ISOTYPE SUBGROUPS OF DIRECT SUMS OF COUNTABLE GROUPS

BY

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I. Introduction

In this paper we shall deal with additively written commutative groups in which each element has finite order. By a theorem whose origin appears to be uncertain [6] such a group G can be decomposed as $G = \sum G_p$ where the summation is over the primes and for each prime p any element of G_p has order a power of p. Thus we may restrict our attention to G_p, that is, there is no loss of generality in assuming that G is primary. If G is a primary group, we define $G[p]$ and pG as follows:

$$G[p] = \{x \in G : px = 0\} \quad \text{and} \quad pG = \{px : x \in G\}.$$

If β is an ordinal, $p^{\beta}G$ is defined inductively by $p^{\beta}G = p(p^{\beta-1}G)$ provided that $\beta - 1$ exists and by $p^{\beta}G = \bigcap_{\alpha<\beta} p^{\alpha}G$ if β is a limit ordinal. The p-primary group G is divisible if $pG = G$ and G is reduced if G does not contain a nontrivial divisible subgroup. A group always decomposes into a divisible part and a reduced part [1]. Since the structure of divisible groups is well known, interest is shifted completely to the reduced part. If G is reduced, there is a smallest ordinal λ such that $p^{\lambda}G = 0$; this λ is called the length of G. For each $\alpha \leqq \lambda$, the dimension $f_G(\alpha)$ of the vector space

$$(p^{\alpha}G \cap G[p])/(p^{\alpha+1}G \cap G[p]),$$

over the prime field of characteristic p, is called the α-th Ulm invariant of G.

It is known that within the class of direct sums of reduced countable primary groups the members are uniquely determined by their Ulm invariants [2], [7]; but subgroups of direct sums of countable groups need not be again direct sums of countable groups [11], [12], [3]. Indeed Nunke has shown in [12] that it is possible for G to be a direct sum of countable reduced primary groups and for H to be nicely embedded in G in the sense that $p^{\alpha}G \cap H = p^{\alpha}H$ for all ordinals α and still H not be a direct sum of countable groups. One of the main results of the present paper is that this can happen only if H has the longest possible length—that length is, of course, Ω. Actually, we prove the following.

THEOREM 1. *Let $G = \sum_I G_i$ be a direct sum of countable primary groups G_i. If H is an isotype subgroup of G having countable length λ, then H is a direct sum of countable groups. Furthermore, if I_0 is a subset of I, then $H \cap \sum_{I_0} G_i$ is a*

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direct summand of H if it is p^λ-pure in H and

$$\{H \cap \textstyle\sum_{I_0} G_i, p^\alpha G\} = \{H, p^\alpha G\} \cap \{\textstyle\sum_{I_0} G_i, p^\alpha G\}$$

for all $\alpha \leqq \lambda$.

A subgroup H of the p-primary group G is called an isotype subgroup of G if $p^\alpha G \cap H = p^\alpha H$ for every ordinal α. If β is an ordinal, we shall say that H is weakly p^β-pure in G if $p^\alpha G \cap H = p^\alpha H$ for all $\alpha \leqq \beta$. For the definition of p^β-purity see [5], [12], or [9]. It is known [5] that weak p^β-purity compares, in the suggested way, with p^β-purity.

Theorem 1 is established in conjunction with the following lemmas.

LEMMA 1. *Suppose that $G = \sum_I G_i$ is a direct sum of countable primary groups G_i and suppose that H is an isotype subgroup of G having countable length λ. Let I_0 be a subset of I such that $H \cap \sum_{I_0} G_i$ is p^λ-pure in H and*

$$\{H \cap \textstyle\sum_{I_0} G_i, p^\alpha G\} = \{H, p^\alpha G\} \cap \{\textstyle\sum_{I_0} G_i, p^\alpha G\}$$

for $\alpha \leqq \lambda$. Let A be a countable subgroup of H. Then there exists a subset I_1 of I containing I_0 such that

(1) *$H \cap \sum_{I_1} G_i$ is p^λ-pure in H,*
(2) *$A \subseteq \sum_{I_1} G_i$,*
(3) *$I_1 - I_0$ is countable,*
(4) *$\{H \cap \sum_{I_1} G_i, p^\alpha G\} = \{H, p^\alpha G\} \cap \{\sum_{I_1} G_i, p^\alpha G\}$ for $\alpha \leqq \lambda$.*

LEMMA 2. *Suppose that $G = \sum_I G_i$ is a direct sum of countable primary groups G_i and let H be an isotype subgroup of G having countable length λ. Suppose that $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_\gamma \subseteq \cdots, \gamma < \delta$, is an ascending chain of subsets of I such that*

(i) *$\{(H \cap \sum_{I_\gamma} G_i), p^\alpha G\} = \{H, p^\alpha G\} \cap \{\sum_{I_\gamma} G_i, p^\alpha G\}$ for each $\alpha \leqq \lambda$ and each $\gamma < \delta$;*
(ii) *$(H \cap \sum_{I_\gamma} G_i)$ is p^λ-pure in H for each $\gamma < \delta$.*

Define $I_\delta = \bigcup_{\gamma<\delta} I_\gamma$. Then conditions (i) *and* (ii) *hold for $\gamma \leqq \delta$.*

We shall prove the lemmas and theorem simultaneously by induction on the ordinal λ. More specifically, we show that the validity of Theorem 1 for all $\lambda < \mu$ implies Lemma 1 and Lemma 2 for $\lambda \leqq \mu$. On the other hand, the two lemmas imply the theorem—λ for λ.

We shall see that Theorem 1 yields a rather strong uniqueness theorem. A consequence of this uniqueness theorem is the following result. If $G/p^\beta G$ is a direct sum of countable groups for a countable limit β, then there exists, upon identifying isomorphic subgroups, a natural correspondence from the pure subgroups of $p^\beta G$ to the pure subgroups of G. The correspondence is $A \to B$ where for a pure subgroup A of $p^\beta G$ the subgroup B is maximal in G with respect to $B \cap p^\beta G = A$.

II. Preliminary Results

Some of the results of this section are implicitly contained in [12]. For completeness, however, we shall in those cases abstract what is needed and provide outlines of proofs.

PROPOSITION 1. *Let G be a primary group and let H be a neat subgroup of G. If $G[p] = \{H[p], p^{\alpha}G[p]\}$ for each $\alpha < \beta$, then H is p^{β}-pure in G.*

Proof. It is easy to show that H is weakly p^{β}-pure in G; a proof is contained in [8]. Since weak p^{β}-purity is equivalent to p^{β}-purity for $\beta \leqq \omega$, we may assume that $\beta > \omega$. The proof now is by induction on β. The induction step is trivial if β is a limit ordinal. Thus assume that $\beta = \alpha + 1 > \omega$. Let $G[p] = H[p] + E$ where $E \subseteq p^{\alpha}G$. Since $\beta > \omega$, G/H is divisible and $\eta = p\xi$ where η is the natural map $G/H \to G/\{H, E\} \to 0$ and ξ is an isomorphism, $0 \to G/H \to G/\{H, E\} \to 0$. From the commutativity of the diagram

$$\begin{array}{lccccc} X_0\colon & H & \rightarrowtail & G & \twoheadrightarrow & G/H \\ & \| & & \downarrow & & \downarrow{\scriptstyle \eta} \\ X_1\colon & H & \rightarrowtail & G/E & \twoheadrightarrow & G/\{H,E\}, \end{array}$$

we have that $X_0 = X_1\eta = X_1 p\xi = X_1 p$. Hence $pX_1 = X_0$ in Ext $(G/H, H)$. It is straightforward to show that

$$(G/E)[p] = \{H[p], p^{\lambda}(G/E)[p]\} \quad \text{if} \quad \lambda < \alpha.$$

Thus $X_1 \in p^{\alpha}$ Ext $(G/H, H)$ by the induction hypothesis, so

$$X_0 \in p^{\beta} \operatorname{Ext}(G/H, H)$$

and H is p^{β}-pure in G.

PROPOSITION 2. *If H is maximal in G with respect to $H \cap p^{\beta}G = 0$, then H is $p^{\beta+1}$-pure in G and $H \cong \{H, p^{\beta}G\}/p^{\beta}G$ is p^{β}-pure in $G/p^{\beta}G$.*

Proof. It is a simple exercise to verify that

$$G[p] = \{H[p], p^{\beta}G[p]\}$$

and

$$(G/p^{\beta}G)[p] = \{(\{H, p^{\beta}G\}/p^{\beta}G)[p], p^{\alpha}(G/p^{\beta}G)[p]\}$$

if $\alpha < \beta$. Since H is neat in G, the conclusion follows by Proposition 1.

A subgroup H of G satisfying the hypothesis of Proposition 2 will be called a β-high subgroup of G (in favor of $p^{\beta}G$-high since p is fixed).

PROPOSITION 3. *If $H/p^{\beta}H$ is p^{β}-pure in $G/p^{\beta}H$, then H is p^{β}-pure in G.*

Proof. The map

$$\phi : \operatorname{Ext}(G/H, p^{\beta}H) \to \operatorname{Ext}(G/H, H)$$

induced by the inclusion map $p^\beta H \to H$ goes into p^β Ext $(G/H, H)$; the proof is given in [5] by induction on β. Thus the complete inverse image of p^β Ext $(G/H, H/p^\beta H)$ under the map

$$\text{Ext}\,(G/H, H) \to \text{Ext}\,(G/H, H/p^\beta H)$$

is precisely p^β Ext $(G/H, H)$, and the proposition is proved.

PROPOSITION 4. *Let H be a subgroup of the primary group G such that $H \cap p^\lambda G = p^\lambda H$. Then H is p^λ-pure in G if and only if $\{H, p^\lambda G\}/p^\lambda G$ is p^λ-pure in $G/p^\lambda G$.*

Proof. Suppose that H is p^λ-pure in G. Then $H/p^\lambda H$ is p^λ-pure in $G/p^\lambda H$. Let $K \supseteq H$ be maximal in G with respect to $K \cap p^\lambda G = p^\lambda H = H \cap p^\lambda G$. Then $K/p^\lambda H$ is λ-high in $G/p^\lambda H$. According to the second half of Proposition 2, $K/p^\lambda H$ is p^λ-pure in $G/p^\lambda G$ under the natural embedding. It follows from $H/p^\lambda H \subseteq K/p^\lambda H \subseteq G/p^\lambda G$ and the transitivity of purity [12] that $H/p^\lambda H$ is p^λ-pure in $G/p^\lambda G$ under the natural embedding, but under this embedding $H/p^\lambda H$ is changed to $\{H, p^\lambda G\}/p^\lambda G$. Conversely, suppose that $\{H, p^\lambda G\}/p^\lambda G$ is p^λ-pure in $G/p^\lambda G$ and let $K/p^\lambda H \supseteq H/p^\lambda H$ be λ-high in $G/p^\lambda H$. Since $\{H, p^\lambda G\}/p^\lambda G$ is p^λ-pure in $\{K, p^\lambda G\}/p^\lambda G$ and since $K \cap p^\lambda G = p^\lambda H$, it follows that $H/p^\lambda H$ is p^λ-pure in $K/p^\lambda H$. Thus $H/p^\lambda H$ is p^λ-pure in $G/p^\lambda H$ because the λ-high subgroup K/p^λH of $G/p^\lambda H$ is p^λ-pure. By Proposition 3, H is p^λ-pure in G.

The next proposition generalizes Theorem 1 in [4]; the formulation is due to Nunke.

PROPOSITION 5. *Let β be an ordinal and G a primary group. Suppose that H is a subgroup of G such that*
(0) *$(G/H)/p^\beta(G/H)$ is p^β-projective,*
(1) *$H \cap p^\beta G = p^\beta H$,*
(2) *$\{H, p^\beta G\}/p^\beta G$ is a direct summand of $G/p^\beta G$,*
(3) *$p^\beta H$ is a direct summand of $p^\beta G$.*
Then H is a direct summand of G.

Proof. Let $G/p^\beta G = \{H, p^\beta G\}/p^\beta G + K/p^\beta G$ and let $p^\beta G = p^\beta H + C$. First observe that $p^\beta(G/H) \subseteq \{H, p^\beta G\}/H$ since $p^\beta(K/p^\beta G) = 0$ and since

$$(G/H)/\{H, p^\beta G\}/H \cong G/\{H, p^\beta G\} \cong (G/p^\beta G)/\{H, p^\beta G\}/p^\beta G \cong K/p^\beta G.$$

Thus $p^\beta(G/H) = \{H, p^\beta G\}/H$ and $G/\{H, p^\beta G\}$ is p^β-projective by (0). Recall that $\{H, p^\beta G\} = \{H, C\}$, so $G/\{H, C\}$ is p^β-projective. Note that $p^\beta(\{H, C\}/C) = \{p^\beta H, C\}/C = p^\beta G/C$, and consider the following exact sequences

$$\text{(A)} \qquad 0 \to \{H, C\}/C \to G/C \to G/\{H, C\} \to 0$$

and

$$\text{(B)} \qquad 0 \to (\{H, C\}/C)/p^\beta G/C \to (G/C)/p^\beta G/C \to G/\{H, C\} \to 0.$$

The latter sequence splits since it is equivalent to

$$0 \to \{H, p^\beta G\}/p^\beta G \to G/p^\beta G \to G/\{H, C\} \to 0$$

and since $\{H, p^\beta G\}/p^\beta G$ is a direct summand of $G/p^\beta G$. Since (B) splits, (A) is p^β-pure, by Proposition 4, and therefore splits as well. Now $G/C = (H + C)/C + L/C$ for some L. It follows that $G = \{H, L\}$ and that $(H + C) \cap L = C$. Thus $H \cap L \subseteq C$. But $p^\beta G \cap H = p^\beta H$ by (1). Therefore

$$C \cap H \subseteq C \cap (p^\beta G \cap H) = C \cap p^\beta H = 0,$$

and $H \cap L = 0$. Hence $G = H + L$.

PROPOSITION 6. *Suppose that $G = \sum_I G_i$ is a direct sum of countable groups G_i and that H is a subgroup of G. Let β be a countable ordinal and let A be a countable subgroup of H. Suppose that J is a subset of I such that*

$$\{H \cap \textstyle\sum_J G_i, p^\alpha G\} = \{H, p^\alpha G\} \cap \{\sum_J G_i, p^\alpha G\}$$

for each $\alpha \leqq \beta$. Then there exists K such that $J \subseteq K \subseteq I$, $K - J$ is countable, $A \subseteq \sum_K G_i$, and

$$\{H \cap \textstyle\sum_K G_i, p^\alpha G\} = \{H, p^\alpha G\} \cap \{\sum_K G_i, p^\alpha G\}$$

for $\alpha \leqq \beta$.

Proof. Let K_0 be the union of J and a countable subset of I such that $A \subseteq \sum_{K_0} G_i$. For each $\alpha \leqq \beta$, choose a set S_0^α of representatives for $\sum_{K_0} G_i \cap \{H, p^\alpha G\}$ modulo $\sum_J G_i \cap \{H, p^\alpha G\}$. Then $S = \bigcup_{\alpha \leqq \beta} S_0^\alpha$ is countable. Now for each element x_α in S_0^α choose one and only one element y_α in $p^\alpha G$ such that $x_\alpha + y_\alpha \in H$. Let $K_1 \supseteq K_0$ be minimal in I such that $\{y_\alpha\}_{\alpha \leqq \beta} \subseteq \sum_{K_1} G_i$. It is easy to verify that

$$\{H, p^\alpha G\} \cap \{\textstyle\sum_{K_0} G_i, p^\alpha G\} \subseteq \{H \cap \sum_{K_1} G_i, p^\alpha G\}.$$

Choose a set S_1^α of representatives for $\sum_{K_1} G_i \cap \{H, p^\alpha G\}$ modulo $\sum_{K_0} G_i \cap \{H, p^\alpha G\}$. For each x_α in S_1^α choose an element $y_\alpha \in p^\alpha G$ such that $x_\alpha + y_\alpha \in H$. Let $K_2 \supseteq K_1$ be minimal in I such that $\{y_\alpha\}_{\alpha \leqq \beta} \subseteq \sum_{K_2} G_i$. Define K_{n+1} in terms of K_n in a similar manner and let $K = \bigcup K_n$.

PROPOSITION 7. *Let $G = \sum_I G_i$ be a direct sum of countable primary groups and suppose that H is an isotype subgroup of G having countable length λ. Let I_0 be a subset of I such that $H \cap \sum_{I_0} G_i$ is isotype in H. If A is a countable subgroup of H, there exists a subject I_1 of I containing I_0 such that $I_1 - I_0$ is countable, $A \subseteq \sum_{I_1} G_i$, and $H \cap \sum_{I_1} G_i$ is isotype in H.*

Proof. It is enough to show that there is a subset J of I containing I_0 such that $J - I_0$ is countable, $A \subseteq \sum_J G_i$, and such that each element of $\{H \cap \sum_{I_0} G_i, A\}$ has the same height in $H \cap \sum_J G_i$ as it does in G. Set $H_0 = H \cap \sum_{I_0} G_i$. For each element $a \in A$ and each ordinal $\alpha \leqq \lambda$, there exists (by an argument similar to Lemma 2 of [4]) a subset $J(\alpha, a)$ of I containing I_0 such that $J(\alpha, a) - I_0$ is countable, $a \in \sum_{J(\alpha,a)} G_i$, and such that each element of $\{H_0, a\}$ that has height at least α in H has height at least α in $H \cap \sum_{J(\alpha,a)} G_i$. Set $J = \bigcup J(\alpha, a)$.

III. Proof of the lemmas and theorem

Proof of Lemma 1. Suppose that $G = \sum_I G_i$, H is an isotype subgroup of G having countable length μ, $H \cap \sum_{I_0} G_i$ is p^μ-pure in H,

$$\{H, p^\lambda G\} \cap \{\textstyle\sum_{I_0} G_i, p^\lambda G\} = \{H \cap \textstyle\sum_{I_0} G_i, p^\lambda G\}$$

for $\lambda \leqq \mu$, and suppose that A is a countable subgroup of H. Assume Theorem 1 for all $\lambda < \mu$.

First consider the case that μ is a limit ordinal. For each $\lambda < \mu$, $\{H, p^\lambda G\}p^\lambda G$ is an isotype subgroup of $G/p^\lambda G = \sum_I \{G_i, p^\lambda G\}p^\lambda G$. According to Theorem 1, for each $\lambda < \mu$, $\{H, p^\lambda G\}/p^\lambda G$ is a direct sum of countable groups. Furthermore,

$$\{H, p^\lambda G\}/p^\lambda G \cap \textstyle\sum_{I_0} \{G_i, p^\lambda G\}/p^\lambda G = \{H \cap \textstyle\sum_{I_0} G_i, p^\lambda G\}p^\lambda G$$

is p^λ-pure in $\{H, p^\lambda G\}p^\lambda G$ since $H \cap \sum_{I_0} G_i$ is p^λ-pure in H. Thus $\{H \cap \sum_{I_0} G_i, p^\lambda G\}/p^\lambda G$ is a direct summand of $\{H, p^\lambda G\}/p^\lambda G$ by Theorem 1; the additional hypothesis of the theorem is easily verified. In view of Proposition 6 it is possible to establish the existence of a subset I_1 such that conditions (2)–(4) of Lemma 1 are satisfied and such that $\{H \cap \sum_{I_1} G_i, p^\lambda G\}/p^\lambda G$ is a direct summand of $\{H, p^\lambda G\}/p^\lambda G$ since $\{H, p^\lambda G\}/p^\lambda G$ is a direct sum of countable groups. By Proposition 7, we may also assume that $H \cap \sum_{I_1} G_i$ is isotype in H. It follows from Proposition 4 that $H \cap \sum_{I_1} G_i$ is p^λ-pure in H for $\lambda < \mu$; consequently, $H \cap \sum_{I_1} G_i$ is p^μ-pure in H.

Now suppose that $\mu - 1$ exists; set $\lambda = \mu - 1$. As before, $\{H, p^\lambda G\}/p^\lambda G$ is a direct sum of countable groups and $\{H \cap \sum_{I_0} G_i, p^\lambda G\}/p^\lambda G$ is a direct summand of $\{H, p^\lambda G\}/p^\lambda G$. There is a subset I_1 of I such that conditions (2)–(4) of Lemma 1 are satisfied, $\{H \cap \sum_{I_1} G_i, p^\lambda G\}/p^\lambda G$ is a direct summand of $\{H, p^\lambda G\}/p^\lambda G$, and $H \cap \sum_{I_1} G_i$ is isotype in H. Since

$$\{H \cap \textstyle\sum_{I_1} G_i, p^\lambda H\}/p^\lambda H \cong \{H \cap \textstyle\sum_{I_1} G_i, p^\lambda G\}/p^\lambda G$$

is a direct summand of $H/p^\lambda H \cong \{H, p^\lambda G\}/p^\lambda G$, then $H \cap \sum_{I_1} G_i$ is a direct summand of H by Proposition 5 since $(H/H \cap \sum_{I_1} G_i)/p^\lambda(H/H \cap \sum_{I_1} G_i)$ is p^λ-projective. In particular, $H \cap \sum_{I_1} G_i$ is p^μ-pure in H, and the lemma is proved.

Proof of Lemma 2. The proof is by induction on λ. It is trivial to verify that condition (i) of Lemma 2 is satisfied for $\gamma = \delta$. Thus we are concerned with proving only that $H \cap \sum_{I_\delta} G_i$ is p^λ-pure in H.

If $\lambda - 1$ exists, set $\beta = \lambda - 1$. Then

$$\{H, p^\beta G\}/p^\beta G \cap \textstyle\sum_{I_\delta} \{G_i, p^\beta G\}/p^\beta G = \{H \cap \sum_{I_\delta} G_i, p^\beta G\}/p^\beta G$$

is p^β-pure in $\{H, p^\beta G\}/p^\beta G$ by the induction hypothesis. Thus

$$\{H \cap \textstyle\sum_{I_\delta} G_i, p^\beta H\}/p^\beta H$$

is p^β-pure in $H/p^\beta H$. Since

$$H/\{H \cap \textstyle\sum_{I_\delta} G_i, p^\beta H\} \cong (\{H, p^\beta G\}/p^\beta G)/(\{H \cap \sum_{I_\delta} G_i, p^\beta G\}/p^\beta G)$$

is a direct sum of countable groups, $H \cap \sum_{I_\delta} G_i$ is a direct summand of H by Proposition 5. Now suppose that λ is a limit ordinal. For each $\beta < \lambda$, $\{H \cap \sum_{I_\delta} G_i, p^\beta H\}/p^\beta H$ is p^β-pure in $H/p^\beta H$. Therefore $H \cap \sum_{I_\delta} G_i$ is p^β-pure in H according to Proposition 4, and the lemma is proved.

Proof of theorem. Suppose that $G = \sum_I G_i$ is a direct sum of countable primary groups G_i. Let H be an isotype subgroup of G having countable length λ. Let I_0 be a subset of I such that $H \cap \sum_{I_0} G_i$ is p^λ-pure in H and

$$\{H, p^\alpha G\} \cap \{\textstyle\sum_{I_0} G_i, p^\alpha G\} = \{H \cap \sum_{I_0} G_i, p^\alpha G\}$$

for $\alpha \leqq \lambda$. It follows from Lemma 1 and Lemma 2 that there is a chain $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_\gamma \subseteq \cdots$ leading up to I such that $I_\gamma = \bigcup_{\beta<\gamma} I_\beta$ if γ is a limit ordinal, $I_{\gamma+1} - I_\gamma$ is countable, $H \cap \sum_{I_\gamma} G_i$ is p^λ-pure in H, and

$$\{H \cap \textstyle\sum_{I_\gamma} G_i, p^\alpha G\} = \{H, p^\alpha G\} \cap \{\sum_{I_\gamma} G_i, p^\alpha G\}$$

for $\alpha \leqq \lambda$. Observe that $(H \cap \sum_{I_{\gamma+1}} G_i)/(H \cap \sum_{I_\gamma} G_i)$ is countable and p^λ-projective. Hence $H \cap \sum_{I_\gamma} G_i$ is a direct summand of $H \cap \sum_{I_{\gamma+1}} G_i$. Thus $H = (H \cap \sum_{I_0} G_i) + \sum_J C_j$ where C_j is countable. The fact that H is a direct sum of countable groups is demonstrated by taking $I_0 = \emptyset$.

IV. Applications and related results

Our first two corollaries of Theorem 1 sharpen results of Nunke [12].

COROLLARY 1. *Suppose that α is a countable ordinal. Let G be a direct sum of countable primary groups and let H be a subgroup of G. If H is weakly p^α-pure in G and if $p^{\alpha+\omega}G$ is countable, then H is a direct sum of countable groups.*

Proof. Let $K = p^\alpha G$. Then K is a direct sum of countable groups and $p^\omega K$ is countable. Hence $K = C + \sum$ cyclics where C is countable, so any subgroup of K is a direct sum of countable groups. Let H be weakly p^α-pure in G. Then $p^\alpha H = p^\alpha G \cap H = K \cap H$ is a direct sum of countable groups. Since $H/p^\alpha H$ is isotype in $G/p^\alpha G$, Theorem 1 implies that $H/p^\alpha H$ is a direct sum of countable groups. Thus H is a direct sum of countable groups [4], [12].

Remark 1. If A is a neat subgroup of $p^\alpha G$ and if $B \supseteq A$ is maximal in G with respect to $B \cap p^\alpha G = A$, then B is $p^{\alpha+1}$-pure in G by Proposition 1 since B is neat in G and since $G[p] = \{B[p], p^\alpha G[p]\}$. If G is a direct sum of reduced countable groups and if $p^\omega G$ is uncountable, then G contains a neat subgroup that is not a direct sum of countable groups; this result is due to Nunke [12], but a particularly simple proof is given in [3]. Thus it follows that if G is a direct sum of reduced countable groups such that $p^{\alpha+\omega}G$ is uncountable, then G contains a p^α-pure subgroup that is not a direct sum of countable groups.

The next result was proved by Nunke for purity rather than weak purity in [12]. His result, Proposition 2.5 in [12], and Theorem 1 yield the stronger form.

COROLLARY 2. *Suppose that α is a countable ordinal. Let G be a direct sum of countable primary groups and let H be a subgroup of G. If H is weakly p^α-pure in G and $p^\beta G$ is countable for some $\beta < \alpha + \omega 2$, then H is p^γ-projective for some countable γ.*

Remark 2. If G is a direct sum of countable groups and has uncountable length, there exist proper subsocles S of G such that $G[p] = \{S, p^\alpha G[p]\}$ for each countable α. In order to verify this, all we need to do is let K be a reduced primary group such that $K/p^\Omega K \cong G$ and $p^\Omega K \neq 0$ and let S be the socle of $\{L, p^\Omega K\}/p^\Omega K$ where L is Ω-high in K. Let S be such a subsocle of G such that $G[p]/S$ is countable and let H be maximal in G with respect to $H[p] = S$. Then H is p^Ω-pure in G; in particular, H is isotype in G and G/H is countable. It is easy to show that H cannot be a direct sum of countable groups, for suppose that $H = \sum_J H_j$ and $G = \sum_I G_i$ where G_i and H_j are countable. There exist countable subsets I_0 and J_0 of I and J, respectively, such that $H \cap \sum_{I_0} G_i = \sum_{J_0} H_j$ and such that $G = \{H, \sum_{I_0} G_i\}$. Now

$$H = (H \cap \textstyle\sum_{I_0} G_i) + \sum_{J-J_0} H_j \quad \text{and} \quad G = \sum_{I_0} G_i + \sum_{J-J_0} H_j\,,$$

which yields a contradiction to the statement that $G[p] = \{H[p], p^\alpha G[p]\}$ for each $\alpha < \Omega$; choose α such that $p^\alpha \sum_{I_0} G_i = 0$, and recall that $H[p] \neq G[p]$.

COROLLARY 3. *Suppose that $G = \sum_{i\in I} G_i + \sum_{i\in J} G_i$ where each G_i is a countable primary group. Let α be a countable ordinal and let H be p^α-pure in $\sum_I G_i$. If K is isotype in G of length α and if*

$$(K, p^\beta G) \cap \{\textstyle\sum_I G_i, p^\beta G\} \subseteq \{H, p^\beta G\}$$

for $\beta \leqq \alpha$, then H is a direct summand of K provided that $K \cap \sum_I G_i = H$.

Proof. Since K is isotype in G and has countable length α, by Theorem 1 it is enough to show that $K \cap \sum_I G_i = H$ is p^α-pure in K because the hypotheses immediately imply that

$$\{K \cap \textstyle\sum_I G_i, p^\beta G\} = \{K, p^\beta G\} \cap \{\sum_I G_i, p^\beta G\}$$

for $\beta \leqq \alpha$. However, $K \cap \sum_I G_i$ is p^α-pure in K since it is p^α-pure in G.

A primary group G is said to be summable if there exists a decomposition $G[p] = \sum S_\alpha$ of the socle of G such that the height of each nonzero element of S_α is precisely α.

As we mentioned in [2], it is not difficult to establish that any countable reduced primary group is summable. Hence a direct sum of such groups is summable. It would be interesting to know the answer to the following question. If G is a direct sum of countable groups and if H is an isotype subgroup of G, must H be a direct sum of countable groups provided that it is summable?

An immediate consequence of Theorem 1 and Nunke's homological characterization of direct sums of countable groups is the following corollary; see Theorem 2.12 in [12].

Corollary 4. *Let the reduced group G be a direct sum of countable primary groups and let H be an isotype subgroup of G. Then H is a direct sum of countable groups if and only if H is p^{Ω}-projective.*

We now state and prove the uniqueness theorem referred to in the introduction.

Theorem 2. *Suppose that the primary group G is such that $G/p^\alpha G$ is a direct sum of countable groups for a countable limit ordinal α. Suppose that each of A and A' is a neat subgroup of $p^\alpha G$. Let $B \supseteq A$ be maximal in G with respect to $B \cap p^\alpha G = A$ and let $B' \supseteq A'$ be maximal with respect to $B' \cap p^\alpha G = A'$. If $A \cong A'$, then $B \cong B'$.*

Proof. As we observed in Remark 1, B and B' are $p^{\alpha+1}$-pure in G. Thus $p^\alpha B = A$ and $p^\alpha B' = A'$. Moreover, $\{B, p^\alpha G\}/p^\alpha G$ and $\{B', p^\alpha G\}/p^\alpha G$ are isotype in $G/p^\alpha G$ with the same Ulm invariants as $G/p^\alpha G$. To verify that $\{B, p^\alpha G\}/p^\alpha G$ and $G/p^\alpha G$ have the same Ulm invariants, notice that B/A is maximal in G/A with respect to $B/A \cap p^\alpha(G/A) = 0$. Hence $\{B, p^\alpha G\}/p^\alpha G \cong B/A$ has the same Ulm invariants as $(G/A)/p^\alpha(G/A) \cong G/p^\alpha G$. We know that B/A and B'/A' are direct sums of countable groups by Theorem 1. Therefore $B/A \cong B'/A'$ since they have the same Ulm invariants, and the proof of the theorem is finished by Hill and Megibben's theorem [4].

An interesting consequence of Theorem 2 is that if the primary group G is a direct sum of countable groups and if A is a neat subgroup of $p^\beta G$ for any $\beta = \omega\alpha$, then, up to isomorphism, there exists only one subgroup B of G that is maximal with respect to $B \cap p^\beta G = A$.

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