + aeu_5,\\
&\mathfrak{h}_9 v = 2a^3u_1 + \left(\frac{3a^4}{e} + e^3 + 18a^2e\right)u_5,\\
&\mathfrak{h}_9 Xv = -4a^3u_1 + \left(\frac{3a^4}{e} - \big(e^3 + 18a^2e\big)\right)u_5,
\end{aligned}$$

it is readily checked that if $H = \sum_{j=1}^9 \alpha_j \mathfrak{h}_j$, then

$$\begin{aligned}
p'(X)[H]v = {}&\alpha_5\frac{1}{\sqrt{2}}\big(9a^2 + e^2\big)u_2 + \alpha_6\frac{1}{\sqrt{2}}\big(a^2 + 9e^2\big)u_3 + \alpha_7\frac{1}{\sqrt{2}}\big(a^2 + 9e^2\big)u_4\\
&+ \alpha_8\big[\big(18a^2 + e^2\big)u_1 + aeu_5\big] + \alpha_9\left(3\frac{a^4}{e} + 3e^3 + 54a^2e\right)u_5
\end{aligned}$$

and hence that

$$\begin{aligned}
&\big\langle p'(X)[H]v, p'(X)[H]v\big\rangle\\
&\quad = \frac{1}{2}\alpha_5^2\big(9a^2 + e^2\big)^2 + \frac{1}{2}\alpha_6^2\big(a^2 + 9e^2\big)^2
\end{aligned}$$

$$+\frac{1}{2}\alpha_7^2\big(a^2+9e^2\big)^2+\alpha_8^2\big(18a^2+e^2\big)^2$$
$$+\left(\alpha_8 ae+\alpha_9\left(3\frac{a^4}{e}+3e^3+54a^2e\right)\right)^2.$$

Now we calculate the δ term of the relaxed Hessian. The preceding formulas for $\mathfrak{h}_j$, $j=1,\dots,9$ and the fact that $Xu_j=u_j$ for $j=1,2$ and $Xu_j=-u_j$ for $j=3,4,5$ imply that if $H=\sum_{j=1}^9\alpha_j\mathfrak{h}_j$, then

$$Hv=\sum_{j=1}^{9}\alpha_j\mathfrak{h}_jv=\sum_{j=1}^{5}\beta_ju_j \quad\text{and}\quad HXv=\sum_{j=1}^{5}\gamma_ju_j$$

for appropriately chosen constants $\beta_1,\dots,\beta_5$ and $\gamma_1,\dots,\gamma_5$; i.e.,

$$\beta_2=-\sqrt{2}ae\alpha_2+\frac{1}{\sqrt{2}}\big(3a^2+e^2\big)\alpha_5,\qquad \beta_3=\sqrt{2}ae\alpha_3+\frac{1}{\sqrt{2}}\big(a^2+3e^2\big)\alpha_6,$$

etc. Thus,

$$\begin{aligned}\big\langle\widetilde{V}(X)[H]v,\widetilde{V}(X)[H]v\big\rangle &= 2v^THHv+v^TXHHXv\\ &=\sum_{j=1}^{5}\big(2\beta_j^2+\gamma_j^2\big).\end{aligned}$$

4. Direct sums

The next lemma expresses a basic principle [CHSY03], [HMV06] that provides a link between the direct sum and the minimum degree (irreducibility) hypotheses in Theorem 1.4. Also in the section is the observation that the full rank condition, as defined in Section 1.3.2, is preserved under direct sums.

Lemma 4.1. *Suppose that the set $\mathcal{S}=\bigcup_{n\ge1}\mathcal{S}_n$ respects direct sums, let N be a given positive integer and let $\mathcal{W}_N$ denote the set of all words of length at most N. Then, either,*

(E) *there exists a positive integer n and a pair $(X,v)\in\mathcal{S}_n$ such that the set*

$$\big\{w(X)v : w\in\mathcal{W}_N\big\}$$

is a linearly independent set of vectors in $\mathbb{R}^n$; or

(O) *there exist real numbers q_w for each $w\in\mathcal{W}_N$ not all of which are zero such that for each $(X,v)\in\mathcal{S}$,*

$$0=\left(\sum_{|w|\le N}q_ww(X)\right)v.$$

It is useful to note that the alternative Lemma 4.1(O) *is equivalent* to saying that there exists a (not necessarily symmetric) nc polynomial q of degree at most N such that $q(X)v=0$ for every choice of $(X,v)\in\mathcal{S}$.

Proof of Lemma 4.1. If condition Lemma 4.1(E) does not hold, then for each positive integer t and each finite set $F=\{(X^1,v^1),\dots,(X^t,v^t)\}$ with $(X^j,v^j)\in\mathcal{S}_{n_j}$ for $j=1,\dots,t$, there is a nonzero function $c_F:\mathcal{W}_N\to\mathbb{R}$ such that

$$0=\sum c_F(w)w(X_F)v_F.$$

Without loss of generality, it may be assumed that $\sum c_F(w)^2=1$ so that c_F can be identified with an element of $\mathbb{B}^L$, the unit ball in $\mathbb{R}^L$, where $L=\sum_0^N g^j$. For each choice of F, let $\mathcal{C}_F\subset\mathbb{B}^L$ denote the collection of all such normalized coefficients c_F (corresponding to the possibly many nc polynomials that annihilate (X,v)) and observe that each $\mathcal{C}_F$ is compact, and of course nonempty by hypothesis. Further, if $F\subset G$, i.e., if $G=\{(X^1,v^1),\dots,(X^t,v^t),(Y,u)\}$ with $(Y,u)\in\mathcal{S}_r$ for some r, then $\mathcal{C}_F\supset\mathcal{C}_G$, from which it follows that the collection

$$\{\mathcal{C}_F : F \text{ is a finite subset of } \mathcal{S}\}$$

satisfies the finite intersection property; i.e., every finite intersection is nonempty. It follows that the whole intersection is nonempty and thus there is a $c\in\mathbb{B}^L$ so that

$$0=\sum c(w)w(X)v \quad \text{for all } (X,v)\in\mathcal{S}. \tag{4.1}$$

□

4.1. Direct sums of full rank points. To show that our main theorem applies to yield Corollary 1.3, we need our full rank assumptions to mesh with the hypotheses of Theorem 1.4. The issue is to show that full rank points respect direct sums.

Lemma 4.2. *Let p be a symmetric nc polynomial in symmetric variables and let $F=\{(X^1,v^1),\dots,(X^t,v^t)\}$ where $(X^j,v^j)\in(\mathbb{R}^{n_j\times n_j}_{\text{sym}})^g\times\mathbb{R}^{n_j}$. If each (X^j,v^j) is a full rank point for p, then so is (X_F,v_F).*

Proof. Let $n=n_1+\cdots+n_t$. Given $w=\operatorname{col}(w_1,\dots,w_t)\in\mathbb{R}^n$ with $w_j\in\mathbb{R}^{n_j}$, there exists $H^j\in(\mathbb{R}^{n_j\times n_j}_{\text{sym}})^g$ so that $p'(X^j)[H^j]v^j=w^j$. This holds because each (X^j,v^j) is a full rank point. Thus, if $H_F=\operatorname{diag}\{H^1,\dots,H^t\}$, then $p'(X_F)[H_F]v_F=w$. □

Given an algebraically open set $\mathcal{O}$ and a symmetric nc polynomial p, let

$$\mathcal{B}(p,\mathcal{O})=\big\{(X,v):(X,v)\text{ is a full rank point, } X\in\mathcal{O}\cap\mathcal{V}(p)\big\}. \tag{4.2}$$

Lemma 4.3. *The set $\mathcal{B}(p,\mathcal{O})$ respects direct sums.*

Proof. This is an immediate consequence of Lemma 4.2 and the fact that both $\mathcal{O}$ and $\mathcal{V}(p)$ respect direct sums. □

5. The middle matrix-border vector representation

Our approach depends heavily upon the border vector-middle matrix representation for non-commutative quadratic functions which we now describe.

A symmetric nc polynomial $f(x)[h]$ in the $2g$ variables $x=(x_1,\dots,x_g)$ and $h=(h_1,\dots,h_g)$ that is of degree s in x and homogeneous of degree two in h admits a representation of the form

$$f(x)[h] = \begin{bmatrix} V_0(x)[h]^T & \cdots & V_s(x)[h]^T \end{bmatrix} Z(x) \begin{bmatrix} V_0(x)[h] \\ \vdots \\ V_s(x)[h] \end{bmatrix}, \tag{5.1}$$

where $Z(x)$ is a square matrix of nc polynomials and the $V_j(x)[h]$ are vectors of nc words of the form $h_l w(x)$ over choices of words w of length j.

In the case that $f(x)[h]$ is the Hessian of a symmetric nc polynomial p, the middle matrix Z takes a rigid form which has been exploited earlier in [HM04], [CHSY03], [DHM07b], [DHM07a] and [DGHM09]:

$$\begin{aligned} p''(x)[h] &= V(x)[h]^T Z(x) V(x)[h] \\ &= \begin{bmatrix} V_0^T, V_1^T, \dots, V_\ell^T \end{bmatrix} \begin{bmatrix} Z_{00} & Z_{01} & \cdots & Z_{0,\ell-1} & Z_{0\ell} \\ Z_{10} & Z_{11} & \cdots & Z_{1,\ell-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ Z_{\ell 0} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_\ell \end{bmatrix}, \end{aligned} \tag{5.2}$$

in which $\ell = d-2$, $V(x)[h]$ is the border vector with vector components $V_j(x)[h]$ of height g^{j+1}, and $Z(x)=[Z_{ij}(x)]$, $i,j=0,\dots,d-2$, the *middle matrix*, is a symmetric matrix polynomial with matrix polynomial entries $Z_{ij}(x)$ of size $g^{i+1}\times g^{j+1}$ and degree no more than $(d-2)-(i+j)$ for $i+j\le d-2$ with $Z_{ij}(x)=0$ for $i+j>d-2$. Since p is symmetric $Z_{ij}=Z_{ji}^T$ and since p has degree d, Z_{ij} is constant when $i+j=d-2$.

The matrix $\mathscr{Z}=Z(0)$, evaluated at $0\in\mathbb{R}^g$, will be called the *scalar middle matrix* of p''. The main conclusions from [DHM07a] that are relevant to this paper are:

(1) $Z(x)$ is polynomially congruent to the scalar middle matrix $\mathscr{Z}=Z(0)$, i.e., there exists a matrix polynomial $B(x)$ with an inverse $B(x)^{-1}$ that is again a matrix polynomial such that

$$\mathscr{Z} = Z(0) = B(x)^T Z(x) B(x). \tag{5.3}$$

(2) $\mu_\pm(\mathscr{Z}) = \sigma_\pm^{\min}(p''(x)[h])$.

(3) If $X\in(\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$, then

$$\mu_\pm\big(Z(X)\big) = n\mu_\pm(\mathscr{Z}). \tag{5.4}$$

(4) The degree d of $p(x)$ is subject to the bound

$$d \le 2\mu_\pm(\mathscr{Z}) + 2. \tag{5.5}$$

Here $\mu_\pm$ are the number of positive/negative eigenvalues of the indicated matrix. Note that item (5.5) bounds the degree of p in terms of the signature (the number of positive, negative and zero eigenvalues) of the middle matrix of its Hessian.

The relaxed Hessian $p''_{\lambda,\delta}$ also has a middle matrix-border vector representation. For the special case where $\delta = 0$, in terms of the notation introduced in (3.1), we have

$$p''_{\lambda,0}(x)[h] = \widetilde{V}(x)[h]^T Z_\lambda(x) \widetilde{V}(x)[h].$$

The polynomial congruence of equation (5.3) extends to Z_λ in that

$$Z_\lambda(x) \sim Z_\lambda(0) =: \mathcal{Z}_\lambda = \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \lambda W \end{bmatrix},$$

where W is a rank one positive matrix and $\sim$ denotes a polynomial congruence which is independent of λ.

PROPOSITION 5.1. *If $X \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$, then*

$$\mu_\pm\big(Z_\lambda(X)\big) = n\mu_\pm(\mathcal{Z}_\lambda).$$

Moreover, if $\lambda > 0$, then

$$\mu_+(\mathcal{Z}_\lambda) = \mu_+(\mathcal{Z}) + 1 \quad \textit{and} \quad \mu_-(\mathcal{Z}_\lambda) = \mu_-(\mathcal{Z});$$

whereas, if $\lambda < 0$, then

$$\mu_+(\mathcal{Z}_\lambda) = \mu_+(\mathcal{Z}) \quad \textit{and} \quad \mu_-(\mathcal{Z}_\lambda) = \mu_-(\mathcal{Z}) + 1.$$

Proof. This is an immediate consequence of the polynomial congruence for $Z_\lambda(x)$ that is described above. $\square$

The middle matrix representation for the relaxed Hessian is

$$p''_{\lambda,\delta}(x)[h] = \widetilde{V}(x)[h]^T Z_{\lambda,\delta}(x) \widetilde{V}(x)[h],$$

where

$$Z_{\lambda,\delta}(x) = Z_\lambda(x) + \delta I.$$

The form of $Z_{\lambda,\delta}$ and the polynomial congruence for Z_λ together yield the following variant of Proposition 5.1, which is needed for this paper.

PROPOSITION 5.2. *Let $X \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$ be given. There exists an $\epsilon > 0$ so that if $0 \le \delta < \epsilon$, then*

$$\mu_-\big(Z_{\lambda,\delta}(X)\big) = n\mu_-(\mathcal{Z}) \quad \textit{for every } \lambda \ge 0.$$

Similarly, there exists an $\epsilon < 0$ so that if $\epsilon < \delta \le 0$, then

$$\mu_+\big(Z_{\lambda,\delta}(X)\big) = n\mu_+(\mathcal{Z}) \quad \textit{for every } \lambda \le 0.$$

Proof. The preceding discussion implies that $Z_{\lambda,\delta}(X)$ is polynomially congruent to a sum of real symmetric matrices of the form

$$A+\delta B \quad \text{where } A\sim Z_{\lambda,0}(X), \text{ and } B\succ 0.$$

Therefore, if the eigenvalues of each of these matrices are indexed in increasing order, i.e., $\lambda_1\le\lambda_2\le\cdots$, it follows readily from the Courant–Fischer theorem that if $\delta\ge 0$, then

$$\lambda_j(A)\le\lambda_j(A+\delta B),$$

i.e., an additive perturbation of A by the positive definite matrix δB shifts the eigenvalues of A to the right. Thus, each nonnegative eigenvalue of A moves into a nonnegative eigenvalue of $A+\delta B$. On the other hand, if $\delta\ge 0$ is kept sufficiently small, so that the shift to the right is small, the negative eigenvalues of A will move into negative eigenvalues of $A+\delta B$. Since

$$\mu_-(A)=\mu_-\big(Z_{\lambda,0}(X)\big)=n\mu_-(\mathcal{Z}) \quad \text{for every } \lambda\ge 0,$$

this completes the proof of the first assertion. The proof of the second is similar. $\square$

5.1. Relaxed Hessian example. The example in this subsection illuminates the middle matrix representation of the relaxed Hessian.

Example 5.3. Let $p(X)=X^3$. Then

$$\begin{aligned} p'(X)[H]&=X^2H+XHX+HX^2 \quad \text{and}\\ p''(X)[H]&=2HXH+2XH^2+2H^2X. \end{aligned}$$

Therefore,

$$\begin{aligned} p''_{\lambda,\delta}(X)[H]=\begin{bmatrix} H\\ HX\\ HX^2\end{bmatrix}^T &\left\{\begin{bmatrix} 2X & 2I & 0\\ 2I & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\lambda\begin{bmatrix} X^2\\ X\\ I\end{bmatrix}\begin{bmatrix} X^2 & X & I\end{bmatrix}\right.\\ &\left.+\,\delta\begin{bmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\right\}\begin{bmatrix} H\\ HX\\ HX^2\end{bmatrix}. \end{aligned}$$

The middle matrix for the relaxed Hessian is inside the braces.

Moreover, since

$$\begin{aligned} &\begin{bmatrix} 2X & 2I & 0\\ 2I & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\lambda\begin{bmatrix} X^2\\ X\\ I\end{bmatrix}\begin{bmatrix} X^2 & X & I\end{bmatrix}\\ &=\begin{bmatrix} I & 0 & X^2\\ 0 & I & X\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix} I & X/2 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix} 0 & 2I & 0\\ 2I & 0 & 0\\ 0 & 0 & \lambda I\end{bmatrix} \end{aligned}$$

$$\times \begin{bmatrix} I & 0 & 0 \\ X/2 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ X^2 & X & I \end{bmatrix}$$

$$= \begin{bmatrix} I & X/2 & X^2 \\ 0 & I & X \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 2I & 0 \\ 2I & 0 & 0 \\ 0 & 0 & \lambda I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ X/2 & I & 0 \\ X^2 & X & I \end{bmatrix},$$

we have

$$p''_{\lambda,\delta}(X)[H] = \begin{bmatrix} H \\ HX \\ HX^2 \end{bmatrix}^T \left\{ \begin{bmatrix} I & X/2 & X^2 \\ 0 & I & X \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 2I & 0 \\ 2I & 0 & 0 \\ 0 & 0 & \lambda I \end{bmatrix} \right.$$
$$\left. \times \begin{bmatrix} I & 0 & 0 \\ X/2 & I & 0 \\ X^2 & X & I \end{bmatrix} + \delta I \right\} \begin{bmatrix} H \\ HX \\ HX^2 \end{bmatrix}.$$

As a more concrete special case, suppose $X = \mathrm{diag}\{I_2, -I_3\}$, $v_1^T = [a \;\; 0]$, $v_2^T = [0 \;\; 0 \;\; e]$, $a > e > 0$ and $v = \mathrm{col}(v_1, v_2)$. Then

$$\left\{ \begin{bmatrix} H \\ HX \\ HX^2 \end{bmatrix} v : H \in \mathbb{R}^{5\times 5}_{\mathrm{sym}} \right\} \tag{5.6}$$
$$= \mathrm{span} \left\{ \begin{bmatrix} u_1 \\ u_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ u_2 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_3 \\ u_3 \\ u_3 \end{bmatrix}, \begin{bmatrix} u_4 \\ u_4 \\ u_4 \end{bmatrix}, \begin{bmatrix} au_5 + eu_1 \\ au_5 - eu_1 \\ au_5 + eu_1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u_4 \\ 0 \end{bmatrix}, \begin{bmatrix} u_5 \\ -u_5 \\ u_5 \end{bmatrix} \right\},$$

where u_j denotes the jth standard basis vector for $\mathbb{R}^5$ for $j = 1, \dots, 5$. A vector in this span is of the form

$$w = \begin{bmatrix} \mathfrak{a}_1 + \mathfrak{b}_1 \\ \mathfrak{a}_2 + \mathfrak{b}_2 \\ \mathfrak{a}_1 + \mathfrak{b}_1 \end{bmatrix},$$

where

$$\mathfrak{a}_1 = (\alpha_1 + \alpha_5 e)u_1 + \alpha_2 u_2, \qquad \mathfrak{a}_2 = (\alpha_1 - \alpha_5 e)u_1 + (\alpha_2 + \alpha_6)u_2,$$
$$\mathfrak{b}_1 = \alpha_3 u_3 + \alpha_4 u_4 + (\alpha_5 a + \alpha_9)u_5 \quad \text{and}$$
$$\mathfrak{b}_2 = (\alpha_3 + \alpha_7)u_3 + (\alpha_4 + \alpha_8)u_4 + (\alpha_5 a - \alpha_9)u_5$$

for some choice of $\alpha_1, \dots, \alpha_9 \in \mathbb{R}$.

Next, since $X\mathfrak{a}_j = \mathfrak{a}_j$, $X\mathfrak{b}_j = -\mathfrak{b}_j$ for $j = 1, 2$, it is readily seen that

$$\begin{bmatrix} I & 0 & 0 \\ X/2 & I & 0 \\ I & X & I \end{bmatrix} \begin{bmatrix} \mathfrak{a}_1 + \mathfrak{b}_1 \\ \mathfrak{a}_2 + \mathfrak{b}_2 \\ \mathfrak{a}_1 + \mathfrak{b}_1 \end{bmatrix} = \begin{bmatrix} \mathfrak{a}_1 + \mathfrak{b}_1 \\ \mathfrak{a}_3 + \mathfrak{b}_3 \\ \mathfrak{a}_4 + \mathfrak{b}_4 \end{bmatrix},$$

where

$$\mathfrak{a}_3 = \mathfrak{a}_2 + (1/2)\mathfrak{a}_1, \qquad \mathfrak{b}_3 = \mathfrak{b}_2 - (1/2)\mathfrak{b}_1,$$
$$\mathfrak{a}_4 = 2\mathfrak{a}_1 + \mathfrak{a}_2, \quad \text{and} \quad \mathfrak{b}_4 = 2\mathfrak{b}_1 - \mathfrak{b}_2.$$

Thus, as

$$\mathfrak{a}_i^T \mathfrak{b}_j = 0 \quad \text{for } i, j = 1, \dots, 4,$$

it is readily seen that $v^T p''_{\lambda,\delta}(X)[H]v$ is of the form

$$\begin{aligned} w^T &\left\{ \begin{bmatrix} I & X/2 & X^2 \\ 0 & I & X \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 2I & 0 \\ 2I & 0 & 0 \\ 0 & 0 & \lambda I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ X/2 & I & 0 \\ X^2 & X & I \end{bmatrix} + \delta I \right\} w \\ &= \mathfrak{a}_1^T \mathfrak{a}_3 + \mathfrak{a}_3^T \mathfrak{a}_1 + \mathfrak{b}_3^T \mathfrak{b}_1 + \mathfrak{b}_1^T \mathfrak{b}_3 + \lambda\big(\mathfrak{a}_4^T \mathfrak{a}_4 + \mathfrak{b}_4^T \mathfrak{b}_4\big) \\ &\quad + \delta\big(2\mathfrak{a}_1^T \mathfrak{a}_1 + \mathfrak{a}_2^T \mathfrak{a}_2 + 2\mathfrak{b}_1^T \mathfrak{b}_1 + \mathfrak{b}_2^T \mathfrak{b}_2\big). \end{aligned}$$

In particular, the term to the right of δ is equal to zero if and only if

$$\alpha_1 = \dots = \alpha_9 = 0.$$

Remark 5.4. We remark that since

$$Q := \begin{bmatrix} I & X/2 & X^2 \\ 0 & I & X \\ 0 & 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -X/2 & -X^2/2 \\ 0 & I & -X \\ 0 & 0 & I \end{bmatrix},$$

the formula for the relaxed Hessian can be re-expressed as

$$\begin{aligned} p''_{\lambda,\delta}(X)[H] = &\begin{bmatrix} H \\ HX \\ HX^2 \end{bmatrix}^T \begin{bmatrix} I & X/2 & X^2 \\ 0 & I & X \\ 0 & 0 & I \end{bmatrix} \left\{ \begin{bmatrix} 0 & 2I & 0 \\ 2I & 0 & 0 \\ 0 & 0 & \lambda I \end{bmatrix} + \delta Q Q^T \right\} \\ &\times \begin{bmatrix} I & 0 & 0 \\ X/2 & I & 0 \\ X^2 & X & I \end{bmatrix} \begin{bmatrix} H \\ HX \\ HX^2 \end{bmatrix}. \end{aligned}$$

6. The CHSY Lemma

The CHSY Lemma (which is based on Lemma 9.5 in [CHSY03]) is the key tool relating the signature of the middle matrix of a quadratic form q with the signature of q. In this section, we develop a version of this lemma that is required for the proof of Theorem 1.4.

Let $(X, v) \in (\mathbb{R}^{n\times n}_{\text{sym}})^g \times \mathbb{R}^n$, let $H \in (\mathbb{R}^{n\times n}_{\text{sym}})^g$ and let

$$\mathcal{R}_s(H) = \begin{bmatrix} V_0(X)[H]v \\ \vdots \\ V_s(X)[H]v \end{bmatrix} \quad \text{and} \quad \mathcal{R}_s(\mathcal{H}) = \big\{\mathcal{R}_s(H) : H \in \mathcal{H}\big\}, \tag{6.1}$$

where the vectors $V_j(x)[h]$ are defined just above (3.1), $\mathcal{H}$ is a subspace of $(\mathbb{R}^{n\times n}_{\text{sym}})^g$ and $\mathcal{R}_s(\mathcal{H})$ is a subspace of $\mathbb{R}^{ng\alpha_s}$.

LEMMA 6.1 (CHSY Lemma). *Given a pair of positive integers g and r, a matrix $X \in (\mathbb{R}^{n\times n}_{\text{sym}})^g$ and a vector $v \in \mathbb{R}^n$, suppose that the set*

$$\{w(X)v : w \text{ is a word with } |w| \le r\}$$

is a linearly independent subset of $\mathbb{R}^n$. Then $\mathcal{R}_s((\mathbb{R}^{n\times n}_{\text{sym}})^g)$ is a subspace of $\mathbb{R}^{ng\alpha_s}$ and

$$\operatorname{codim} \mathcal{R}_s\big((\mathbb{R}^{n\times n}_{\text{sym}})^g\big) \le ng(\alpha_s - \alpha_r) + g\alpha_r \frac{\alpha_r - 1}{2} \quad \text{if } s \ge r. \tag{6.2}$$

If $s = r$, then the codimension of $\mathcal{R}_r((\mathbb{R}^{n\times n}_{\text{sym}})^g)$ is independent of n and

$$\operatorname{codim} \mathcal{R}_r\big((\mathbb{R}^{n\times n}_{\text{sym}})^g\big) = g\alpha_r \frac{\alpha_r - 1}{2}. \tag{6.3}$$

REMARK 6.2. It is important to bear in mind that if $\mathcal{H}$ is a subspace of $(\mathbb{R}^{n\times n}_{\text{sym}})^g$, then

$$\operatorname{codim}(\mathcal{H}) = g\frac{n(n+1)}{2} - \dim(\mathcal{H}),$$

whereas

$$\operatorname{codim}\big(\mathcal{R}_s(\mathcal{H})\big) = ng\alpha_s - \dim\big(\mathcal{R}_s(\mathcal{H})\big).$$

LEMMA 6.3. *If $\mathcal{H}$ and $\mathcal{H}^c$ are complementary subspaces of $(\mathbb{R}^{n\times n}_{\text{sym}})^g$, i.e., if*

$$\big(\mathbb{R}^{n\times n}_{\text{sym}}\big)^g = \mathcal{H} \dot{+} \mathcal{H}^c,$$

then $\mathcal{R}_s(\mathcal{H})$ is a subspace of $\mathbb{R}^{ng\alpha_s}$ and

$$\operatorname{codim} \mathcal{R}_s(\mathcal{H}) \le \operatorname{codim} \mathcal{R}_s\big((\mathbb{R}^{n\times n}_{\text{sym}})^g\big) + \operatorname{codim} \mathcal{H},$$
$$\operatorname{codim} \mathcal{R}_s\big(\mathcal{H}^c\big) \le \operatorname{codim} \mathcal{R}_s\big((\mathbb{R}^{n\times n}_{\text{sym}})^g\big) + \dim \mathcal{H}.$$

Proof. If $\mathcal{H}^c$ is a complementary subspace to $\mathcal{H}$ in $(\mathbb{R}^{n\times n}_{\text{sym}})^g$, then

$$\begin{aligned}\dim \mathcal{R}_s\big((\mathbb{R}^{n\times n}_{\text{sym}})^g\big) &= \dim \mathcal{R}_s(\mathcal{H}) + \dim \mathcal{R}_s\big(\mathcal{H}^c\big)\\ &\le \dim \mathcal{R}_s(\mathcal{H}) + \dim \mathcal{H}^c.\end{aligned}$$

The first asserted inequality now follows easily upon re-expressing the last inequality in terms of codimensions.

The second asserted inequality follows from the first by replacing $\mathcal{H}$ by $\mathcal{H}^c$ and noting that the dimension of $\mathcal{H}$ is equal the codimension of $\mathcal{H}^c$. □

LEMMA 6.4. *Let $Z(x)$ denote the middle matrix in the representation (5.1) of a symmetric nc polynomial $f(x)[h]$ of degree s in the g symmetric variables $x = (x_1, \dots, x_g)$ that is homogeneous of degree two in the g symmetric variables $h = (h_1, \dots, h_g)$, and let $\mathcal{H}$ and $\mathcal{G}$ denote subspaces of $(\mathbb{R}^{n\times n}_{\text{sym}})^g$ such that*

$$\begin{aligned}&\big\langle f(X)[H]v, v\big\rangle < 0 \quad \text{if } H \in \mathcal{H} \text{ and } H \ne 0 \quad \text{and}\\ &\big\langle f(X)[H]v, v\big\rangle \ge 0 \quad \text{if } H \in \mathcal{G}.\end{aligned}$$

Then

$$\operatorname{codim}\mathcal{R}_s(\mathcal{G}) \geq \mu_-\big(Z(X)\big) \geq \dim\mathcal{R}_s(\mathcal{H}), \tag{6.4}$$

$$\operatorname{codim}\mathcal{R}_s(\mathcal{H}) \geq \mu_+\big(Z(X)\big) + \mu_0\big(Z(X)\big) \geq \dim\mathcal{R}_s(\mathcal{G}) \tag{6.5}$$

and

$$\dim\mathcal{R}_s(\mathcal{H}) \geq \dim\mathcal{H}. \tag{6.6}$$

Proof. The lower bounds in (6.4) and (6.5) are self-evident. The upper bounds then follow from the identities

$$\begin{aligned}\mu_-\big(Z(X)\big) + \mu_0\big(Z(X)\big) + \mu_+\big(Z(X)\big)&\\ = \dim\mathcal{R}_s(\mathcal{H}) + \operatorname{codim}\mathcal{R}_s(\mathcal{H})&\\ = \dim\mathcal{R}_s(\mathcal{G}) + \operatorname{codim}\mathcal{R}_s(\mathcal{G}),&\end{aligned}$$

upon re-expressing the two lower bounds in terms of codimensions.

To verify (6.6), it suffices to note that if $H_1, \dots, H_k$ is a basis for $\mathcal{H}$, then the vectors $\mathcal{R}_s(H_1), \dots, \mathcal{R}_s(H_k)$ must be linearly independent because of the presumed strict negativity of $\mathcal{H}$. □

LEMMA 6.5. *If the subspaces $\mathcal{G}$ and $\mathcal{H}$ considered in Lemma 6.4 are such that $\mathcal{G} = \mathcal{H}^c$ is complementary to $\mathcal{H}$, then*

$$\operatorname{codim}\mathcal{R}_s\big(\mathcal{H}^c\big) \geq \mu_-\big(Z(X)\big) \geq \dim\mathcal{H} \tag{6.7}$$

and

$$\dim\mathcal{R}_s\big(\mathcal{H}^c\big) \leq \mu_+\big(Z(X)\big) + \mu_0\big(Z(X)\big) \leq ng\alpha_s - \dim\mathcal{H}. \tag{6.8}$$

Proof. This is an immediate consequence of Lemma 6.4. □

LEMMA 6.6. *Let $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and let $\mathcal{U}$ be a maximal strictly negative subspace of $\mathbb{R}^n$ with respect to the quadratic form $\langle Au, u\rangle$. Then there exists a complementary subspace $\mathcal{V}$ of $\mathbb{R}^n$ such that $\langle Av, v\rangle \geq 0$ for every $v \in \mathcal{V}$.*

Proof. Let $U \in \mathbb{R}^{n\times n}$ be an orthogonal matrix and $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix such that $AU = UD$ and assume that $\mu_-(A) = k_1 > 0$, $\mu_+(A) = k_2 > 0$ and $\mu_0(A) = k_3 > 0$. Then we may assume that

$$U = [U_1 \quad U_2 \quad U_3] \quad \text{and} \quad D = \text{diagonal}\{D_1, D_2, D_3\},$$

where

$$AU_i = U_i D_i \quad \text{for } i = 1, 2, 3, D_1 \prec 0, D_2 \succ 0 \text{ and } D_3 = 0.$$

Now let $u_1, \dots, u_{k_1}$ be a basis for $\mathcal{U}$. Then, since the columns of U span $\mathbb{R}^n$, there exists a matrix $M \in \mathbb{R}^{n\times k_1}$ with blocks $M_{i1} \in \mathbb{R}^{k_i\times k_1}$ for $i = 1, 2, 3$ such that

$$[u_1 \quad \cdots \quad u_{k_1}] = UM = U_1M_{11} + U_2M_{21} + U_3M_{31}.$$

The next step is to check that M_{11} is invertible. But, if $M_{11}c=0$ for some vector $c\in\mathbb{R}^{k_1}$ with components $c_1,\dots,c_{k_1}$, then

$$\begin{aligned}\Big\langle A\sum c_iu_i,\sum c_iu_i\Big\rangle &= \langle AUMc, UMc\rangle\\ &= \langle U_2D_2M_{21}c+U_3D_3M_{31}c, U_2M_{21}c+U_3M_{31}c\rangle\\ &= \langle D_2M_{21}c, M_{21}c\rangle \geq 0.\end{aligned}$$

Therefore, since $\mathcal{U}$ is a strictly negative subspace, it follows that $c=0$. Thus, M_{11} is invertible. Let $M_{ij}\in\mathbb{R}^{k_i\times k_j}$ for $i=2,3$ and $j=1,2,3$ with M_{22} and M_{33} invertible, and let

$$\mathcal{V}=\operatorname{span}\left\{U\begin{bmatrix}0 & 0\\ M_{22} & 0\\ M_{32} & M_{33}\end{bmatrix}b : b\in\mathbb{R}^{k_2+k_3}\right\}.$$

Then it is readily checked that $\mathcal{V}$ is a complementary subspace to $\mathcal{U}$ in $\mathbb{R}^n$ and that $\langle Av,v\rangle\geq 0$ for every $v\in\mathcal{V}$. □

The following proposition ties this section in with Section 5, by relating the number of negative eigenvalues of $Z(X)$ to the dimension of a maximal negative subspace of the clamped second fundamental form.

PROPOSITION 6.7. *Let $\mathcal{Z}$ be the scalar middle matrix of the Hessian p'' of a symmetric nc polynomial p in symmetric variables and let $(X,v)\in(\mathbb{R}^{n\times n}_{\text{sym}})^g\times\mathbb{R}^n$. There is an $\epsilon>0$ such that if $0<\delta<\epsilon$, $\lambda>0$ and $\mathcal{H}$ is a maximal strictly negative subspace for the quadratic form*

$$\left(\mathbb{R}^{n\times n}_{\text{sym}}\right)^g \ni H\mapsto \left\langle p''_{\lambda,\delta}(X)[H]v,v\right\rangle$$

(*based on the relaxed Hessian*), *then*

$$\dim\mathcal{H}\leq n\mu_-(\mathcal{Z})\leq \dim\mathcal{H}+\operatorname{codim}\mathcal{R}_{d-1}\Big(\left(\mathbb{R}^{n\times n}_{\text{sym}}\right)^g\Big). \tag{6.9}$$

Proof. Let d denote the degree of p and g the number of variables. Choose $\epsilon>0$ as in Proposition 5.2. Then for $0<\delta<\epsilon$ and $\lambda>0$

$$\mu_-\big(Z_{\lambda,\delta}(X)\big)=n\mu_-(\mathcal{Z}). \tag{6.10}$$

Since $\mathcal{H}$ is a maximal strictly negative subspace, the space $\mathcal{G}$ considered in Lemma 6.4 can be chosen to coincide with a complementary subspace $\mathcal{H}^c$ to $\mathcal{H}$ (thanks to Lemma 6.6) and hence, by Lemmas 6.5 and 6.3,

$$\begin{aligned}\dim\mathcal{H}\leq\mu_-\big(Z_{\lambda,\delta}(X)\big) &\leq \operatorname{codim}\mathcal{R}_{d-1}\big(\mathcal{H}^c\big)\\ &= \operatorname{codim}\mathcal{R}_{d-1}\Big(\left(\mathbb{R}^{n\times n}_{\text{sym}}\right)^g\Big)+\dim\mathcal{R}_{d-1}(\mathcal{H})\\ &\leq \dim\mathcal{H}+\operatorname{codim}\mathcal{R}_{d-1}\Big(\left(\mathbb{R}^{n\times n}_{\text{sym}}\right)^g\Big).\end{aligned}$$

The rest follows from (6.10). □

REMARK 6.8. If the relaxed Hessian is negative definite, then the lower bound in (6.9) applied to $\mathcal{H} = (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$ implies that

$$\dim\big(\mathbb{R}^{n\times n}_{\mathrm{sym}}\big)^g = \frac{n(n+1)}{2} g \le n\mu_-(\mathcal{Z}).$$

Therefore, since

$$\mu_-(\mathcal{Z}) \le g + g^2 + \cdots + g^{d-1},$$

the relaxed Hessian cannot be negative definite if $n > 2(1+g+\cdots+g^{d-2}) - 1$.

7. Proof of Theorem 1.4 and related results

In this section, we prove Theorem 1.4, Corollary 1.3 and some variations thereof. The first subsection contains a proof of the existence of the limit $C_\pm(\mathcal{S})$; the second verifies the inequality in (1.10); the remaining parts of the theorem and Corollary 1.3 are proved in the third subsection. Some supplementary results are given in the fourth and final subsection.

7.1. The existence of $C_\pm$. We shall need the following result:

LEMMA 7.1. *Suppose p is a symmetric nc polynomial in symmetric variables. Let $(X, v) \in (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g \times \mathbb{R}^n$ be given. If k is a positive integer and*

$$Y = \operatorname{diag}\{X, \dots, X\} \quad \textit{and} \quad w = \operatorname{col}(v, \dots, v) \quad k \textit{ times},$$

then

$$c_\pm^{kn}(Y, w; p) \ge k c_\pm^n(X, v; p).$$

Proof. Recall that $c_+^n(X, v; p)$ is the maximum dimension of a strictly positive subspace of $\mathcal{T}^n \subset (\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$ with respect to the quadratic form,

$$\langle H, K\rangle = \big\langle p''(X)[H][K]v, v\big\rangle,$$

where

$$\mathcal{T}^n = \big\{H \in \big(\mathbb{R}^{n\times n}_{\mathrm{sym}}\big)^g : p'(X)[H]v = 0\big\}$$

and the superscript n has been added because the size of the matrices under consideration is now an issue. Similarly, $c_+^{kn}(Y, w; p)$ is the dimension of a maximal positive subspace of $\mathcal{T}^{nk}$ relative to the form

$$\langle H, K\rangle = \big\langle p''(Y)[H][K]w, w\big\rangle, \tag{7.1}$$

where

$$\mathcal{T}^{kn} = \big\{H \in \big(\mathbb{R}^{nk\times nk}_{\mathrm{sym}}\big)^g : p'(Y)[H]w = 0\big\}.$$

Let $\mathcal{P}_n$ denote a positive subspace of $(\mathbb{R}^{n\times n}_{\mathrm{sym}})^g$ with $\dim \mathcal{P}_n = c_+^n(X, v; p)$ and let

$$\mathcal{Q}_{nk} = \big\{\operatorname{diag}\big\{H^1, \dots, H^k\big\} : H^j \in \mathcal{P}_n\big\}.$$

Then $\mathcal{Q}_{nk} \subseteq \mathcal{T}^{nk}$ and $\mathcal{Q}_{nk}$ is positive relative to the form in equation (7.1). Therefore,

$$c_+^{kn}(Y, w; p) \ge k c_+^n(X, v; p),$$

since the dimension of $\mathcal{Q}_{nk}$ is k times the dimension of $\mathcal{P}_n$. The verification of the analogous inequality with $-$ instead of $+$ is similar. $\square$

Proof of (i) *in Theorem* 1.4. The bound

$$\frac{c_\pm^n(X,v;p)}{n} \leq \mu_\pm(\mathcal{Z}) + \mu_0(\mathcal{Z})$$

guarantees that

$$\Gamma_\pm = \sup_n \left(\sup \left\{ \frac{c_\pm^n(X,v;p)}{n} : (X,v) \in \mathcal{S}_n \right\} \right)$$

is finite.

Let

$$\beta_\pm^n = \sup \left\{ \frac{c_\pm^n(X,v;p)}{n} : (X,v) \in \mathcal{S}_n \right\}. \tag{7.2}$$

We shall prove that $\beta_\pm^n \to \Gamma_\pm$ as $n \uparrow \infty$. by showing that given any $\varepsilon > 0$, there exists an $N > 0$ such that

$$\Gamma_\pm \geq \beta_\pm^n \geq \Gamma_\pm - \varepsilon$$

for every integer $n \geq N$. Since the upper bound $\Gamma_\pm \geq \beta_\pm^n$ is clear, it suffices to verify the lower bound when $\Gamma_\pm - \varepsilon > 0$. Under this assumption, there exists a positive integer t and a pair $(X,v) \in (\mathbb{R}^{t\times t}_{\text{sym}})^g \times \mathbb{R}^t$ such that

$$\frac{c_\pm^t(X,v;p)}{t} \geq \Gamma_\pm - \varepsilon/2.$$

Let $k_0 \geq \frac{2\Gamma_\pm}{\varepsilon}$. Then for any integer $n > k_0 t$, there exists an integer $k \geq k_0$ such that

$$kt < n \leq (k+1)t.$$

Here we use the hypothesis that $\mathcal{S}_1$, and hence $\mathcal{S}_m$ for every m, is nonempty. Since, by hypothesis, $\mathcal{S}_{n-kt}$ is nonempty, there is a pair $(Z,u) \in \mathcal{S}_{n-kt}$. Let $(Y,w) = \bigoplus_1^k (X,v) \oplus (Z,u)$ Then, since $Y \in (\mathbb{R}^{n\times n}_{\text{sym}})^g$ and $w \in \mathbb{R}^n$,

$$c_\pm^n(Y,w;p) \geq k c_\pm^t(X,v;p) + c_\pm^n(Z,u;p) \geq k c_\pm^t(X,v;p).$$

Therefore,

$$\begin{aligned}
\Gamma_\pm \geq \beta_\pm^n \geq \frac{c_\pm^n(Y,w;p)}{n} &\geq \frac{k c_\pm^t(X,v;p)}{n} \geq \frac{k c_\pm^t(X,v;p)}{(k+1)t} \\
&\geq \frac{k}{(k+1)} \frac{c_\pm^t(X,v;p)}{t} \\
&\geq \frac{k_0}{(k_0+1)}(\Gamma_\pm - \varepsilon/2) \quad (\text{since } k \geq k_0) \\
&\geq \frac{2\Gamma_\pm/\varepsilon}{1 + 2\Gamma_\pm/\varepsilon}(\Gamma_\pm - \varepsilon/2) \quad (\text{since } k_0 \geq 2\Gamma_\pm/\varepsilon) \\
&\geq \Gamma_\pm - \varepsilon.
\end{aligned}$$

$\square$

7.2. Proof of inequality (1.10). The representation formula (5.2) for the Hessian $p''(x)[h]$ of p and formulas (5.3) and (5.4) imply that

$$n\mu_-(\mathcal{Z}) \geq e_-^n\big(X,v;p'',(\mathbb{R}^{n\times n}_{\text{sym}})^g\big) \geq e_-^n\big(X,v;p'',\mathcal{T}\big) = c_-^n(X,v;p) \tag{7.3}$$

and hence, in view of (7.2) that

$$\beta_-^n \leq \mu_-(\mathcal{Z}).$$

The next lemma is a step in the proof of (1.10) that contains information of independent interest.

LEMMA 7.2. *Let p be a symmetric nc polynomial in symmetric variables and let $(X,v)\in(\mathbb{R}^{n\times n}_{\text{sym}})^g\times\mathbb{R}^n$ be given. If there is no nonzero nc polynomial q (not necessarily symmetric) of degree less than d such that $q(X)v=0$, then*

$$\frac{c_\pm^n(X,v;p)}{n} \leq \mu_\pm(\mathcal{Z}) \leq \frac{g\alpha_{d-1}(\alpha_{d-1}-1)}{2n} + \frac{c_\pm^n(X,v;p)}{n}. \tag{7.4}$$

In particular, if $c_-^n(X,v;p)=0$ and $2n > g\alpha_{d-1}(\alpha_{d-1}-1)$, then $\mu_-(\mathcal{Z})=0$.

Proof. Let d denote the degree of p and g the number of variables. By Lemma 6.1 with $r=s=d-1$,

$$\operatorname{codim}\mathcal{R}_{d-1}\big((\mathbb{R}^{n\times n}_{\text{sym}})^g\big) = g\alpha_{d-1}\frac{\alpha_{d-1}-1}{2}.$$

By Theorem 3.2, there is a $\delta_0>0$ such that for each $0<\delta\leq\delta_0$ there exists a $\lambda>0$ such that

$$e_-^n\big(X,v;p''_{\delta,\lambda},(\mathbb{R}^{n\times n}_{\text{sym}})^g\big) = c_-^n(X,v;p).$$

With $\epsilon>0$ as in Proposition 5.2 and $0<\delta<\epsilon$ (as well as $\delta<\delta_0$), the second inequality in Proposition 6.7 implies that

$$n\mu_-(\mathcal{Z}) \leq c_-^n(X,v;p) + g\alpha_{d-1}(\alpha_{d-1}-1)/2. \tag{7.5}$$

Thus, in view of (7.3), the inequalities in (7.4) hold for the numbers $c_-^n(X,v;p)$ and $\mu_-(\mathcal{Z})$. However, since

$$c_\pm^n(X,v;p) = c_\mp^n(X,v;-p) \quad\text{and}\quad \mu_\pm(\mathcal{Z}) = \mu_\mp(-\mathcal{Z}), \tag{7.6}$$

these inequalities also hold for the numbers $c_+^n(X,v;p)$ and $\mu_+(\mathcal{Z})$. $\square$

Returning to the proof of inequality (1.10), since $\mathcal{S}$ respects direct sums and p is a minimum degree defining polynomial for $\mathcal{S}$, the condition Lemma 4.1(O) with $N=d-1$ can not hold. Thus, Lemma 4.1 guarantees that there is an integer j and a pair $(Y,w)\in\mathcal{S}_j$ such that $\{m(Y)w : |m|<d\}$ is linearly independent. For a fixed positive integer k, let $(X,v)=\bigoplus_1^k(Y,w)$. Then $(X,v)\in\mathcal{S}_{jk}$ and $\{m(X)v : |m|<d\}$ is linearly independent. This linear independence is equivalent to the hypothesis of Lemma 7.2 and hence, in view of (7.3),

$$\mu_-(\mathcal{Z}) \leq \frac{g\alpha_{d-1}(\alpha_{d-1}-1)}{2n} + \frac{c_-^n(X,v;p)}{n}. \tag{7.7}$$

Consequently,

$$\beta_-^n \le \mu_-(\mathcal{Z}) \le \frac{g\alpha_{d-1}(\alpha_{d-1}-1)}{2n} + \beta_-^n$$

and hence

$$\limsup_{n\uparrow\infty} \beta_-^n \le \mu_-(\mathcal{Z}) \le \liminf_{n\uparrow\infty} \beta_-^n.$$

Therefore,

$$C_-(\mathcal{S}) = \lim_{n\uparrow\infty} \beta_-^n = \mu_-(\mathcal{Z}). \tag{7.8}$$

By a similar argument, or by exploiting (7.6) and (7.8),

$$C_+(\mathcal{S}) = \lim_{n\uparrow\infty} \beta_+^n = \mu_+(\mathcal{Z}). \tag{7.9}$$

The bounds (1.10) follow easily from the identifications (7.8) and (7.9) and the bounds (5.5).

Remark 7.3. If p is a k-minimum degree defining polynomial, then the argument proving (7.7) can be modified as follows:

$\{w(Y)u : |w| < d-k\}$ is linearly independent. For a fixed positive integer ℓ, let $(X,v) = \bigoplus_1^\ell (Y,u)$. Then $(X,v) \in \mathcal{S}_{j\ell}$ and $\{w(X)v : |w| < d-k\}$ is linearly independent. By Lemma 6.1 with $n = j\ell$ and $s = d-1$ and $r = d-1-k$, the codimension of $\mathcal{R}_{d-1}((\mathbb{R}^{n\times n}_{\mathrm{sym}})^g)$ is at most

$$ng(\alpha_{d-1} - \alpha_{d-1-k}) + g\alpha_{d-1-k}(\alpha_{d-1-k}-1)/2.$$

Hence, by the second inequality in Proposition 6.7,

$$n\mu_-(\mathcal{Z}) - ng(\alpha_{d-1} - \alpha_{d-1-k}) - \frac{g\alpha_{d-1-k}(\alpha_{d-1-k}-1)}{2} \le c_-(X,v;p). \tag{7.10}$$

7.3. Proof of (B)–(D) in Theorem 1.4 and Corollary 1.3. Returning to the assumption that p is a minimum degree defining polynomial for $\mathcal{S}$, if $C_-(\mathcal{S}) = 0$, then, by equation (7.8), $\mu_-(\mathcal{Z}) = 0$; i.e., the middle matrix for $p''(x)[h]$ is positive (resp., negative) semi-definite and thus also constant (i.e., $Z(x) = Z(0) = \mathcal{Z}$). The factorization of $Z(x) = \mathcal{Z}$ as W^*W in the positive semi-definite case shows that $p(x) = L(x) + \Lambda(x)^T\Lambda(x)$, where $L(x)$ has degree at most one and $\Lambda(x)$ is either equal to zero or to a homogeneous polynomial of degree one. In particular, p is convex.

A similar argument prevails in the case that $C_+(\mathcal{S}) = 0$.

For the converse, if p is convex, then

$$p''(X)[H]$$

is positive semi-definite for all X, H and thus $c_-(X,v;p) = 0$ (for all X and v).

Note that the verification of the equalities $C_\pm(\mathcal{S}) = \mu_\pm(\mathcal{Z})$ from equations (7.9) and (7.8) in the proof of Lemma 7.2 depends only upon S being a nonempty set which is closed with respect to direct sums and for which p is a minimum degree defining polynomial. Consequently, Theorem 1.4(D) holds.

Proof of Corollary 1.3. The set of full rank points of p in $\mathcal{V}(p) \cap \mathcal{O}$ respects direct sums (see Lemma 4.2). If p has positive curvature on $\mathcal{S}$, then for each $(X,v) \in \mathcal{S}_n$, we have $c_+(X,v;p) = 0$ and hence $C_+(\mathcal{S}) = 0$. The conclusion now follows from statement (B) in Theorem 1.4. □

7.4. Determining sets for the signature of $\mathcal{V}(p)$. This subsection explores and expands upon the principle (mentioned in the Introduction, see Proposition 1.5) that the signature of $\mathcal{V}(p)$ is determined on any subset $\mathcal{S}$ respecting direct sums, which is large enough so that p is a minimal defining polynomial for $\mathcal{S}$.

THEOREM 7.4. *Let p be a symmetric nc polynomial of degree d in g symmetric variables. Suppose*

$$n > \frac{1}{2} g \alpha_{d-1}(\alpha_{d-1} - 1), \tag{7.11}$$

$X \in (\mathbb{R}^{n \times n}_{\text{sym}})^g$, $v \in \mathbb{R}^n$, *and that*

(a) $(X,v) \in \mathcal{V}(p)$; *and*
(b) *there is no nonzero nc polynomial q (not necessarily symmetric) of degree less than d such that $q(X)v = 0$; and*
(c) *$\mathcal{S}$ is any subset of $\mathcal{V}(p)$ which is nonempty, closed with respect to direct sums and for which p is a minimum degree defining polynomial (so that hypotheses of Theorem* 1.4(A)(i) *and* (ii) *hold),*

then

$$C_\pm(\mathcal{S}) = \left\lceil \frac{c_\pm^n(X,v;p)}{n} \right\rceil$$

were $\lceil r \rceil$ is the ceiling function; i.e., the the smallest integer bigger than or equal to r.

Proof of Theorem 7.4. This follows from the inequalities (7.4), formulas (7.8) and (7.9) (which also serve to identify $C_\pm(\mathcal{S})$ as integers) and the bound (7.11), which insures that

$$1 > \frac{1}{2n} g \alpha_{d-1}(\alpha_{d-1} - 1).$$ □

Proof of Proposition 1.5. The first part of the proposition is an immediate consequence of Theorem 7.4. The existence of a pair (X,v) with the properties claimed in the second part of the proposition follows from the the second half of the proof of (1.10) in Section 7.2. □

8. Positive curvature and convex sets

If $p = p(x_1, \ldots, x_g)$ is a concave polynomial in g commuting variables, then for each of $\alpha \in \mathbb{R}$, the superlevel set

$$L_\alpha = \{x : p(x) \geq \alpha\}$$

is either empty or a convex subset of $\mathbb{R}^g$. The converse is false. However, in the classical setting, a defining polynomial for a convex set $\mathcal{C}$ with smooth boundary has second derivative which, when restricted to the tangent plane $T\mathcal{C}$ at each point of $\partial\mathcal{C}$ is negative semi-definite, or, in the language of differential geometry, this is the same as saying that the second fundamental form is positive semi-definite. This section discusses a non-commutative analog. The article [Po06] takes nc varieties in another direction.

Let p denote a symmetric nc polynomial of degree d in g symmetric variables. Assume that $p(0) \succ 0$; i.e., the constant term of p is strictly positive. The *positivity domain* of such a p in dimension n, denoted $\mathcal{D}_p^n$, is the closure of the component of 0 of the set

$$\mathcal{P}_p^n = \{X \in (\mathbb{R}_{\text{sym}}^{n\times n})^g : p(X) \succ 0\}.$$

As usual, let $\mathcal{D}_p = \bigcup_n \mathcal{D}_p^n$. Let $\partial\mathcal{D}_p^n$ denote the boundary of the set $\mathcal{D}_p^n$ and $\partial\mathcal{D}_p = \bigcup_n \partial\mathcal{D}_p^n$. If $X \in \partial\mathcal{D}_p$, then $\mathcal{K}_X$, the kernel of $p(X)$, is non-zero.

Given a smooth curve $X(t) \in (\mathbb{R}_{\text{sym}}^{n\times n})^g$, the derivative $G'(t)$ of the function $G(t) = p(X(t))$ is equal to

$$G'(t) = p'\big(X(t)\big)\big[X'(t)\big]. \tag{8.1}$$

The second derivative $G''(t)$ is described in terms of the directional derivative and Hessian of p by

$$G''(t) = p''\big(X(t)\big)\big[X'(t)\big] + p'\big(X(t)\big)\big[X''(t)\big], \tag{8.2}$$

an identity which leads to the following property of convex sets.

Lemma 8.1 (Lemma 2.1 of [DHM07b]). *Suppose $\mathcal{D}_p$ is convex and let $(X, v) \in (\mathbb{R}_{\text{sym}}^{n\times n})^g \times \mathbb{R}^n$ be given. If*

(i) $X \in \partial\mathcal{D}_p$;
(ii) $v \neq 0$ *and* $p(X)v = 0$; *and*
(iii) $(-\delta, \delta) \ni t \mapsto X(t)$ *is a smooth curve in* $\partial\mathcal{D}_p$ *for which* $X(0) = X$ *and* $p(X(t))v = 0$,

then

$$\big\langle p''\big(X(0)\big)\big[X'(0)\big]v, v\big\rangle \le 0.$$

Next, fix X in $\partial\mathcal{D}_p$ and assume that $\mathcal{K}_X$ is one-dimensional and spanned by v and that X is a smooth point (full rank point) of $\partial\mathcal{D}_p$. Under these hypothesis, if H is in the clamped tangent plane,

$$\mathcal{T} = \{H \in (\mathbb{R}_{\text{sym}}^{n\times n})^g : p'(X)[H]v = 0\},$$

to $\partial\mathcal{D}_p$ at X, then (by the implicit function theorem) there is a smooth curve $X(t)$ in $\partial\mathcal{D}_p$ such that $X(0) = X$ and $X'(0) = H$. This establishes the hypothesis of the lemma. Thus, the clamped second fundamental form is negative semi-definite at X, for more details see Section 2 of [DHM07b].

Next, note that $\partial\mathcal{P}(p)$ is closed with respect to direct sums. In [DHM07b] smoothness and irreducibility type conditions implied that $\partial\mathcal{P}(p)$ generically, but not universally, has positive curvature. Indeed, positive curvature could not be guaranteed at points $(X,v) \in \partial\mathcal{P}(p)$ where the dimension of the kernel $p(X)$ exceeds one, which is the case for a direct sum of two points $(X,v),(Y,w) \in \partial\mathcal{P}(p)$. The trouble with higher dimensional kernels is that the implicit function theorem argument applied near such a point produces a curve which lies in $\mathcal{V}(p)$, but perhaps not in the smaller set $\partial\mathcal{P}(p)$.

On the other hand, under some additional fairly natural assumptions in [DHM07b], convexity of $\mathcal{P}(p)$ implies positive curvature at many points in $\partial\mathcal{P}(p)$. This plus Corollary 1.3 implies that if p is a 0-minimal degree defining polynomial for $\partial\mathcal{D}_p$, then p has degree at most two. This conclusion is stronger than the conclusion p has degree at most four that was obtained in [DHM07b]. This sharpening of the bound depends upon the stronger irreducibility hypothesis that is imposed here and the extra mileage obtained from the introduction and careful analysis of the relaxed Hessian. More precisely, in the current terminology (see Section 1.3.5), the minimum degree hypothesis in [DHM07b] is that p is a 1-minimal degree polynomial, whereas Theorem 1.4 in this paper assumes that p is a 0-minimal degree polynomial. Another recent result in this line that is obtained without assuming irreducibility states that if $\mathcal{D}_p$ is both bounded and convex, then there is a positive integer ℓ and a set of real symmetric $\ell \times \ell$ matrices $A_1,\dots,A_\ell$ such that $X \in \mathcal{D}_p$ if and only if

$$I_\ell \otimes I_n - \sum A_j \otimes X_j \succ 0.$$

See [HM] for details where it is proved with separating hyperplane techniques.

References

[BCR91] J. Bochnak, M. Coste and M.-F. Roy, *Real algebraic geometry*, Springer, Berlin, 1998. MR 1659509

[CHSY03] J. F. Camino, J. W. Helton, R. E. Skelton and J. Ye, *Matrix inequalities: A symbolic procedure to determine convexity automatically*, Integral Equations Operator Theory **46** (2003), 399–454. MR 1997979

[DHM07a] H. Dym, J. W. Helton and S. A. McCullough, *The Hessian of a non-commutative polynomial has numerous negative eigenvalues*, J. Anal. Math. **102** (2007), 29–76. MR 2346553

[DHM07b] H. Dym, J. W. Helton and S. A. McCullough, *Irreducible non-commutative defining polynomials for convex sets have degree four or less*, Indiana Univ. Math. J. **56** (2007), 1189–1231. MR 2333470

[DGHM09] H. Dym, J. Greene, J. W. Helton and S. McCullough, *Classification of all non-commutative polynomials whose Hessian has negative signature one and a non-commutative second fundamental form*, J. Anal. Math. **108** (2009), 19–59. MR 2544753

[HM04] J. W. Helton and S. McCullough, *Convex non-commutative polynomials have degree two or less*, SIAM J. Matrix Anal. **25** (2004), 1124–1139. MR 2081134

[HM] J. W. Helton and S. McCullough, *Every free basic semi-algebraic set has an LMI representation*, Ann. of Math. (2) **176** (2012), 979–1013.

[HMV06] J. W. Helton, S. McCullough and V. Vinnikov, *Non-commutative convexity arises from linear matrix inequalities*, J. Funct. Anal. **240** (2006), 105–191. MR 2259894

[HP07] J. W. Helton and M. Putinar, *Positive polynomials in scalar and matrix variables, the spectral theorem and optimization*, Operator theory, structured matrices, and dilations, Theta Ser. Adv. Math., vol. 7, Theta, Bucharest, 2007, pp. 229–306. MR 2389626

[KVV06] D. S. Kalyuzhnyi-Verbovetskiĭ and V. Vinnikov, *Non-commutative positive kernels and their matrix evaluations*, Proc. Amer. Math. Soc. **134** (2006), 805–816. MR 2180898

[Po06] G. Popescu, *Operator theory on non-commutative varieties*, Indiana Univ. Math. J. **55** (2006), 389–442. MR 2225440

[Vo06] D. Voiculescu, *Symmetries arising from free probability theory*, Frontiers in number theory, physics, and geometry, vol. I Springer, Berlin, 2006, pp. 231–243. MR 2261097

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