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2-LOCAL AMALGAMS FOR THE SIMPLE GROUPS $GL(5,2)$, M_{24} AND He

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Abstract. We elaborate on a method of G. Michler [9] to construct finite groups with a prescribed involution centralizer H using compatible pairs and amalgamation of H and another naturally arising 2-local subgroup. Here we deal with the particular set-up leading to the simple groups $GL(5,2), M_{24}$ and He.

1. Introduction

In [9] G. Michler described a method to construct the finite simple groups G satisfying the following hypothesis:

Hypothesis 1.1. G contains a 2-central involution z such that:

(1) $\mathbb{C}_G(z) \cong H$ for some given group H.
(2) $G = \langle \mathbb{C}_G(z), N\rangle$, where $N := \mathbb{N}_G(A)$ for some elementary abelian normal subgroup A of maximal order $|A| \geq 4$ of a Sylow 2-subgroup S of H.
(3) For some prime $p < |H|^2 - 1$ not dividing $|H| \cdot |N|$ the group G has an irreducible p-modular representation, the restriction of which to H is multiplicity free.

In this paper we shall elaborate on this for the case where $H \cong 2^{1+6} : L_3(2)$. As is well known, this will lead to the simple groups $GL_5(2), M_{24}$ and He. In fact, all finite groups having H as an involution centralizer are known.

Theorem 1.2. *Let H_0 be isomorphic to the centralizer of a 2-central involution of $GL(5,2)$, i.e., $H_0 \cong 2^{1+6} : L_3(2)$. Moreover, suppose that G is a finite group containing an involution z such that $\mathbb{C}_G(z) \cong H_0$. Then one of the following holds:*

(1) $G = O(G) : \mathbb{C}_G(z)$.
(2) $G \cong \mathrm{Hol}(2^4) \cong 2^4 : GL(4,2)$.

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(3) *G is isomorphic to $GL(5,2), M_{24}$ or He.*

Proof. Theorem 2 of [10] gives the structure of $G/O(G)$ as indicated above. If $G \neq O(G) : \mathbb{C}_G(z)$, then G contains a fours-subgroup the involutions of which are conjugate to z. An easy application of a well known fixed point formula of R. Brauer (see, e.g., (12.6) in [7]) now shows that $O(G) = 1$, and thus yields the claim. □

For the sake of convenience we recall the main steps of the construction method mentioned above. To this end we assume that H is the prescribed centralizer of an involution z and that S is a Sylow 2-subgroup of H.

Step 1: Determine the set $\mathcal{A}$ of all elementary abelian normal subgroups A of S such that $D := D(A) := \mathbb{N}_H(A)$ is a proper subgroup of H.

Step 2: For each $A \in \mathcal{A}$ determine the possible structure of $\mathbb{N}_G(A)$, i.e., determine a group $N := N(A)$ together with an embedding $\phi : D \to D_1 \leq N$ such that $D_1 = \mathbb{C}_N(\phi(z))$ is a proper subgroup of odd index in N.

Step 3: For each quadruple (H, D, ϕ, N) found so far determine the set Π of so-called compatible pairs $(\chi, \psi) \in \mathrm{char}(H) \times \mathrm{char}(N)$ of complex valued characters such that both χ and ψ are faithful with $\chi_{|D} = \psi_{|D_1}$ and χ multiplicity free.

Step 4: For each quadruple (H, D, ϕ, N) and each admissible compatible pair $(\chi, \psi) \in \Pi$ and for each prime $p < b := |H|^2 - 1$ coprime to $|H| \cdot |N|$ construct (up to isomorphism) all possible amalgams $\chi(H) *_{\chi(D)} \psi(N) \leq GL(n, F)$, where $n := \deg(\chi) = \deg(\psi)$ and F is a finite splitting field of characteristic p for the groups H, D and N. (Here we have abused notation in so far as $\chi(H)$ and $\psi(N)$ denote the matrix representations of H and N inside $GL(n, F)$ corresponding to χ and ψ, respectively.)

Step 5: Among the amalgamated products constructed in Step 4 determine those in which H actually does occur as the centralizer of a 2-central involution.

Remarks 1.3.

(1) The condition in Step 3 that χ is multiplicity free of course ensures that the resulting set Π is finite and that in Step 4 the amalgamation process becomes somewhat easier. Since without the condition of multiplicity freeness the resulting set Π is infinite, it is reasonable to have some bound on the degree $n = \deg(\chi) = \deg(\psi)$.

(2) The bound b in Step 4 is large and is due to the fact that the simple groups to be constructed are known to embed into the alternating group of degree b. However, in some instances the bound can be improved; see Remark 3.8 in [9].

(3) The choice of the field F is more subtle than indicated in Step 4. In general, when the resulting finite simple groups are not known, one has to enlarge F to be a splitting field for the alternating group Alt_b and all its subgroups in order not to miss a group satisfying the given hypothesis. If however the orders of the simple target groups are known at Step 4, then a considerably smaller field F can be chosen.

(4) The purpose of this paper is to show how the method outlined above works if are several simple target groups to be expected, and thereby give a constructive existence proof of the sporadic groups M_{24} and He starting from scratch with a prescribed involution centralizer. Furthermore, we shall demonstrate—under certain circumstances—how the number of different amalgams and the number of associated finite completions to be considered may be kept within a manageable order of magnitude.

2. Some preliminaries and notation

We start this section by collecting some well known and helpful facts concerning amalgams of rank 2.

THEOREM 2.1. *Let* $\mathcal{A} := (\varphi_i : B \to P_i; i = 1, 2)$ *be an amalgam of rank* 2 *and let* $A_i^* := \{\alpha \in \mathrm{Aut}(P_i) | \alpha \text{ normalizes } \mathrm{Im}(\varphi_i)\}$. *Moreover, for* $i \in \{1, 2\}$ *define a homomorphism* $\alpha_i \in \mathrm{Hom}(A_i^*, \mathrm{Aut}(B))$ *by* $\alpha_i(\eta) := \varphi_i \eta \varphi_i^{-1}$ *for* $\eta \in A_i^*$ *and put* $A_i := \mathrm{Im}(\alpha_i) \leq \mathrm{Aut}(B)$. *Then there is a* $1-1$ *correspondence between the isomorphism classes of amalgams having the same type as* $\mathcal{A}$ *and the* (A_1, A_2) *double cosets in* $\mathrm{Aut}(B)$*; more precisely, for any double coset representative* β *of* A_1 *and* A_2 *in* $\mathrm{Aut}(B)$ *the corresponding amalgam is given by* $(\varphi_1 : B \to P_1, \beta\varphi_2 : B \to P_2)$.

Proof. See [4]. □

THEOREM 2.2. *Let* F *be a finite field and* $n \in \mathbb{N}$*; furthermore, let* A *and* B *be subgroups of* $G := GL(n, F)$ *with* $D := A \cap B$. *Then the isomorphism classes of amalgams of type* $A *_D B$ *in* G *are in* $1-1$ *correspondence with the* $(\mathbb{C}_G(A), \mathbb{C}_G(B))$ *double cosets in* $\mathbb{C}_G(D)$.

Proof. See [15]. □

THEOREM 2.3. *Let* X *be a finite group containing an involution* j *such that* $F^*(\mathbb{C}_X(j))$ *is extraspecial of width at least* 2. *If* $O(X) = 1$ *then one of the following holds:*

(1) $j \in \mathbb{Z}(X)$.
(2) $F^*(X) = \langle j^X \rangle$ *is isomorphic to the* m*-dimensional unitary group* $U_m(2)$ *over* $GF(2)$ *or to the second Conway group* Co_2 *or to the linear group* $L_4(3)$.
(3) j *is* X*-conjugate to some noncentral involution in* $F^*(\mathbb{C}_X(j))$.

Proof. The statement follows from the main results in [1] and [12]. □

The notation used will be standard. In addition we shall make use of the following convention.

NOTATION 2.4. If X is a group acting on a finite set S such that s_i^X with $i \in \{1, \ldots, r\}$ are the mutually different X-orbits on S and $|s_i^X| = n_i$, then we shall express this by

$$(X \Downarrow S) = n_1 \cdot s_1 + \cdots + n_r \cdot s_r.$$

Next we fix notation for elements and subgroups of the prescribed involution centralizer $H \cong 2^{1+6} : L(3,2)$ as it occurs in $GL(5,2)$.

NOTATION 2.5.
(1) Clearly, H is isomorphic to the group $H_0 := \mathbb{C}_{GL(5,2)}(r_0)$ where

$$r_0 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

An easy calculation reveals that $H_0 = \langle a, b, c \rangle$ where

$$a := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$c := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

are elements of $GL(5,2)$ of orders $8, 12$ and 12, respectively.
(2) Let T denote the set of lower triangular matrices of $GL(5,2)$; so T is a Sylow 2-subgroup of both $GL(5,2)$ and H_0. Furthermore, define ten involutory generators of T by the 5×5-scheme

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t_1 & 1 & 0 & 0 & 0 \\ d_1 & s_1 & 1 & 0 & 0 \\ r_1 & r_3 & s_2 & 1 & 0 \\ r_0 & r_2 & d_2 & t_2 & 1 \end{pmatrix},$$

where a symbol in the (i,j)-position represents the involution of T having exactly one nonzero entry in position (i,j) off the diagonal. Straightforward matrix calculations reveal the following identities:

$$r_0 = a^4,\ r_1 = a^4\cdot(a^2\cdot b^2)^3,\ r_2 = (a^2\cdot b^3)^2,\ r_3 = a\cdot b\cdot a\cdot b^5\cdot a\cdot b^4\cdot a^7,$$
$$d_1 = a\cdot b\cdot a\cdot b^2\cdot a\cdot b\cdot a^5, d_2 = a^3\cdot b^3\cdot a\cdot b^9,$$
$$s_1 = a\cdot b^4\cdot a^3\cdot b^2\cdot a\cdot b^7,\ s_2 = (a\cdot b^5)^2,$$
$$t_1 = a\cdot b^5\cdot a^3\cdot b^5\cdot a^6,\ t_2 = (a^2\cdot b^2)^3\cdot b^3.$$

(3) Put $E_i := \langle r_0, r_i, d_i, t_i\rangle$ for $i\in\{1,2\}$ and $Q := \langle E_1, E_2\rangle$. Then E_i is a self-centralizing elementary abelian normal subgroup of order 2^4 of H and $Q := E_1E_2 = O_2(H) \cong 2^{1+6}$.

(4) Put $R_i := \langle r_0, r_1, r_2, r_3, d_i, s_i\rangle$ and $H_i := \mathbb{N}_{H_0}(R_i)$ for $i\in\{1,2\}$ as well as

$$h_1 := t_1\cdot b^4 = \begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\quad h_2 := c^4 = \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}.$$

Then R_i is a self-centralizing elementary abelian normal subgroup of order 2^6 of T and $H_i = Q : P_i = R_i : K_i$, where $P_i := \langle r_3, s_i, h_j, s_j\rangle \cong \mathrm{Sym}_4$ and $K_i := \langle t_i, t_j, d_j, h_j, s_j\rangle \cong \mathbb{Z}_2\times\mathrm{Sym}_4$ with $i\neq j\in\{1,2\}$.

Finally we record some basic facts concerning the group H_0 and some of its subgroups.

Lemma 2.6. *The following hold:*

1. *For each $x\in H_0$ let x^α denote the reflection of x^{-1} along the antidiagonal. Then α induces an involutory outer automorphism of H_0 with $E_1^\alpha = E_2$ and $R_1^\alpha = R_2$ as well as $H_1^\alpha = H_2$ and $T^\alpha = T$.*
2. *R_1 and R_2 are the only elementary abelian normal subgroups of maximal order 2^6 in T.*
3. *$(H_0 \Downarrow Q^\sharp) = 1r_0 + 14r_1 + 14r_2 + 42r_1r_2 + 56r_1t_2$ with $(r_1t_2)^2 = r_0$.*
4. *$(H_2 \Downarrow R_2^\sharp) = 1r_0 + 2r_1 + 6r_2 + 6r_1r_2 + 12r_3 + 12r_0r_3 + 24r_3d_2$. Furthermore, we have $V_1 := \langle r_1^{H_2}\rangle = \langle r_0, r_1\rangle$, $V_2 := \langle r_2^{H_2}\rangle = \langle r_0, r_2, d_2\rangle$, $V_3 := \langle (r_1r_2)^{H_2}\rangle = \langle r_0, r_1, r_2, d_2\rangle$ and $\langle r_3^{H_2}\rangle = \langle (zr_3)^{H_2}\rangle = \langle (r_3d_2)^{H_2}\rangle = R_2$. In particular, H_2 acts irreducibly on the section $R_2/V_3 \cong 2^2$.*

Proof. All claims can be verified by means of easy matrix calculations. □

3. The second 2-local subgroup N

In view of Hypothesis 1.1 and Lemma 2.6 we shall assume from now on the following hypothesis:

Hypothesis 3.1. *G is a finite group containing an involution z such that $H := \mathbb{C}_G(z) \cong H_0$ and $G \neq O(G)H$ as well as $G = \langle H, N\rangle$ with $N := \mathbb{N}_G(R_2)$.*

For the sake of convenience we shall also identify H with H_0 and use the same notation for elements and subgroups of H as for H_0 as introduced in 2.5 and 2.6; so, in particular, $z = r_0$.

Lemma 3.2. *The following hold:*

(1) *T is a Sylow 2-subgroup of G with $|T| = 2^{10}$ and $\mathbb{Z}(T) = \langle z\rangle$. In particular, $\mathbb{N}_G(T) = \mathbb{N}_H(T) = T$.*
(2) *The groups R_1 and R_2 are not conjugate within G.*
(3) *z is G-conjugate to at least one element of $\{r_1, r_2, r_1r_2\}$.*
(4) *The group N controls G-fusion in R_2.*
(5) *$\mathbb{C}_G(R_2) = R_2$ and N/R_2 is isomorphic to a subgroup of $GL(6, 2)$.*
(6) *D is a proper subgroup of N with $|N : D|$ dividing $3^3 \cdot 5 \cdot 7^2 \cdot 31$; moreover, $D/R_2 \cong \mathbb{Z}_2 \times \mathrm{Sym}_4$.*

Proof. The claims in (1) are obvious. Since R_1 and R_2 are the only elementary abelian normal subgroups of order 64 of T, $\mathbb{N}_G(T)$ controls the G-fusion of R_1 and R_2; therefore, (2) follows from (1).

Next observe that (3) is an easy consequence of Lemma 2.6 and Theorem 2.3. All remaining claims are immediate now. □

Lemma 3.3. *If N acts reducibly on R_2, then $\langle z^N\rangle \in \{V_1, V_2\}$ and $O_2(G) = \mathbb{C}_G(O_2(G)) \in \{E_1, E_2\}$ and $G \cong \mathrm{Hol}(2^4) \cong 2^4 : GL(4, 2)$.*

Proof. Put $V := \langle z^N\rangle$ and note that $V \in \{V_1, V_2, V_3\}$ by Lemma 2.6; moreover, N normalizes $C := \mathbb{C}_G(V)$.

Assume first that $V = V_1$. Then $C = \mathbb{C}_H(V_1) = E_1R_2 : \langle h_1, s_1\rangle$ and E_1 is the only elementary abelian normal subgroup of order 16 in $O_2(C) = E_1R_2$ not contained in R_2; hence E_1 is normal in N.

If $V = V_2$ then $C = \mathbb{C}_H(V_2) = E_2R_2$ and E_2 is the only elementary abelian normal subgroup of order 16 in C not contained in R_2; hence E_2 is normal in N.

Finally assume that $V = V_3$. Thus $C = \mathbb{C}_H(V_3) = R_2$. Moreover, z must be N-conjugate to r_1r_2 and thus $|z^N| \in \{7, 7 + 2, 7 + 6, 7 + 2 + 6\}$, i.e., $|N/R_2| \in \{2^4 \cdot 3 \cdot 7, 2^4 \cdot 3^3, 2^4 \cdot 3 \cdot 13, 2^4 \cdot 3^2 \cdot 5\}$. On the other hand, N/R_2 is isomorphic to a subgroup of $\mathrm{Aut}(V) \cong GL(4, 2)$ and so has order dividing $2^6 \cdot 3^2 \cdot 5 \cdot 7$; therefore $|N/R_2| \in \{2^4 \cdot 3 \cdot 7, 2^4 \cdot 3^2 \cdot 5\}$. An inspection of the subgroup structure of $GL(4, 2)$ now shows that $|N/R_2| = 2^4 \cdot 3^2 \cdot 5$ and $N/R_2 \cong \mathrm{Sym}_6$. As this contradicts the fact that N/R_2 acts irreducibly on the section R_2/V_3, we are done. □

In view of this result we may and shall assume from now on the following hypothesis:

HYPOTHESIS 3.4. The group N acts irreducibly on R_2.

As an immediate consequence of 3.4 we obtain the following result:

LEMMA 3.5. *The following hold:*

(1) $O_2(N) = R_2$.
(2) *Each element of* $\{z, r_1, r_2, r_1r_2\}$ *has an* N*-conjugate in* $\{r_3, zr_3, r_3d_2\}$.
(3) $|z^N| = |N : D| \in \{15, 21, 27, 31, 45, 49, 63\}$.

Proof. The claims in (1) and (2) are obvious. Next, observe that $|z^N| = |N : D|$ is a sum of some D-orbit lengths given in Lemma 2.6, i.e., a sum of some of the numbers $1, 2, 6, 6, 12, 12, 24$. Since $|N : D|$ also divides $3^3 \cdot 5 \cdot 7^2 \cdot 31$, an easy inspection using (2) now yields the claim in (3). $\square$

Before we are able to determine the structure of N we need to collect some information on the structure of some subgroups of $GL(6, 2)$.

LEMMA 3.6. *Let* X *be a subgroup of* $GL(6, 2)$ *and* $Y := \mathbb{N}_{GL(6,2)}(X)$. *Then the following hold:*

(1) *If* $|X| = 7^2$, *then* $Y \cong ((7 : 3) \times (7 : 3)) : 2$.
(2) *If* $|X| = 7$, *then* Y *is isomorphic to one of the groups* $(7 : 3) \times L(3, 2), (7 \times L(2, 8)) : 3$ *or* $(7 \times 7) : 6$.
(3) *If* $X \cong \mathbb{Z}_5$, *then* $Y \cong (\mathbb{Z}_{15} : 4) \times \mathrm{Sym}_3$.
(4) *If* $X \cong \mathbb{Z}_9$, *then* $Y \cong \mathbb{Z}_{63} : 6$.
(5) *If* $|X| = 3^3$, *then one of the following holds:*
    (a) X *is extraspecial of exponent* 3 *and* $Y \cong 3^{1+2} : GL(2, 3)$.
    (b) X *is extraspecial of exponent* 9 *and* $Y \cong 3^{1+2} : \mathrm{Sym}_3$.
    (c) X *is elementary abelian and* $Y \cong \mathrm{Sym}_3 \wr \mathrm{Sym}_3$; *moreover,* Y *induces four orbits of lengths* $1, 9, 27$ *and* 27 *on the underlying* 6*-dimensional* $GF(2)$*-space.*

Proof. All claims can be verified by straightforward calculations in the group $GL(6, 2)$. $\square$

LEMMA 3.7. N *is a split extension of* R_2 *and a group* K *such that exactly one of the following two cases occurs:*

(1) $|N : D| = 21$, $K \cong \mathrm{Sym}_3 \times L(3, 2)$ *and* $(N \Downarrow R_2^\sharp) = 21z + 42r_3d_2$.
(2) $|N : D| = 45$, $K \cong \hat{3}\mathrm{Sym}_6$ *and* $(N \Downarrow R_2^\sharp) = 45z + 18x$ *for a suitable* $x \in R_2$.

Moreover, in either case the group N *is determined uniquely up to isomorphism.*

Proof. Put $I := |N : D|$ and $\bar{N} := N/R_2$, and recall that $I \in \{15, 21, 27, 31, 45, 49, 63\}$ and $\bar{D} \cong \mathbb{Z}_2 \times \mathrm{Sym}_4$.

Assume that $I = 31$. Then N induces orbits of lengths $1, 31$ and 32 on R_2, contradicting the fact that 32 does not divide $|\bar{N}|$. Therefore $I \neq 31$.

We know already that $O_2(\bar{N}) = 1$. By Lemma 3.6 we easily see that $O_5(\bar{N}) = O_7(\bar{N}) = 1$.

Assume next that $\bar{N}$ is solvable. Inspecting the order of $\bar{N}$ we get $F(\bar{N}) = O_3(\bar{N})$. Since $GL(2,3)$ does not involve a section isomorphic to $\bar{D}$, application of parts (4) and (5) of 3.6 now shows that $F(\bar{N})$ is elementary abelian of order 3^3 and that $\bar{N} = F(\bar{N}) : \bar{D} \cong \mathrm{Sym}_3 \wr \mathrm{Sym}_3$ induces orbits of lengths $1, 9, 27, 27$ on R_2. But this contradicts the fact that $|\mathbb{Z}(T)| = 2$.

We have shown that $\bar{N}$ is nonsolvable. In particular, $I \neq 27$. Furthermore, $\bar{N}$ has a nonabelian simple composition factor, say L.

Assume next that $I = 63$ and thus $|\bar{N}| = 2^4 \cdot 3^3 \cdot 7$. Moreover, suppose that $L \cong L(2,8)$. Since now L has trivial Schur multiplier and since $\mathrm{Out}(L) \cong \mathbb{Z}_3$, $\bar{N}$ must contain a subgroup isomorphic to $\mathbb{Z}_2 \times L(2,8)$. As this conflicts with the structure of a Sylow 2-subgroup of $\bar{N}$, we conclude that $L \cong L(3,2)$. Since neither $\mathrm{Aut}(L)$ nor the double cover of L has a Sylow 2-subgroup isomorphic to $\mathbb{Z}_2 \times D_8$, we get $\bar{N} = C \times L$, where $C := \mathbb{C}_{\bar{N}}(L)$ has order $2 \cdot 3^2$. By part (2) of 3.6 we see that $O_3(C) \cong \mathbb{Z}_9$. As this conflicts with part (4) of 3.6, we finally conclude that $I \neq 63$.

If $I = 49$ and thus $|\bar{N}| = 2^4 \cdot 3 \cdot 7^2$, we easily see that $L \cong L(3,2)$. Similar arguments as above now yield $\bar{N} \cong (7:2) \times L(3,2)$, contrary to $O_7(\bar{N}) = 1$. Therefore, $I \neq 49$.

Assume next that $I = 15$ and hence $|\bar{N}| = 2^4 \cdot 3^2 \cdot 5$. Since the groups $\mathbb{Z}_2 \times \mathrm{Alt}_6, \mathrm{Sym}_6, PGL(2,9)$ and M_{10} do not have an irreducible $GF(2)$-representation of dimension 6 and since the double cover of Alt_6 has Sylow 2-subgroups not isomorphic to $\mathbb{Z}_2 \times D_8$, Alt_6 cannot be involved in $\bar{N}$. So we conclude that $L \cong \mathrm{Alt}_5$.

Since L has Schur multiplier of order 2 and since $O_2(\bar{N}) = 1$ as well as $\mathrm{Out}(L) \cong \mathbb{Z}_2$, the 2-structure of $\bar{N}$ now implies that $\bar{N} \cong \mathrm{Sym}_3 \times \mathrm{Sym}_5$. We have derived a contradiction, because the group $\mathrm{Sym}_3 \times \mathrm{Sym}_5$ has no irreducible $GF(2)$-representation of dimension 6. Hence $I \neq 15$.

Assume now that $I = 21$ and hence $|\bar{N}| = 2^4 \cdot 3^2 \cdot 7$. If $L \cong L(2,8)$, similar arguments as above yield $\bar{N} \cong \mathbb{Z}_2 \times L(2,8)$, a contradiction. Therefore we have $L \cong L(3,2)$.

As $O_2(\bar{N}) = 1$ and since the Sylow 2-subgroups of $\mathrm{Aut}(L)$ are dihedral of order 16, we easily verify now that $\bar{N} \cong \mathrm{Sym}_3 \times L(3,2)$.

Observe next that the group $\bar{N}$ has exactly two nonisomorphic irreducible $GF(2)$-modules of dimension 6 and that exactly one of these two modules admits $\bar{D}$ as the stabilizer of a nontrivial module element. Also note that as a $GF(2)\bar{N}$-module R_2 is isomorphic to a tensor product of a 2-dimensional $GF(2)\mathrm{Sym}_3$-module and a 3-dimensional $GF(2)L(3,2)$-module. Therefore, elements of order 7 of N act fixed point freely on $R_2^\sharp$; consequently N induces orbits of lengths $1, 21$ and 42 on R_2. Finally note that N splits over R_2

because T does. Therefore, N is determined uniquely up to isomorphism in case (1).

We are left with the case $I = 45$ and $|\bar{N}| = 2^4 \cdot 3^3 \cdot 5$. Assume first that $L \cong \mathrm{Alt}_5$. Since $\mathrm{Aut}(L) \cong \mathrm{Sym}_5$ and since L has Schur multiplier of order 2 and since $O_2(\bar{N}) = 1$, the presence of $\bar{D} \cong \mathbb{Z}_2 \times \mathrm{Sym}_4$ now implies that $\bar{N} \cong \mathrm{Sym}_3 \times ((\mathbb{Z}_3 \times \mathrm{Alt}_5) : \mathbb{Z}_2)$ with N inducing orbits of lengths $3, 15$ and 45 on $R_2^\sharp$. But this contradicts the fact that $|\mathbb{Z}(T)| = 2|$. Therefore we have $L \cong \mathrm{Alt}_6$.

Now observe that $\mathrm{Out}(L)$ is a 2-group and that $\mathbb{Z}_3 \times \mathrm{Alt}_6$ has no irreducible $GF(2)$-representation of dimension 3 or 6; therefore $\bar{N}'$ is isomorphic to the triple cover $\hat{3}\mathrm{Alt}_6$. Since $\mathbb{Z}(\bar{D}) = \langle \bar{t}_2 \rangle$ with $|[R_2, t_2]| = 2^3$ we then get $\bar{N} \cong \hat{3}\mathrm{Sym}_6$ with an involution of $\bar{N} - \bar{N}'$ acting invertingly on $O_3(\bar{N}')$. By part (2) of Lemma 3.5 we also see that N induces orbits of lengths $1, 18$ and 45 on R_2. Clearly, N splits over R_2 for the same reasons as above.

Finally observe that $\bar{N}$ has exactly two nonisomorphic irreducible $GF(2)$-modules of dimension 6 and that exactly one of these two modules admits $\bar{D}$ as the stabilizer of a nontrivial module element. Hence, N is determined uniquely up to isomorphism also in case (2). □

Before we describe the possibilities for the group N more explicitly we give a presentation for the group D, which will be helpful in later arguments.

Lemma 3.8. *Define a set $\mathcal{R}(D)$ of relations as follows:*

$$a^8,\ b^{12},\ a^4 \cdot b^6,\ (a \cdot b)^4,\ (a \cdot b^{-1})^4,\ (a \cdot b^{-2} \cdot a \cdot b)^2,\ (a \cdot b^2 \cdot a^{-2} \cdot b)^2,$$
$$a^2 \cdot b^3 \cdot a^{-2} \cdot b \cdot a^{-1} \cdot b^{-3} \cdot a \cdot b^{-1},\ a^2 \cdot b \cdot a^2 \cdot b^{-1} \cdot a^{-2} \cdot b \cdot a^{-2} \cdot b^{-1}.$$

Then $\mathcal{R}(D)$ is a set of defining relations for the group D.

Proof. The claims can easily be verified by a straightforward coset enumeration, e.g., by means of MAGMA [2]. □

Lemma 3.9. *The following hold:*

(1) *If $N \cong 2^6 : (\mathrm{Sym}_3 \times L(3,2))$, then we may identify N without loss with the group $N_1 := \langle a, b, d \rangle \leq GL(5,2)$, where*

$$d = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(2) *If $N \cong 2^6 : \hat{3}\mathrm{Sym}_6$, then we may identify N without loss with the group $N_2 := \langle a', b', f \rangle \leq GL(7,2)$, where*

$$a' = \begin{pmatrix} 1&0&0&1&0&0&1\\ 0&1&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&1&0&0&1&0&0\\ 0&1&1&0&1&1&0\\ 0&0&1&1&0&1&1 \end{pmatrix}, \quad b' = \begin{pmatrix} 1&1&0&0&1&0&0\\ 0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0\\ 0&0&1&1&0&0&0\\ 0&1&0&0&1&0&0\\ 0&1&0&1&1&0&1\\ 0&0&1&1&0&1&1 \end{pmatrix},$$

$$f = \begin{pmatrix} 1&0&1&0&1&0&0\\ 0&0&0&1&0&1&1\\ 0&1&1&0&0&1&1\\ 0&0&0&0&1&0&1\\ 0&0&1&1&0&1&0\\ 0&0&1&1&1&0&1\\ 0&1&0&1&1&0&1 \end{pmatrix}.$$

Furthermore, $(a,b) \to (a',b')$ *gives an embedding of* D *into* N_2.

Proof. Clearly, in case (1) the group N can be embedded into the group $GL(5,2)$ in view of the tensor product decomposition of R_2 as an N/R_2-module (see the proof of Lemma 3.7). So we simply have to extend the group D to a maximal parabolic of $GL(5,2)$, which of course is straightforward.

Using MAGMA [2] all claims in part (2) can easily be verified. It is obvious that the split extension $N = R_2 : K$ can be embedded into $GL(7,2)$. In order to do this we first represent the action of elements of D on the normal subgroup R_2 by suitable (6×6)-matrices over $GF(2)$ with respect to the ordered basis $\{r_0, r_2, d_2, r_1, r_3, s_2\}$ and then extend the resulting subgroup of $GL(6,2)$ by a suitable element to a subgroup isomorphic to $\hat{3}\mathrm{Sym}_6$. With this we form the split extension with R_2 inside $GL(7,2)$ and then find the appropriate generators as given above. Finally observe, that the pair (a',b') satisfies the presentation of D given in Lemma 3.8. $\square$

4. The amalgams

In this section we are going to determine representatives of the isomorphism classes of amalgams of type (H,D,N). To this end we need some information about the automorphism groups of the groups involved.

Lemma 4.1. *Put* $b_1 := b^{(a \cdot b^3 \cdot a^3)}$. *Then the following hold:*

(1) *The pairs* $(a,b), (a,b^7), (a,b_1)$ *and* (a,b_1^7) *are representatives of the four different* D*-classes of pairs satisfying the defining relations* $\mathcal{R}(D)$ *of* D.

(2) *The maps* $\delta_1 : (a,b) \to (a,b^7)$, $\delta_2 : (a,b) \to (a,b_1)$, *and* $\delta_3 : (a,b) \to (a,b_1^7)$ *define automorphisms of* D *such that* $\delta_1^2 = \delta_2^2 = \delta_3^2 = \mathrm{id}_D$ *and* $\delta_1\delta_2 = \delta_2\delta_1 = \delta_3$ *as well as* $\mathrm{Aut}(D) = \mathrm{Inn}(D) : \langle \delta_1, \delta_2 \rangle$ *and* $|\mathrm{Aut}(D) : \mathrm{Inn}(D)| = 4$.

Proof. Since D has only one class of elements of order 8 and only one class of elements order 12, an easy search in $a^D \times b^D$—e.g., by means of MAGMA [2]—yields the claims in (1). Similarly we show that $\delta_1^2 = \delta_2^2 = \delta_3^2 = \mathrm{id}_D$. Now all remaining claims in (2) are obvious. □

LEMMA 4.2. *Define a set $\mathcal{R}(H)$ of relations as follows:*

$$\begin{aligned}
&a^8,\ b^{12},\ c^{12},\\
&a^4 \cdot b^6,\ (a \cdot b)^4,\ (a \cdot b^{-1})^4,\ (a \cdot b^{-2} \cdot a \cdot b)^2,\ (a \cdot b^2 \cdot a^{-2} \cdot b)^2,\\
&a^2 \cdot b^3 \cdot a^{-2} \cdot b \cdot a^{-1} \cdot b^{-3} \cdot a \cdot b^{-1},\ a^2 \cdot b \cdot a^2 \cdot b^{-1} \cdot a^{-2} \cdot b \cdot a^{-2} \cdot b^{-1},\\
&a^4 \cdot c^6,\ (a \cdot c)^4,\ (a \cdot c^{-1})^4,\ (a \cdot c^{-2} \cdot a \cdot c)^2,\ (a \cdot c^2 \cdot a^{-2} \cdot c)^2,\\
&a^2 \cdot c^3 \cdot a^{-2} \cdot c \cdot a^{-1} \cdot c^{-3} \cdot a \cdot c^{-1},\ a^2 \cdot c \cdot a^2 \cdot c^{-1} \cdot a^{-2} \cdot c \cdot a^{-2} \cdot c^{-1},\\
&(b \cdot c^{-1})^4,\ (c \cdot a^{-1} \cdot b)^3,\ (a^2 \cdot c^{-1} \cdot b^{-1} \cdot c)^2,\ a \cdot c \cdot b^2 \cdot a^{-1} \cdot b^{-1} \cdot c^{-2},\\
&a \cdot c \cdot b^{-1} \cdot c \cdot b^{-2} \cdot a^{-1} \cdot c^{-1} \cdot b^{-1}.
\end{aligned}$$

Then the following hold:

(1) *$\mathcal{R}(H)$ is a set of defining relations for the group H.*
(2) *(a,b,c) and (a,c,b^a) are representatives of the two H-classes of triples satisfying the defining relations $\mathcal{R}(H)$ of H.*
(3) *Put $A := \mathrm{Aut}(H)$. Then $|A : \mathrm{Inn}(H)| = 2$ and H has an outer automorphism of order 2 interchanging the two maximal parabolics $\langle a,b\rangle$ and $\langle a,c\rangle$ of H. In particular, $\mathrm{Aut}_A(D) := \mathbb{N}_A(D)/\mathbb{C}_A(D)$ is isomorphic to $\mathrm{Inn}(D) \cong D/\langle z\rangle$.*

Proof. An easy check using MAGMA [2] yields all claims in parts (1) and (2). The claims in (3) now follow immediately from the fact that $\mathbb{N}_H(D) = D$. □

LEMMA 4.3. *Define a set $\mathcal{R}(N_1)$ of relations as follows:*

$$\begin{aligned}
&a^8,\ b^{12},\ a^4 \cdot b^6,\ (a \cdot b)^4,\ (a \cdot b^{-1})^4,\ (a \cdot b^{-2} \cdot a \cdot b)^2,\ (a \cdot b^2 \cdot a^{-2} \cdot b)^2,\\
&a^2 \cdot b^3 \cdot a^{-2} \cdot b \cdot a^{-1} \cdot b^{-3} \cdot a \cdot b^{-1},\ a^2 \cdot b \cdot a^2 \cdot b^{-1} \cdot a^{-2} \cdot b \cdot a^{-2} \cdot b^{-1},\\
&d^{21},\ (a \cdot d^{-2})^2,\ d^2 \cdot b^{-1} \cdot a^{-3} \cdot b^{-2} \cdot d \cdot a \cdot b^{-1},\\
&a^{-1} \cdot b \cdot a \cdot d \cdot a^2 \cdot d^{-1} \cdot b \cdot a \cdot b.
\end{aligned}$$

Then the following hold:

(1) *$\mathcal{R}(N_1)$ is a set of defining relations for the group N_1.*
(2) *(a,b,d) represents the single N_1-class of triples satisfying the relations $\mathcal{R}(N_1)$. In particular, $\mathrm{Aut}(N_1) = \mathrm{Inn}(N_1) \cong N_1$.*
(3) *Put $A := \mathrm{Aut}(N_1)$. Then $\mathrm{Aut}_A(D) := \mathbb{N}_A(D)/\mathbb{C}_A(D) \cong \mathrm{Inn}(D)$.*

Proof. Easy computational exercise. □

Lemma 4.4. *Define a set $\mathcal{R}(N_2)$ of relations as follows:*

$$a^8,\ b^{12},\ a^4\cdot b^6,\ (a\cdot b)^4,\ (a\cdot b^{-1})^4,\ (a\cdot b^{-2}\cdot a\cdot b)^2,\ (a\cdot b^2\cdot a^{-2}\cdot b)^2,$$
$$a^2\cdot b^3\cdot a^{-2}\cdot b\cdot a^{-1}\cdot b^{-3}\cdot a\cdot b^{-1}, a^2\cdot b\cdot a^2\cdot b^{-1}\cdot a^{-2}\cdot b\cdot a^{-2}\cdot b^{-1},$$
$$f^{15},\ (a\cdot b^{-2}\cdot f\cdot a)^2,\ b^{-1}\cdot f^{-1}\cdot a^{-3}\cdot f^2\cdot b^{-1}\cdot a,$$
$$b^{-2}\cdot f\cdot b^{-2}\cdot a\cdot f\cdot b^2\cdot a^{-1},\ f^{-1}\cdot b^{-1}\cdot f\cdot a^{-2}\cdot b^{-1}\cdot f\cdot a\cdot b^{-1}.$$

Then the following hold:

(1) *$\mathcal{R}(N_2)$ is a set of defining relations for the group N_2.*
(2) *(a', b', f) represents the single N_2-class of triples satisfying the relations $\mathcal{R}(N_2)$. In particular,* $\mathrm{Aut}(N_2) = \mathrm{Inn}(N_2) \cong N_2$.
(3) *Put $A := \mathrm{Aut}(N_2)$. Then $\mathrm{Aut}_A(D) := \mathbb{N}_A(D)/\mathbb{C}_A(D) \cong \mathrm{Inn}(D)$.*

Proof. Easy computational exercise. □

Now we are in a position to describe all relevant amalgams.

Lemma 4.5. *Let $i \in \{1, 2\}$. Then there are exactly four different isomorphism classes of amalgams of type (H, D, N_i); these classes are represented by the amalgams $\mathcal{A}_{i,j} := (D \le H, \nu_{i,j} : D \to N_i)$, $j \in \{0, 1, 2, 3\}$, where the embeddings $\nu_{i,j}$ of D into N_i are defined by*

$$\nu_{i,0} : (a, b) \to (\alpha, \beta), \quad \nu_{i,1} : (a, b) \to (\alpha, \beta^7),$$
$$\nu_{i,2} : (a, b) \to (\alpha, \beta_1), \quad \nu_{i,3} : (a, b) \to (\alpha, \beta_1^7)$$

with $(\alpha, \beta) = (a, b)$ in case $i = 1$ and $(\alpha, \beta) = (a', b')$ in case $i = 2$ and with $\beta_1 := \beta^{(\alpha\cdot\beta^3\cdot\alpha^3)}$.

Proof. The claims are an immediate consequence of Theorem 2.1 and the earlier results in this chapter. □

5. Compatible pairs and associated completions

In this section we determine compatible pairs for the rank 2 amalgams $\mathcal{A}_{i,j}$ defined in Lemma 4.5 and study the associated finite completions over a suitable field. In order to do this we need to know the irreducible characters (over $\mathbb{C}$) of the groups involved; in particular, we need to know how the irreducible characters of H and N_i restrict to the amalgamated subgroup. Once this information is available, we can employ an algorithm due to M. Kratzer [8] to compute compatible pairs for the amalgams $\mathcal{A}_{i,j}$.

Lemma 5.1. *The irreducible complex characters of the groups D, H, N_1 and N_2 are as given in the tables (I)–(IV) in the appendix.*

Proof. Clearly, if need be, the character tables in question can be worked out by hand. However, it is more convenient to use GAP [11] or MAGMA [2] to calculate these tables. □

REMARKS 5.2.

(1) By an algorithm of M. Kratzer [8] canonical representatives of the conjugacy classes of the groups D, H, N_1 and N_2 can be determined as words in the respective generators. These canonical class representatives have been used in the process of calculating the character tables in Lemma 5.1 above by means of GAP [11].

(2) It will also be advantageous to have the canonical class representatives at our disposal when determining the fusion pattern of the conjugacy classes of D in H and of $\nu_{i,j}(D)$ in N_i, respectively. In the next result we shall give these fusion patterns without actually listing the canonical class representatives, because the fusion patterns carry enough information to work out the restrictions of the irreducible characters of H (resp. N_i) to the group D (resp. $\nu_{i,j}(D)$).

LEMMA 5.3. *Let F_0 denote the fusion pattern of D in H; i.e., F_0 is a sequence of positive integers such that $F_0[i] = j$ if and only if the i-th conjugacy class of D is contained in the j-th conjugacy class of H; here the numbering of classes refers to the character tables mentioned in 5.1. Similarly, let $F_{i,j}$ denote the fusion pattern of $\nu_{i,j}(D)$ in N_i for $j \in \{0,1,2,3\}$. Then the following hold:*

$$\begin{aligned}
F_0 = [&1,2,3,4,5,4,3,6,5,7,8,9,6,5,7,9,8,10,11,12,11,11,15,16,15,\\
&14,13,14,15,14,12,13,17,19,18,19,16,19,20,21,22,23,24,27,28];\\
F_{1,0} = [&1,2,2,2,4,3,5,2,6,4,4,6,5,7,6,6,7,9,11,12,13,14,16,12,12,\\
&14,14,13,16,11,16,15,12,19,14,18,20,17,17,18,21,23,21,27,28],\\
F_{1,1} = [&1,2,2,2,4,3,5,2,6,4,4,6,6,7,5,6,7,9,11,12,13,14,16,12,12,\\
&14,14,13,16,11,16,15,14,19,12,18,20,17,17,18,21,23,21,27,28],\\
F_{1,2} = [&1,2,2,2,4,3,6,2,5,4,4,6,5,7,6,6,7,9,11,12,13,14,16,14,12,\\
&14,12,13,16,11,16,18,12,20,14,15,19,17,17,18,21,23,21,27,28],\\
F_{1,3} = [&1,2,2,2,4,3,6,2,5,4,4,6,6,7,5,6,7,9,11,12,13,14,16,14,12,\\
&14,12,13,16,11,16,18,14,20,12,15,19,17,17,18,21,23,21,27,28];\\
F_{2,0} = [&1,3,3,3,2,6,4,3,7,2,3,7,4,5,7,7,6,9,11,12,14,13,14,12,13,\\
&12,13,17,11,17,17,15,13,20,12,19,18,16,17,19,22,26,22,27,29],\\
F_{2,1} = [&1,3,3,3,2,6,4,3,7,2,3,7,7,5,4,7,6,9,11,12,14,13,14,12,13,\\
&12,13,17,11,17,17,15,12,20,13,19,18,16,17,19,22,26,22,27,29],
\end{aligned}$$

$$F_{2,2} = [1,3,3,3,2,6,7,3,4,2,3,7,4,5,7,7,6,9,11,12,14,13,14,13,13,\\ 12,12,17,11,17,17,19,13,18,12,15,20,16,17,19,22,26,22,27,29],$$
$$F_{2,3} = [1,3,3,3,2,6,7,3,4,2,3,7,7,5,4,7,6,9,11,12,14,13,14,13,13,\\ 12,12,17,11,17,17,19,12,18,13,15,20,16,17,19,22,26,22,27,29].$$

Proof. Straight forward calculations yield the claims. □

Equipped with the information of this lemma and the character tables for the groups H, D and N_i we can now work out the restrictions of the irreducible characters of H to D as well as the restrictions of the irreducible characters of N_i to $\nu_{i,j}(D)$. This in turn is used as input for the algorithm of Kratzer [8] to compute compatible pairs (χ, ψ) for the amalgams $\mathcal{A}_{i,j}$. In order to describe the results of these computations we shall use the following convention:

The irreducible characters of H will be denoted by $\chi_1, \chi_2, \dots$ according to the ordering used in the character table of H. Similarly, the irreducible characters of N_i will be denoted by $\psi_1, \psi_2, \dots$ and those of D will be denoted by $\delta_1, \delta_2, \dots$.

Lemma 5.4. *The following hold:*

(1) $\mathcal{A}_{1,0}$ *has the following compatible pairs* (χ, ψ)*:*
  (a) $(\chi_4 + \chi_5 + \chi_6 + \chi_{11}, \psi_2 + \psi_8 + \psi_{20})$ *of degree* 28 *with* $\chi_{|D} = \delta_1 + \delta_2 + \delta_6 + \delta_7 + \delta_{18} + \delta_{22} + \delta_{26} + \delta_{36}$*;*
  (b) $(\chi_1 + \chi_4 + \chi_5 + \chi_6 + \chi_{11}, \psi_3 + \psi_8 + \psi_{20})$ *of degree* 29 *with* $\chi_{|D} = 2 \cdot \delta_1 + \delta_2 + \delta_6 + \delta_7 + \delta_{18} + \delta_{22} + \delta_{26} + \delta_{36}$*;*
  (c) $(\chi_1 + \chi_4 + \chi_5 + \chi_6 + \chi_{11}, \psi_1 + \psi_2 + \psi_8 + \psi_{20})$ *of degree* 29 *with* $\chi_{|D} = 2 \cdot \delta_1 + \delta_2 + \delta_6 + \delta_7 + \delta_{18} + \delta_{22} + \delta_{26} + \delta_{36}$.

(2) $\mathcal{A}_{1,1}$ *has no compatible pairs* (χ, ψ) *such that* χ *is multiplicity free.*

(3) $\mathcal{A}_{1,2}$ *has no compatible pairs* (χ, ψ) *such that* χ *is multiplicity free.*

(4) $\mathcal{A}_{1,3}$ *has the following compatible pairs* (χ, ψ)*:*
  (a) $(\chi_5 + \chi_8 + \chi_{11}, \psi_2 + \psi_{22})$ *of degree* 22 *with* $\chi_{|D} = \delta_2 + \delta_{16} + \delta_{21} + \delta_{26} + \delta_{36}$*;*
  (b) $(\chi_1 + \chi_5 + \chi_8 + \chi_{11}, \psi_3 + \psi_{22})$ *of degree* 23 *with* $\chi_{|D} = \delta_1 + \delta_2 + \delta_{16} + \delta_{21} + \delta_{26} + \delta_{36}$*;*
  (c) $(\chi_{19} + \chi_i, \psi_j + \psi_{26})$ *of degree* 45 *with* $i \in \{22, 23\}, j \in \{4, 5\}$ *and* $\chi_{|D} = \delta_{13} + \delta_{32} + \delta_{40} + \delta_{45}$.

(5) $\mathcal{A}_{2,0}$ *has the following compatible pairs* (χ, ψ)*:*
  (a) $(\chi_1 + \chi_6 + \chi_9 + \chi_{11}, \psi_4 + \psi_{16})$ *of degree* 23 *with* $\chi_{|D} = \delta_1 + \delta_3 + \delta_{18} + \delta_{22} + \delta_{30} + \delta_{36}$*;*
  (b) $(\chi_{18} + \chi_i, \psi_{22})$ *of degree* 45 *with* $i \in \{22, 23\}$ *and* $\chi_{|D} = \delta_{12} + \delta_{24} + \delta_{38} + \delta_{45}$.

(6) $\mathcal{A}_{2,1}$ *has no compatible pairs* (χ, ψ) *such that* χ *is multiplicity free.*

(7) $\mathcal{A}_{2,2}$ *has no compatible pairs* (χ, ψ) *such that* χ *is multiplicity free.*

(8) *$\mathcal{A}_{2,3}$ has the compatible pairs $(\chi, \psi) = (\chi_4 + \chi_{14} + \chi_i, \psi_1 + \psi_6 + \psi_{19})$ of degree 51 with $i \in \{22, 23\}$ and $\chi_{|D} = \delta_1 + \delta_6 + \delta_7 + \delta_{14} + \delta_{24} + \delta_{40} + \delta_{45}$.*

Furthermore, in cases (1), (4), (5) and (8) the compatible pairs presented also include the smallest nontrivial compatible pairs.

Remark 5.5. In each of the four cases of Lemma 5.4 in which there is no compatible pair (χ, ψ) such that χ is multiplicity free the smallest nontrivial compatible pairs have degree 273 and the multiplicities that do occur are at most 2; moreover, $\chi_{|D}$ has at least seven different irreducible constituents with multiplicity 2.

Next we investigate finite completions of the amalgams $\mathcal{A}_{i,j}$ associated with the compatible pairs (χ, ψ) described in Lemma 5.4. Having decided on the underlying finite field F we proceed to construct the irreducible FH-modules and FN_i-modules corresponding to the irreducible constituents of χ and ψ, respectively; this is done by standard techniques such as chopping up permutation modules, tensor products, exterior powers or symmetric powers. Taking appropriate direct sums we then construct the FH-module X and the FN_i-module Y corresponding to χ and ψ, respectively. Next, we have to ensure that $X_{|D} = Y_{|\nu_{i,j}(D)}$. Finally, we investigate the various amalgams induced by X and Y inside the linear group $GL(n, F)$, where $n = \deg(\chi) = \deg(\psi)$.

All this is an easy and straightforward task, in particular, as the embeddings $\nu_{i,j}(D)$ of D in N_i are already known at this stage. The only limitating factor may be the number of different amalgams induced by X and Y in $GL(n, F)$; of course, this number can be determined by Theorem 2.2 well before the actual amalgamation is carried out.

Remarks 5.6.

(1) Note that the smallest nontrivial irreducible representation of the group $GL(5,2)$ over a field of odd characteristic has dimension 29 and that this representation arises as the heart of the permutation module on the cosets of a maximal parabolic subgroup of index 31 over a field of characteristic 31. Since we expect $GL(5,2)$ to be a finite completion of one of the amalgams $\mathcal{A}_{1,j}$ ($j \in \{0,1,2,3\}$) it is reasonable to investigate the completions of $\mathcal{A}_{1,j}$ associated with the compatible pairs given in Lemma 5.4 over the field $GF(31)$, even if the irreducible constituents can be realized over smaller fields.

(2) Clearly, 11 is the smallest prime not dividing $|H| \cdot |N_2|$; furthermore, all irreducible constituents of the compatible pairs of $\mathcal{A}_{2,j}$ given in Lemma 5.4 can be realized over the field $GF(11)$. Hence, it is reasonable to study the associated completions of $\mathcal{A}_{2,j}$ over the field $GF(11)$.

Theorem 5.7. *The following hold:*

(1) *The compatible pair* $(\chi_5+\chi_8+\chi_{11}, \psi_2+\psi_{22})$ *of degree* 22 *of* $\mathcal{A}_{1,3}$ *gives rise to* 30 *different amalgams inside* $GL(22,31)$*, all but two of which generate the group* $SL(22,31)$*; the two exceptions generate a subgroup isomorphic to* $O^-(22,31)$.
(2) *The compatible pair* $(\chi_1+\chi_5+\chi_8+\chi_{11}, \psi_3+\psi_{22})$ *of degree* 23 *of* $\mathcal{A}_{1,3}$ *gives rise to* 30 *different amalgams inside* $GL(23,31)$*;* 28 *of these amalgams generate* $SL(23,31)$*, one generates the alternating group* Alt_{24} *and one generates a simple group* $S_{1,3}$ *of order* $2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$ *having* H *as an involution centralizer, i.e.,* $S_{1,3}\cong M_{24}$.
(3) *The compatible pair* $(\chi_1+\chi_6+\chi_9+\chi_{11}, \psi_4+\psi_{16})$ *of degree* 23 *of* $\mathcal{A}_{2,0}$ *gives rise to* 10 *different amalgams inside* $GL(23,11)$. *Eight of these amalgams generate* $SL(23,11)$*, one generates a group isomorphic to* $O(23,11)$ *and one generates a simple group* $S_{2,0}$ *of order* $2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$ *having* H *as an involution centralizer, i.e.,* $S_{2,0}\cong M_{24}$.
(4) *Let* $i\in\{22,23\}$. *Then the compatible pair* $(\chi_4+\chi_{14}+\chi_i, \psi_1+\psi_6+\psi_{19})$ *of degree* 51 *of* $\mathcal{A}_{2,3}$ *gives rise to* 100 *different amalgams inside* $GL(51,11)$*, all but one of which generate the group* $SL(51,11)$. *The remaining exceptional amalgam generates a simple group* $S_{2,3}$ *of order* $2^{10}\cdot 3^3\cdot 5^2\cdot 7^3\cdot 17$ *having* H *as an involution centralizer, i.e.,* $S_{2,3}\cong He$.

Proof. We have already outlined the general strategy how to construct the finite completions associated with a given compatible pair. In particular, Theorem 2.2 tells us how to get hold of the various isomorphism types inside the ambient linear group. So all claims can easily be verified by computational means. Note that for the identification of the alternating groups we use results of [14]. The groups M_{24} and He can be identified using results in [5], [6] and [13]. □

Remark 5.8. By Theorem 2.2 the number of isomorphism types of finite completions of $\mathcal{A}_{1,0}$ in $GL(29,31)$ in cases (1b) and (1c) of Lemma 5.4 can be estimated to be in the range of $|GL(2,31)| = 892800$ and $|SL(2,31)| = 29760$, respectively. In particular, the first estimate explains why we did not consider completions of $\mathcal{A}_{1,0}$ in the preceding theorem. Instead we shall refine the constructive approach outlined so far.

In order to avoid having to consider too many isomorphism types of finite completions of $\mathcal{A}_{1,j}$ we have to modify the construction, and therefore recall the following relevant standing hypothesis.

Hypothesis 5.9. G is a finite group containing an involution z such that $\mathbb{C}_G(z) = H$ and $G = \langle H, N\rangle$ with $N \cong N_1 \cong 2^6 : (\mathrm{Sym}_3 \times L(3,2))$ where H and N are amalgamated over the subgroup D via the embedding $\nu_{1,j} : D \to N$.

Next, recall that E_i $(i \in \{1,2\})$ is an elementary abelian normal subgroup of H contained in D. Since $z \in E_i$, clearly $\mathbb{C}_G(E_i) = \mathbb{C}_H(E_i) = E_i$. Moreover, $H/E_i \cong 2^3 : L(3,2)$ is a maximal subgroup of $\mathrm{Aut}(E_i) \cong GL(4,2)$; therefore, either $\mathbb{N}_G(E_i) = H$ or $\mathbb{N}_G(E_i) \cong 2^4 : GL(4,2)$. In view of this observation we want to determine the group $\mathbb{N}_N(\nu_{1,j}(E_i))$ for $j \in \{0,1,2,3\}$. Before stating the result we recall that in any case $\nu_{1,j}(D) = \langle a,b\rangle \leq N = \langle a,b,d\rangle$ with N satisfying the presentation $\mathcal{R}(N_1)$ given in Lemma 4.3.

LEMMA 5.10. *The following hold:*

(1) $\nu_{1,0}(E_1) = \nu_{1,1}(E_1) \neq \nu_{1,2}(E_1) = \nu_{1,3}(E_1)$ *with* $|\mathbb{N}_N(\nu_{1,0}(E_1)) : D| = 3$ *and* $\mathbb{N}_N(\nu_{1,2}(E_1)) = D$.

(2) $\nu_{1,j}(E_2) = \nu_{1,0}(E_2)$ *for* $j \in \{1,2,3\}$ *with* $|\mathbb{N}_N(\nu_{1,0}(E_2)) : D| = 7$ *and* $\mathbb{N}_N(\nu_{1,0}(E_2)) = \langle a,b,d^3\rangle \cong 2^4 : 2^3 : L_3(2)$.

Proof. The claims can be verified by easy computations using the known presentation for N. □

In view of this lemma and the preceding discussion the next result is immediate.

COROLLARY 5.11. *Assume Hypothesis 5.9 and put* $H^* := \mathbb{N}_G(E_2)$ *as well as* $D^* := \mathbb{N}_N(E_2) = H^* \cap N$. *Then the following hold:*

(1) $G = \langle H,N\rangle = \langle H^*,N\rangle$ *where* $H^* = \langle H,D^*\rangle \cong 2^4 : GL(4,2)$ *and* $D^* \cong 2^4 : 2^3 : L(3,2)$ *with* $\mathbb{Z}(D^*) = 1$, *i.e.,* D^*/E_2 *is a maximal parabolic of* $\mathrm{Aut}(E_2) \cong GL(4,2)$ *stabilizing a hyperplane of* E_2.

(2) *Without loss of generality the group* H^* *can be identified with the maximal parabolic subgroup of* $GL(5,2)$ *generated by* H_0 *and the element*

$$e := d^3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence $H^* = \langle a,b,c,e\rangle$ *and* $D^* = \langle D,e\rangle = \langle a,b,e\rangle$.

6. Play it again

In view of the discussion in the preceding section we have to subject the triple (H^*, D^*, N_1) to the same procedure as we did with the triples (H, D, N_i), $i \in \{1,2\}$. So the first aim is to find representatives of the different isomorphism types of amalgams associated with (H^*, D^*, N_1).

LEMMA 6.1. *Define a set* $\mathcal{R}(D^*)$ *of relations as follows:*

$a^8,\ b^{12},\ a^4 \cdot b^6,\ (a \cdot b)^4,\ (a \cdot b^{-1})^4,\ (a \cdot b^{-2} \cdot a \cdot b)^2,\ (a \cdot b^2 \cdot a^{-2} \cdot b)^2,$

$$a^2 \cdot b^3 \cdot a^{-2} \cdot b \cdot a^{-1} \cdot b^{-3} \cdot a \cdot b^{-1},\ a^2 \cdot b \cdot a^2 \cdot b^{-1} \cdot a^{-2} \cdot b \cdot a^{-2} \cdot b^{-1}.$$
$$e^7,\ (a^{-1} \cdot b \cdot e^{-1} \cdot b^{-1})^2,\ (a \cdot e^{-1} \cdot a^{-1} \cdot b^{-1})^2,$$
$$e^2 \cdot a \cdot b^{-1} \cdot a \cdot e \cdot b^{-1} \cdot e,\ a^{-1} \cdot e \cdot b \cdot a^{-2} \cdot e \cdot b \cdot e^{-1} \cdot b.$$

Then the following hold:

(1) *$\mathcal{R}(D^*)$ is a set of defining relations for D^*.*
(2) *D^* has exactly two classes of triples satisfying the defining relations $\mathcal{R}(D^*)$; representatives of these classes are (a, b, e) and (a, b_1^7, e_1) with $e_1 := (e^2)^{b^2 e^4}$.*

Proof. Easy computational exercise. □

Lemma 6.2. *There are exactly two different isomorphism classes of amalgams of type (H^*, D^*, N_1); these classes are represented by $\mathcal{A}_j^* := (D^* \leq H^*, \nu_j : D^* \to N_1)$ with $j \in \{0, 1\}$, where the embeddings ν_j of D^* into N_1 are defined as follows:*

$$\nu_0 : (a, b, e) \to (a, b, e), \quad \nu_1 : (a, b, e) \to (a, b_1^7, e_1).$$

Proof. First of all note that $\mathrm{Aut}(H^*) \cong H^*$ and $\mathbb{N}_{H^*}(D^*) = D^*$. Moreover, $\nu_j(D^*) = \langle a, b, e\rangle$ is a maximal, nonnormal subgroup of index 3 in N_1, and thus is self-normalizing in N_1. An easy application of Theorem 2.1 together with the results in Lemma 4.3 and Lemma 6.1 now yields the claims. □

Lemma 6.3. *The irreducible complex characters of the groups D^* and H^* are as given in the tables (V) and (VI) in the appendix.*

Proof. Similar as for Lemma 5.1. □

Remark 6.4. The remarks made in 5.2 apply equally well to the groups D^* and H^*.

Lemma 6.5. *Let F_0^* denote the fusion pattern of D^* in H^*, where the numbering of classes refers to the character tables mentioned in 6.3. Similarly, let $F_{1,j}^*$ denote the fusion pattern of $\nu_j(D^*)$ in N_1 for $j \in \{0, 1\}$. Then the following hold:*

$$\begin{aligned} F_0^* = [&1, 2, 2, 3, 4, 3, 4, 5, 4, 7, 8, 9, 8, 8, 11, 9, 10, 10,\\ &12, 11, 12, 12, 13, 15, 15, 16, 18, 19, 20, 21, 22, 23].\\ F_{1,0}^* = [&1, 2, 3, 2, 4, 5, 6, 6, 7, 9, 11, 12, 13, 14, 12, 16, 14,\\ &15, 17, 20, 18, 19, 18, 21, 23, 21, 25, 26, 27, 28, 31, 30].\\ F_{1,1}^* = [&1, 2, 3, 2, 4, 6, 5, 6, 7, 9, 11, 12, 13, 14, 14, 16, 12,\\ &18, 17, 19, 15, 20, 18, 21, 23, 21, 25, 26, 27, 28, 31, 30].\end{aligned}$$

Proof. Again, straightforward calculations yield the claims. □

Now we are in a position to determine compatible pairs for $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$. We shall use the same convention as for Lemma 5.4, only with D and H replaced by D^* and H^*, respectively.

LEMMA 6.6. *The following hold:*

(1) $\mathcal{A}_0^*$ *has the following compatible pairs* (χ, ψ) *of degree* 29*:*
   (a) $(\chi_3+\chi_4, \psi_1+\psi_2+\psi_8+\psi_{20})$ *with* $\chi_{|D^*} = \delta_1+\delta_2+\delta_8+\delta_{12}+\delta_{20}$*;*
   (b) $(\chi_3+\chi_4, \psi_3+\psi_8+\psi_{20})$ *with* $\chi_{|D^*} = \delta_1+\delta_2+\delta_8+\delta_{12}+\delta_{20}$.
(2) $\mathcal{A}_1^*$ *has the following compatible pairs* (χ, ψ)*:*
   (a) $(\chi_2+\chi_4, \psi_2+\psi_{22})$ *of degree* 22 *with* $\chi_{|D^*} = \delta_2+\delta_{14}+\delta_{20}$*;*
   (b) $(\chi_1+\chi_2+\chi_4, \psi_3+\psi_{22})$ *of degree* 23 *with* $\chi_{|D^*} = \delta_1+\delta_2+\delta_{14}+\delta_{20}$.

Proof. By computational means, similar as for Lemma 5.4. □

Finally we are ready to investigate the finite completions of $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ arising from the compatible pairs in the preceding lemma.

THEOREM 6.7. *The following hold:*

(1) *The compatible pair* $(\chi_3+\chi_4, \psi_1+\psi_2+\psi_8+\psi_{20})$ *of degree* 29 *of* $\mathcal{A}_0^*$ *gives rise to just one amalgam inside* $GL(29,31)$ *and this generates a subgroup isomorphic to* Alt_{30}.
(2) *The compatible pair* $(\chi_3+\chi_4, \psi_3+\psi_8+\psi_{20})$ *of degree* 29 *of* $\mathcal{A}_0^*$ *gives rise to* 30 *different amalgams inside* $GL(29,31)$*;* 28 *of these amalgams generate* $SL(29,31)$*, one generates the alternating group* Alt_{31} *and one generates a simple subgroup* S_0^* *of order* $2^{10}\cdot 3^2\cdot 5\cdot 7\cdot 31$ *isomorphic to* $GL(5,2)$.
(3) *The compatible pair* $(\chi_2+\chi_4, \psi_2+\psi_{22})$ *of degree* 22 *of* $\mathcal{A}_1^*$ *gives rise to just one amalgam inside* $GL(22,31)$ *and this generates a subgroup isomorphic to* $O^-(22,31)$.
(4) *The compatible pair* $(\chi_1+\chi_2+\chi_4, \psi_3+\psi_{22})$ *of degree* 23 *of* $\mathcal{A}_1^*$ *gives rise to just one amalgam inside* $GL(23,31)$ *and this generates a simple group* S_1^* *isomorphic to* M_{24}.

Proof. By computational means, similar as for Theorem 5.7. □

REMARK 6.8. Occasionally—maybe for computational reasons—there is a need to present a matrix group as a group of permutations. In case this should happen for $X \in \{S_{1,3}, S_{2,0}, S_{2,3}, S_0^*, S_1^*\}$ we give generators for a maximal subgroup U of smallest index so that X can be represented via the action on the cosets of U.

Recall that $H = \langle a,b,c\rangle \cong 2^{1+6} : L(3,2)$, $N_1 = \langle a,b,d\rangle \cong 2^6 : (\mathrm{Sym}_3 \times L(3,2))$, $N_2 = \langle a,b,f\rangle \cong 2^6 : \hat{3}\mathrm{Sym}_6$ and $H^* = \langle a,b,c,e\rangle \cong 2^4 : GL(4,2)$. The appropriate maximal subgroup U of X can then be generated as follows:

(1) In case (2) of Theorem 5.7 with $S_{1,3} = \langle a, b, c, d\rangle \cong M_{24}$ we have $U := \langle ab, cd\rangle \cong M_{23}$.
(2) In case (3) of Theorem 5.7 with $S_{2,0} = \langle a, b, c, f\rangle \cong M_{24}$ we have $U := \langle ab^2, c^8 f\rangle \cong M_{23}$.
(3) In case (4) of Theorem 5.7 with $S_{2,3} = \langle a, b, c, f\rangle \cong He$ we have $U := \langle (a^2b^2)^3(a^2b^3)^2(ab)^2(ba)^2, cf^4\rangle \cong Sp(4, 4) : 2$.
(4) In case (2) of Theorem 6.7 with $S_0^* = \langle a, b, c, e, d\rangle = \langle a, b, c, d\rangle \cong GL(5, 2)$ we have $U := H^*$.
(5) In case (4) of Theorem 6.7 with $S_1^* = \langle a, b, c, e, d\rangle = \langle a, b, c, d\rangle \cong M_{24}$ we have $U := \langle ab, cd\rangle \cong M_{23}$.

Using these facts it is now an easy exercise to work out the character tables for the groups $GL(5, 2), M_{24}$ and He by means of standard procedures available in GAP [11] or MAGMA [2] and thereby verify the tables given in [3].

The interested reader can find generating matrices for the groups M_{24} and He obtained by means of the procedure discussed in this paper at the addresses `http://www.exp-math.uni-essen.de/~lempken/M24mats` and `http://www.exp-math.uni-essen.de/~lempken/HEmats`, respectively.

We conclude this paper with a few observations.

Remarks 6.9.

(1) In Theorems 5.7 and 6.7 we have encountered all the groups to be expected in view of Theorem 1.2. So there is no need to investigate further compatible pairs and associated finite completions for the amalgams $\mathcal{A}_{i,j}$ and $\mathcal{A}_i^*$.

(2) We have seen that the nontrivial compatible pair of smallest degree does not necessarily give rise to at least one of the groups implicitly given by the initial prescription of the involution centralizer. So in general one should have a fairly good idea in which dimensions to hunt for the groups in question.

(3) We have seen that it pays to switch from a given triple (H, D, N) to a 'larger' triple (H^*, D^*, N) in order to reduce the number of isomorphism types of amalgams as well as the number of isomorphism types of associated finite completions to be considered. This appears to be a natural process, in particular, if H^* can be chosen to be another (maybe maximal) 2-local containing H properly.

Clearly, a similar effect may be achieved if the group N is replaced by some suitable overgroup N^*.

	1a	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	2k	2l	2m	2n	2o	2p	3a	4a	4b	4c	4d	4e
2	10	10	9	9	9	7	8	8	8	8	7	7	7	7	7	6	6	3	7	7	7	7	6
3	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	−1	1	1	1	1	1	1	1	−1	1	1	−1	1	−1	1	−1	1	−1
χ_3	1	1	1	1	1	−1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	−1	1	−1	1	−1
χ_4	1	1	1	1	1	1	1	1	1	1	1	1	−1	1	−1	−1	−1	1	1	1	1	1	1
χ_5	2	2	2	2	2	−2	2	2	2	2	2	2	.	−2	.	.	.	−1	−2	2	−2	2	−2
χ_6	2	2	2	2	2	2	2	2	2	2	2	2	.	2	.	.	.	−1	2	2	2	2	2
χ_7	3	3	3	3	3	3	3	−1	3	−1	−1	−1	1	3	1	1	1	.	3	−1	3	3	−1
χ_8	3	3	3	3	3	−3	3	−1	3	−1	−1	−1	1	−3	1	1	−1	.	−3	−1	−3	3	1
χ_9	3	3	3	3	3	−3	3	−1	3	−1	−1	−1	−1	−3	−1	−1	1	.	−3	−1	−3	3	1
χ_{10}	3	3	3	3	3	3	3	−1	3	−1	−1	−1	−1	3	−1	−1	−1	.	3	−1	3	3	−1
χ_{11}	3	3	3	3	3	−3	−1	−1	−1	−1	−1	3	−1	1	−1	−1	1	.	−3	3	1	−1	−3
χ_{12}	3	3	3	3	3	−3	−1	−1	−1	−1	−1	3	1	1	1	1	−1	.	−3	3	1	−1	−3
χ_{13}	3	3	3	3	3	−3	−1	3	−1	3	3	−1	−1	1	−1	−1	1	.	−3	−1	1	−1	1
χ_{14}	3	3	3	3	3	−3	−1	3	−1	3	3	−1	1	1	1	1	−1	.	−3	−1	1	−1	1
χ_{15}	3	3	3	3	3	3	−1	−1	−1	−1	−1	3	−1	−1	−1	−1	−1	.	3	3	−1	−1	3
χ_{16}	3	3	3	3	3	3	−1	−1	−1	−1	−1	3	1	−1	1	1	1	.	3	3	−1	−1	3
χ_{17}	3	3	3	3	3	3	−1	3	−1	3	3	−1	−1	−1	−1	−1	−1	.	3	−1	−1	−1	−1
χ_{18}	3	3	3	3	3	3	−1	3	−1	3	3	−1	1	−1	1	1	1	.	3	−1	−1	−1	−1
χ_{19}	4	4	−4	4	−4	−4	.	.	.	.	.	.	−2	.	−2	2	2	1	4	.	.	.	.
χ_{20}	4	4	−4	4	−4	−4	.	.	.	.	.	.	2	.	2	−2	−2	1	4	.	.	.	.
χ_{21}	4	4	−4	4	−4	4	.	.	.	.	.	.	−2	.	−2	2	−2	1	−4	.	.	.	.
χ_{22}	4	4	−4	4	−4	4	.	.	.	.	.	.	2	.	2	−2	2	1	−4	.	.	.	.
χ_{23}	6	6	6	−2	−2	.	−2	2	6	2	−2	2	−2	.	−2	−2	.	.	.	−2	.	−2	.
χ_{24}	6	6	6	−2	−2	.	−2	2	6	2	−2	2	2	.	2	2	.	.	.	−2	.	−2	.
χ_{25}	6	6	6	−2	−2	.	6	2	−2	2	−2	2	−2	.	−2	−2	.	.	.	−2	.	−2	.
χ_{26}	6	6	6	−2	−2	.	6	2	−2	2	−2	2	2	.	2	2	.	.	.	−2	.	−2	.
χ_{27}	6	6	6	6	6	−6	−2	−2	−2	−2	−2	−2	.	2	.	.	.	.	−6	−2	2	−2	2
χ_{28}	6	6	6	6	6	6	−2	−2	−2	−2	−2	−2	.	−2	.	.	.	.	6	−2	−2	−2	−2
χ_{29}	6	6	6	−2	−2	.	6	−2	−2	−2	2	−2	.	.	.	.	−2	.	.	2	.	−2	.
χ_{30}	6	6	6	−2	−2	.	6	−2	−2	−2	2	−2	.	.	.	.	2	.	.	2	.	−2	.
χ_{31}	6	6	6	−2	−2	.	−2	−2	6	−2	2	−2	.	.	.	.	−2	.	.	2	.	−2	.
χ_{32}	6	6	6	−2	−2	.	−2	−2	6	−2	2	−2	.	.	.	.	2	.	.	2	.	−2	.
χ_{33}	8	8	−8	8	−8	−8	.	.	.	.	.	.	.	.	.	.	.	−1	8	.	.	.	.
χ_{34}	8	8	−8	8	−8	8	.	.	.	.	.	.	.	.	.	.	.	−1	−8	.	.	.	.
χ_{35}	8	−8	.	.	.	.	.	4	.	−4	.	.	−4	.	4	.	.	2	.	.	.	.	.
χ_{36}	8	−8	.	.	.	.	.	4	.	−4	.	.	4	.	−4	.	.	2	.	.	.	.	.
χ_{37}	12	12	−12	−4	4	.	.	.	.	.	.	.	−2	−4	−2	2	2	.	.	.	4	.	.
χ_{38}	12	12	−12	−4	4	.	.	.	.	.	.	.	−2	4	−2	2	−2	.	.	.	−4	.	.
χ_{39}	12	12	−12	−4	4	.	.	.	.	.	.	.	2	−4	2	−2	−2	.	.	.	4	.	.
χ_{40}	12	12	−12	−4	4	.	.	.	.	.	.	.	2	4	2	−2	2	.	.	.	−4	.	.
χ_{41}	12	12	12	−4	−4	.	−4	4	−4	4	−4	−4	.	.	.	.	.	.	.	4	.	4	.
χ_{42}	12	12	12	−4	−4	.	−4	−4	−4	−4	4	4	.	.	.	.	.	.	.	−4	.	4	.
χ_{43}	16	−16	.	.	.	.	.	8	.	−8	.	.	.	.	.	.	.	−2	.	.	.	.	.
χ_{44}	24	−24	.	.	.	.	.	−4	.	4	.	.	4	.	−4	.	.	.	.	.	.	.	.
χ_{45}	24	−24	.	.	.	.	.	−4	.	4	.	.	−4	.	4	.	.	.	.	.	.	.	.

TABLE I. Character Table of D

2	6	6	6	6	6	6	6	6	5	5	5	5	5	5	5	4	4	3	2	2	4	2
3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	1
	$4f$	$4g$	$4h$	$4i$	$4j$	$4k$	$4l$	$4m$	$4n$	$4o$	$4p$	$4q$	$4r$	$4s$	$4t$	$4u$	$4v$	$6a$	$6b$	$6c$	$8a$	$12a$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	−1	1	1	−1	1	−1	−1
χ_3	1	−1	−1	1	1	1	−1	1	−1	−1	1	−1	−1	1	−1	1	−1	1	−1	1	1	−1
χ_4	1	−1	−1	1	−1	−1	1	−1	−1	−1	−1	−1	−1	−1	1	−1	−1	1	1	1	−1	1
χ_5	2	.	.	2	.	.	−2	.	.	.	.	.	.	.	−2	.	.	−1	1	−1	.	1
χ_6	2	.	.	2	.	.	2	.	.	.	.	.	.	.	2	.	.	−1	−1	−1	.	−1
χ_7	−1	1	1	−1	1	1	−1	1	1	−1	1	−1	1	1	−1	−1	−1	.	.	.	−1	.
χ_8	−1	1	1	−1	−1	−1	1	−1	1	−1	−1	−1	1	−1	1	1	−1	.	.	.	1	.
χ_9	−1	−1	−1	−1	1	1	1	1	−1	1	1	1	−1	1	1	−1	1	.	.	.	−1	.
χ_{10}	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	−1	1	−1	−1	−1	1	1	.	.	.	1	.
χ_{11}	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1	−1	1	−1	−1	.	.	.	1	.
χ_{12}	−1	1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	1	1	1	1	.	.	.	−1	.
χ_{13}	−1	−1	−1	−1	1	1	−3	1	1	−1	−1	−1	1	−1	1	1	1	.	.	.	−1	.
χ_{14}	−1	1	1	−1	−1	−1	−3	−1	−1	1	1	1	−1	1	1	−1	−1	.	.	.	1	.
χ_{15}	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	−1	.	.	.	−1	.
χ_{16}	−1	1	1	−1	1	1	−1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	.	.	.	1	.
χ_{17}	−1	−1	−1	−1	−1	−1	3	−1	1	−1	1	−1	1	1	−1	−1	1	.	.	.	1	.
χ_{18}	−1	1	1	−1	1	1	3	1	−1	1	−1	1	−1	−1	−1	1	−1	.	.	.	−1	.
χ_{19}	.	2	−2	.	2	−2	.	−2	.	.	.	.	.	.	.	.	.	1	−1	−1	.	1
χ_{20}	.	−2	2	.	−2	2	.	2	.	.	.	.	.	.	.	.	.	1	−1	−1	.	1
χ_{21}	.	2	−2	.	−2	2	.	2	.	.	.	.	.	.	.	.	.	1	1	−1	.	−1
χ_{22}	.	−2	2	.	2	−2	.	−2	.	.	.	.	.	.	.	.	.	1	1	−1	.	−1
χ_{23}	2	2	2	−2	.	.	.	.	2	.	.	.	−2	.	.	.	.	.	.	.	.	.
χ_{24}	2	−2	−2	−2	.	.	.	.	−2	.	.	.	2	.	.	.	.	.	.	.	.	.
χ_{25}	−2	2	2	2	.	.	.	.	−2	.	.	.	2	.	.	.	.	.	.	.	.	.
χ_{26}	−2	−2	−2	2	.	.	.	.	2	.	.	.	−2	.	.	.	.	.	.	.	.	.
χ_{27}	2	.	.	2	.	.	2	.	.	.	.	.	.	.	−2	.	.	.	.	.	.	.
χ_{28}	2	.	.	2	.	.	−2	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.
χ_{29}	2	.	.	−2	2	2	.	−2	.	.	2	.	.	−2	.	.	.	.	.	.	.	.
χ_{30}	2	.	.	−2	−2	−2	.	2	.	.	−2	.	.	2	.	.	.	.	.	.	.	.
χ_{31}	−2	.	.	2	2	2	.	−2	.	.	−2	.	.	2	.	.	.	.	.	.	.	.
χ_{32}	−2	.	.	2	−2	−2	.	2	.	.	2	.	.	−2	.	.	.	.	.	.	.	.
χ_{33}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	1	1	.	−1
χ_{34}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	−1	1	.	1
χ_{35}	.	.	.	.	.	.	.	.	.	2	.	−2	.	.	.	.	.	−2	.	.	.	.
χ_{36}	.	.	.	.	.	.	.	.	.	−2	.	2	.	.	.	.	.	−2	.	.	.	.
χ_{37}	.	−2	2	.	−2	2	.	−2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{38}	.	−2	2	.	2	−2	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{39}	.	2	−2	.	2	−2	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{40}	.	2	−2	.	−2	2	.	−2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{41}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{42}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{43}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.
χ_{44}	.	.	.	.	.	.	.	.	.	2	.	−2	.	.	.	.	.	.	.	.	.	.
χ_{45}	.	.	.	.	.	.	.	.	.	−2	.	2	.	.	.	.	.	.	.	.	.	.

TABLE I. Character Table of D (continued)

2	10	10	9	9	9	8	8	7	7	3	7	7	6	6	6	6	5
3	1	1	1	1	.	.	.	.	.	1	1	.	.	.	.	.	.
7	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$2h$	$3a$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	3	3	3	−1	−1	−1	−1	.	3	−1	−1	−1	−1	−1	1
χ_3	3	3	3	3	3	−1	−1	−1	−1	.	3	−1	−1	−1	−1	−1	1
χ_4	6	6	6	6	6	2	2	2	2	.	6	2	2	2	2	2	.
χ_5	7	7	7	−1	−1	3	3	−1	3	1	−1	−1	3	−1	−1	−1	1
χ_6	7	7	−1	7	−1	3	3	3	−1	1	−1	−1	−1	3	−1	−1	1
χ_7	7	7	7	7	7	−1	−1	−1	−1	1	7	−1	−1	−1	−1	−1	−1
χ_8	7	7	−1	7	−1	−1	−1	−1	3	1	−1	3	−1	−1	3	−1	−1
χ_9	7	7	7	−1	−1	−1	−1	3	−1	1	−1	3	−1	−1	−1	3	−1
χ_{10}	8	8	8	8	8	.	.	.	.	−1	8	.	.	.	.	.	.
χ_{11}	8	−8	.	.	.	4	−4	.	.	2	.	.	.	.	.	.	−2
χ_{12}	14	14	14	−2	−2	2	2	2	2	−1	−2	2	2	−2	−2	2	.
χ_{13}	14	14	−2	14	−2	2	2	2	2	−1	−2	2	−2	2	2	−2	.
χ_{14}	21	21	−3	−3	5	5	5	1	1	.	−3	−3	−3	−3	1	1	1
χ_{15}	21	21	21	−3	−3	1	1	−3	1	.	−3	−3	1	1	1	−3	−1
χ_{16}	21	21	−3	21	−3	1	1	1	−3	.	−3	−3	1	1	−3	1	−1
χ_{17}	21	21	−3	−3	5	−3	−3	1	1	.	−3	5	1	1	−3	−3	1
χ_{18}	21	21	−3	−3	5	1	1	−3	5	.	−3	1	−3	1	−3	1	−1
χ_{19}	21	21	−3	−3	5	1	1	5	−3	.	−3	1	1	−3	1	−3	−1
χ_{20}	21	21	−3	21	−3	−3	−3	−3	1	.	−3	1	1	−3	1	1	1
χ_{21}	21	21	21	−3	−3	−3	−3	1	−3	.	−3	1	−3	1	1	1	1
χ_{22}	24	−24	.	.	.	−4	4	.	.	.	.	.	.	.	.	.	−2
χ_{23}	24	−24	.	.	.	−4	4	.	.	.	.	.	.	.	.	.	−2
χ_{24}	28	28	−4	−4	−4	4	4	−4	−4	1	4	4	.	.	.	.	.
χ_{25}	28	28	−4	−4	−4	−4	−4	4	4	1	4	−4	.	.	.	.	.
χ_{26}	42	42	−6	−6	10	−2	−2	−2	−2	.	−6	−2	2	2	2	2	.
χ_{27}	48	−48	.	.	.	8	−8	.	.	.	.	.	.	.	.	.	.
χ_{28}	56	56	−8	−8	−8	.	.	.	.	−1	8	.	.	.	.	.	.
χ_{29}	56	−56	.	.	.	−4	4	.	.	2	.	.	.	.	.	.	2
χ_{30}	64	−64	.	.	.	.	.	.	.	−2	.	.	.	.	.	.	.

TABLE II. Character Table of H. Here $A = -1 - b7$.

	5	5	4	4	3	2	2	1	1	4	2	1	1
	.	.	.	.	1	1	1	.	.	.	1	.	.
	.	.	.	.	.	.	.	1	1	.	.	1	1
	$4h$	$4i$	$4j$	$4k$	$6a$	$6b$	$6c$	$7a$	$7b$	$8a$	$12a$	$14a$	$14b$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	1	.	.	.	A	$/A$	1	.	A	$/A$
χ_3	1	−1	1	1	.	.	.	$/A$	A	1	.	$/A$	A
χ_4	.	2	.	.	.	.	.	−1	−1	.	.	−1	−1
χ_5	1	−1	−1	1	1	−1	1	.	.	−1	−1	.	.
χ_6	1	−1	1	−1	1	1	−1	.	.	−1	−1	.	.
χ_7	−1	−1	−1	−1	1	1	1	.	.	−1	1	.	.
χ_8	−1	−1	−1	1	1	1	−1	.	.	1	−1	.	.
χ_9	−1	−1	1	−1	1	−1	1	.	.	1	−1	.	.
χ_{10}	.	.	.	.	−1	−1	−1	1	1	.	−1	1	1
χ_{11}	2	.	.	.	−2	.	.	1	1	.	.	−1	−1
χ_{12}	.	−2	.	.	−1	1	−1	.	.	.	1	.	.
χ_{13}	.	−2	.	.	−1	−1	1	.	.	.	1	.	.
χ_{14}	1	1	−1	−1	.	.	.	.	.	1	.	.	.
χ_{15}	−1	1	1	−1	.	.	.	.	.	1	.	.	.
χ_{16}	−1	1	−1	1	.	.	.	.	.	1	.	.	.
χ_{17}	1	1	−1	−1	.	.	.	.	.	1	.	.	.
χ_{18}	−1	1	1	1	.	.	.	.	.	−1	.	.	.
χ_{19}	−1	1	1	1	.	.	.	.	.	−1	.	.	.
χ_{20}	1	1	1	−1	.	.	.	.	.	−1	.	.	.
χ_{21}	1	1	−1	1	.	.	.	.	.	−1	.	.	.
χ_{22}	2	.	.	.	.	.	.	A	$/A$	.	.	$-A$	$-/A$
χ_{23}	2	.	.	.	.	.	.	$/A$	A	.	.	$-/A$	$-A$
χ_{24}	.	.	.	.	1	−1	−1	.	.	.	1	.	.
χ_{25}	.	.	.	.	1	−1	−1	.	.	.	1	.	.
χ_{26}	.	−2	.	.	.	.	.	.	.	.	.	.	.
χ_{27}	.	.	.	.	.	.	.	−1	−1	.	.	1	1
χ_{28}	.	.	.	.	−1	1	1	.	.	.	−1	.	.
χ_{29}	−2	.	.	.	−2	.	.	.	.	.	.	.	.
χ_{30}	.	.	.	.	2	.	.	1	1	.	.	−1	−1

TABLE II. Character Table of H (continued). Here $A = -1 - b7$.

2	10	10	7	9	8	8	7	3	3	2	7	7	7	7	5	6
3	2	1	1	1	1	.	.	2	2	2	1	.	.	.	1	.
7	1	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.
	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$3a$	$3b$	$3c$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	1	1	-1	1	1	1	-1	1	-1	1	1	-1
χ_3	2	2	.	2	2	2	.	-1	2	-1	.	2	.	2	2	.
χ_4	3	3	-3	3	-1	-1	1	3	.	.	-3	-1	1	-1	1	1
χ_5	3	3	-3	3	-1	-1	1	3	.	.	-3	-1	1	-1	1	1
χ_6	3	3	3	3	-1	-1	-1	3	.	.	3	-1	-1	-1	1	-1
χ_7	3	3	3	3	-1	-1	-1	3	.	.	3	-1	-1	-1	1	-1
χ_8	6	6	6	6	2	2	2	6	.	.	6	2	2	2	.	2
χ_9	6	6	-6	6	2	2	-2	6	.	.	-6	2	-2	2	.	-2
χ_{10}	6	6	.	6	-2	-2	.	-3	.	.	.	-2	.	-2	2	.
χ_{11}	6	6	.	6	-2	-2	.	-3	.	.	.	-2	.	-2	2	.
χ_{12}	7	7	7	7	-1	-1	-1	7	1	1	7	-1	-1	-1	-1	-1
χ_{13}	7	7	-7	7	-1	-1	1	7	1	1	-7	-1	1	-1	-1	1
χ_{14}	8	8	8	8	.	.	.	8	-1	-1	8	.	.	.	.	.
χ_{15}	8	8	-8	8	.	.	.	8	-1	-1	-8	.	.	.	.	.
χ_{16}	12	12	.	12	4	4	.	-6	.	.	.	4	.	4	.	.
χ_{17}	14	14	.	14	-2	-2	.	-7	2	-1	.	-2	.	-2	-2	.
χ_{18}	16	16	.	16	.	.	.	-8	-2	1	.	.	.	.	.	.
χ_{19}	21	5	-7	-3	9	1	-3	.	3	.	1	-3	-3	1	3	1
χ_{20}	21	5	7	-3	9	1	3	.	3	.	-1	-3	3	1	3	-1
χ_{21}	21	5	-7	-3	-3	5	1	.	3	.	1	1	1	-3	-3	-3
χ_{22}	21	5	7	-3	-3	5	-1	.	3	.	-1	1	-1	-3	-3	3
χ_{23}	42	10	-14	-6	6	6	-2	.	-3	.	2	-2	-2	-2	.	-2
χ_{24}	42	10	14	-6	6	6	2	.	-3	.	-2	-2	2	-2	.	2
χ_{25}	42	-6	.	2	6	-2	-4	.	.	3	.	2	4	-2	.	.
χ_{26}	42	-6	.	2	6	-2	4	.	.	3	.	2	-4	-2	.	.
χ_{27}	63	15	-21	-9	3	-5	-1	.	.	.	3	-1	-1	3	-3	3
χ_{28}	63	15	21	-9	3	-5	1	.	.	.	-3	-1	1	3	-3	-3
χ_{29}	63	15	-21	-9	-9	-1	3	.	.	.	3	3	3	-1	3	-1
χ_{30}	63	15	21	-9	-9	-1	-3	.	.	.	-3	3	-3	-1	3	1
χ_{31}	84	-12	.	4	12	-4	.	.	.	-3	.	4	.	-4	.	.
χ_{32}	126	-18	.	6	-6	2	-4	.	.	.	.	-2	4	2	.	.
χ_{33}	126	-18	.	6	-6	2	4	.	.	.	.	-2	-4	2	.	.

TABLE III. Character Table of N_1. Here $A = -1 - b7$ and $B = -1 - i7$.

	5	5	5	5	3	3	2	2	1	1	4	2	2	1	1	.	.
	.	.	.	.	1	1	1	1	1	1	.	1	1	.	.	1	1
	.	.	.	.	.	.	.	.	1	1	.	.	.	1	1	1	1
	$4g$	$4h$	$4i$	$4j$	$6a$	$6b$	$6c$	$6d$	$7a$	$7b$	$8a$	$12a$	$12b$	$14a$	$14b$	$21a$	$21b$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	−1	1	−1	−1	1	1	−1	1	1	1	−1	−1	1	−1	−1	1	1
χ_3	.	2	.	.	2	−1	.	−1	2	2	.	.	−1	.	.	−1	−1
χ_4	1	1	−1	−1	.	−1	.	.	A	$/A$	−1	.	1	$-/A$	$-A$	$/A$	A
χ_5	1	1	−1	−1	.	−1	.	.	$/A$	A	−1	.	1	$-A$	$-/A$	A	$/A$
χ_6	−1	1	1	1	.	−1	.	.	A	$/A$	1	.	1	$/A$	A	$/A$	A
χ_7	−1	1	1	1	.	−1	.	.	$/A$	A	1	.	1	A	$/A$	A	$/A$
χ_8	2	.	.	.	.	2	.	.	−1	−1	.	.	.	−1	−1	−1	−1
χ_9	−2	.	.	.	.	2	.	.	−1	−1	.	.	.	1	1	−1	−1
χ_{10}	.	2	.	.	.	1	.	.	B	$/B$	.	.	−1	.	.	$-/A$	$-A$
χ_{11}	.	2	.	.	.	1	.	.	$/B$	B	.	.	−1	.	.	$-A$	$-/A$
χ_{12}	−1	−1	−1	−1	1	−1	1	1	.	.	−1	1	−1	.	.	.	.
χ_{13}	1	−1	1	1	1	−1	−1	1	.	.	1	−1	−1	.	.	.	.
χ_{14}	.	.	.	.	−1	.	−1	−1	1	1	.	−1	.	1	1	1	1
χ_{15}	.	.	.	.	−1	.	1	−1	1	1	.	1	.	−1	−1	1	1
χ_{16}	.	.	.	.	.	−2	.	.	−2	−2	.	.	.	.	.	1	1
χ_{17}	.	−2	.	.	2	1	.	−1	.	.	.	.	1	.	.	.	.
χ_{18}	.	.	.	.	−2	.	.	1	2	2	.	.	.	.	.	−1	−1
χ_{19}	1	−1	−1	−1	−1	.	−1	.	.	.	1	1	.	.	.	.	.
χ_{20}	−1	−1	1	1	−1	.	1	.	.	.	−1	−1	.	.	.	.	.
χ_{21}	1	1	1	1	−1	.	−1	.	.	.	−1	1	.	.	.	.	.
χ_{22}	−1	1	−1	−1	−1	.	1	.	.	.	1	−1	.	.	.	.	.
χ_{23}	2	.	.	.	1	.	1	.	.	.	.	−1	.	.	.	.	.
χ_{24}	−2	.	.	.	1	.	−1	.	.	.	.	1	.	.	.	.	.
χ_{25}	.	.	2	−2	.	.	.	−1	.	.	.	.	.	.	.	.	.
χ_{26}	.	.	−2	2	.	.	.	−1	.	.	.	.	.	.	.	.	.
χ_{27}	−1	1	1	1	.	.	.	.	.	.	−1	.	.	.	.	.	.
χ_{28}	1	1	−1	−1	.	.	.	.	.	.	1	.	.	.	.	.	.
χ_{29}	−1	−1	−1	−1	.	.	.	.	.	.	1	.	.	.	.	.	.
χ_{30}	1	−1	1	1	.	.	.	.	.	.	−1	.	.	.	.	.	.
χ_{31}	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.
χ_{32}	.	.	−2	2	.	.	.	.	.	.	.	.	.	.	.	.	.
χ_{33}	.	.	2	−2	.	.	.	.	.	.	.	.	.	.	.	.	.

TABLE III. Character Table of N_1 (continued). Here $A = -1 - b7$ and $B = -1 - i7$.

2	10	9	10	8	7	7	8	3	3	3	7	7	7	7	5	5
3	3	1	1	1	1	1	.	3	2	2	1	.	.	.	1	1
5	1	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.
	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$3a$	$3b$	$3c$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	−1	1	1	1	1	−1	1	1	−1	1	−1
χ_3	5	5	5	1	1	−3	1	5	2	−1	−3	1	1	1	−1	1
χ_4	5	5	5	1	−1	3	1	5	2	−1	3	1	1	−1	−1	−1
χ_5	5	5	5	1	−3	1	1	5	−1	2	1	1	1	−3	−1	−3
χ_6	5	5	5	1	3	−1	1	5	−1	2	−1	1	1	3	−1	3
χ_7	6	6	6	−2	.	.	−2	−3	.	.	.	−2	−2	.	2	.
χ_8	6	6	6	−2	.	.	−2	−3	.	.	.	−2	−2	.	2	.
χ_9	9	9	9	1	−3	−3	1	9	.	.	−3	1	1	−3	1	−3
χ_{10}	9	9	9	1	3	3	1	9	.	.	3	1	1	3	1	3
χ_{11}	10	10	10	−2	−2	2	−2	10	1	1	2	−2	−2	−2	.	−2
χ_{12}	10	10	10	−2	2	−2	−2	10	1	1	−2	−2	−2	2	.	2
χ_{13}	12	12	12	4	.	.	4	−6	.	.	.	4	4	.	.	.
χ_{14}	16	16	16	.	.	.	.	16	−2	−2	.	.	.	.	.	.
χ_{15}	18	−6	2	6	.	−4	−2	.	3	.	4	2	−2	.	.	.
χ_{16}	18	−6	2	6	.	4	−2	.	3	.	−4	2	−2	.	.	.
χ_{17}	18	18	18	2	.	.	2	−9	.	.	.	2	2	.	2	.
χ_{18}	30	30	30	−2	.	.	−2	−15	.	.	.	−2	−2	.	−2	.
χ_{19}	45	5	−3	9	7	3	1	.	.	3	3	−3	1	−1	3	−1
χ_{20}	45	5	−3	9	−7	−3	1	.	.	3	−3	−3	1	1	3	1
χ_{21}	45	5	−3	−3	−5	3	5	.	.	3	3	1	−3	3	−3	−1
χ_{22}	45	5	−3	−3	5	−3	5	.	.	3	−3	1	−3	−3	−3	1
χ_{23}	72	−24	8	.	.	8	.	.	3	.	−8	.	.	.	.	.
χ_{24}	72	−24	8	.	.	−8	.	.	3	.	8	.	.	.	.	.
χ_{25}	90	−30	10	6	.	4	−2	.	−3	.	−4	2	−2	.	.	.
χ_{26}	90	−30	10	6	.	−4	−2	.	−3	.	4	2	−2	.	.	.
χ_{27}	90	10	−6	6	2	6	6	.	.	−3	6	−2	−2	2	.	−2
χ_{28}	90	10	−6	6	−2	−6	6	.	.	−3	−6	−2	−2	−2	.	2
χ_{29}	108	−36	12	−12	.	.	4	.	.	.	.	−4	4	.	.	.
χ_{30}	135	15	−9	−9	−3	−3	−1	.	.	.	−3	3	−1	5	3	−3
χ_{31}	135	15	−9	−9	3	3	−1	.	.	.	3	3	−1	−5	3	3
χ_{32}	135	15	−9	3	−9	3	−5	.	.	.	3	−1	3	−1	−3	3
χ_{33}	135	15	−9	3	9	−3	−5	.	.	.	−3	−1	3	1	−3	−3

TABLE IV. Character Table of N_2. Here $A = -1 - b15$.

	6	5	5	5	2	3	3	3	2	2	4	2	2	2	2	.	.
	.	.	.	.	1	1	1	1	1	1	.	.	1	1	1	1	1
	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	1	1
	$4g$	$4h$	$4i$	$4j$	$5a$	$6a$	$6b$	$6c$	$6d$	$6e$	$8a$	$10a$	$12a$	$12b$	$12c$	$15a$	$15b$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	−1	−1	1	−1	1	1	1	1	−1	−1	−1	1	−1	−1	1	1	1
χ_3	−3	−1	−1	−1	.	2	1	−1	1	.	−1	.	.	1	−1	.	.
χ_4	3	1	−1	1	.	2	1	−1	−1	.	1	.	.	−1	−1	.	.
χ_5	1	−1	−1	−1	.	−1	1	2	.	1	−1	.	1	.	−1	.	.
χ_6	−1	1	−1	1	.	−1	1	2	.	−1	1	.	−1	.	−1	.	.
χ_7	.	.	2	.	1	.	1	.	.	.	.	1	.	.	−1	A	$/A$
χ_8	.	.	2	.	1	.	1	.	.	.	.	1	.	.	−1	$/A$	A
χ_9	−3	1	1	1	−1	.	1	.	.	.	1	−1	.	.	1	−1	−1
χ_{10}	3	−1	1	−1	−1	.	1	.	.	.	−1	−1	.	.	1	−1	−1
χ_{11}	2	.	.	.	.	1	−2	1	1	−1	.	.	−1	1	.	.	.
χ_{12}	−2	.	.	.	.	1	−2	1	−1	1	.	.	1	−1	.	.	.
χ_{13}	.	.	.	.	2	.	−2	.	.	.	.	2	.	.	.	−1	−1
χ_{14}	.	.	.	.	1	−2	.	−2	.	.	.	1	.	.	.	1	1
χ_{15}	.	−2	.	2	3	−1	.	.	.	−1	.	−1	1	.	.	.	.
χ_{16}	.	2	.	−2	3	−1	.	.	.	1	.	−1	−1	.	.	.	.
χ_{17}	.	.	2	.	−2	.	−1	.	.	.	.	−2	.	.	−1	1	1
χ_{18}	.	.	−2	.	.	.	1	.	.	.	.	.	.	.	1	.	.
χ_{19}	−1	1	−1	1	.	.	.	−1	1	.	−1	.	.	−1	.	.	.
χ_{20}	1	−1	−1	−1	.	.	.	−1	−1	.	1	.	.	1	.	.	.
χ_{21}	−1	−1	1	−1	.	.	.	−1	1	.	1	.	.	−1	.	.	.
χ_{22}	1	1	1	1	.	.	.	−1	−1	.	−1	.	.	1	.	.	.
χ_{23}	.	.	.	.	−3	−1	.	.	.	−1	.	1	1	.	.	.	.
χ_{24}	.	.	.	.	−3	−1	.	.	.	1	.	1	−1	.	.	.	.
χ_{25}	.	−2	.	2	.	1	.	.	.	1	.	.	−1	.	.	.	.
χ_{26}	.	2	.	−2	.	1	.	.	.	−1	.	.	1	.	.	.	.
χ_{27}	−2	.	.	.	.	.	.	1	−1	.	.	.	.	1	.	.	.
χ_{28}	2	.	.	.	.	.	.	1	1	.	.	.	.	−1	.	.	.
χ_{29}	.	.	.	.	3	.	.	.	.	.	.	−1	.	.	.	.	.
χ_{30}	1	1	−1	1	.	.	.	.	.	.	−1	.	.	.	.	.	.
χ_{31}	−1	−1	−1	−1	.	.	.	.	.	.	1	.	.	.	.	.	.
χ_{32}	−1	1	1	1	.	.	.	.	.	.	−1	.	.	.	.	.	.
χ_{33}	1	−1	1	−1	.	.	.	.	.	.	1	.	.	.	.	.	.

TABLE IV. Character Table of N_2 (continued). Here $A = -1 - b15$.

2	10	10	7	9	9	8	8	7	7	3	7	7	7	7	6	6
3	1	1	1	1	.	.	.	.	.	1	1	.	.	.	.	.
7	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$2h$	$3a$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	1	1	1	1	1	−1	1	−1	1	−1	1	1	−1
χ_3	3	3	−3	3	3	−1	−1	−1	1	.	−3	−1	1	−1	−1	1
χ_4	3	3	−3	3	3	−1	−1	−1	1	.	−3	−1	1	−1	−1	1
χ_5	3	3	3	3	3	−1	−1	−1	−1	.	3	−1	−1	−1	−1	−1
χ_6	3	3	3	3	3	−1	−1	−1	−1	.	3	−1	−1	−1	−1	−1
χ_7	6	6	−6	6	6	2	2	2	−2	.	−6	2	−2	2	2	−2
χ_8	6	6	6	6	6	2	2	2	2	.	6	2	2	2	2	2
χ_9	7	7	−7	7	7	−1	−1	−1	1	1	−7	−1	1	−1	−1	1
χ_{10}	7	7	7	7	7	−1	−1	−1	−1	1	7	−1	−1	−1	−1	−1
χ_{11}	7	7	−7	−1	−1	3	3	−1	−3	1	1	−1	−3	3	−1	1
χ_{12}	7	7	7	−1	−1	3	3	−1	3	1	−1	−1	3	3	−1	−1
χ_{13}	7	7	−7	−1	−1	−1	−1	3	1	1	1	3	1	−1	−1	−3
χ_{14}	7	7	7	−1	−1	−1	−1	3	−1	1	−1	3	−1	−1	−1	3
χ_{15}	8	8	−8	8	8	.	.	.	.	−1	−8	.	.	.	.	.
χ_{16}	8	8	8	8	8	.	.	.	.	−1	8	.	.	.	.	.
χ_{17}	14	14	−14	−2	−2	2	2	2	−2	−1	2	2	−2	2	−2	−2
χ_{18}	14	14	14	−2	−2	2	2	2	2	−1	−2	2	2	2	−2	2
χ_{19}	14	−2	.	6	−2	−2	6	2	.	2	.	−2	.	−2	2	.
χ_{20}	14	−2	.	6	−2	6	−2	2	.	2	.	−2	.	−2	−2	.
χ_{21}	21	21	−21	−3	−3	1	1	−3	−1	.	3	−3	−1	1	1	3
χ_{22}	21	21	21	−3	−3	1	1	−3	1	.	−3	−3	1	1	1	−3
χ_{23}	21	21	−21	−3	−3	−3	−3	1	3	.	3	1	3	−3	1	−1
χ_{24}	21	21	21	−3	−3	−3	−3	1	−3	.	−3	1	−3	−3	1	1
χ_{25}	28	−4	.	12	−4	4	4	4	.	−2	.	−4	.	−4	.	.
χ_{26}	42	−6	.	18	−6	−6	2	−2	.	.	.	2	.	2	2	.
χ_{27}	42	−6	.	18	−6	2	−6	−2	.	.	.	2	.	2	−2	.
χ_{28}	42	−6	.	−6	2	−2	6	−2	−4	.	.	2	4	−2	−2	.
χ_{29}	42	−6	.	−6	2	−2	6	−2	4	.	.	2	−4	−2	−2	.
χ_{30}	42	−6	.	−6	2	6	−2	−2	−4	.	.	2	4	−2	2	.
χ_{31}	42	−6	.	−6	2	6	−2	−2	4	.	.	2	−4	−2	2	.
χ_{32}	84	−12	.	−12	4	−4	−4	4	.	.	.	−4	.	4	.	.

TABLE V. Character Table of D^*. Here $A = -1 - b7$.

	6	5	5	5	5	5	4	3	2	2	1	1	4	2	1	1
	.	.	.	.	.	.	.	1	1	1	.	.	.	1	.	.
	.	.	.	.	.	.	.	.	.	.	1	1	.	.	1	1
	$4g$	$4h$	$4i$	$4j$	$4k$	$4l$	$4m$	$6a$	$6b$	$6c$	$7a$	$7b$	$8a$	$12a$	$14a$	$14b$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	1	-1	1	1	-1	1	1	1	-1	-1	-1	-1
χ_3	-1	1	1	-1	1	-1	1	.	.	.	A	$/A$	-1	.	$-A$	$-/A$
χ_4	-1	1	1	-1	1	-1	1	.	.	.	$/A$	A	-1	.	$-/A$	$-A$
χ_5	-1	1	-1	1	1	1	1	.	.	.	A	$/A$	1	.	A	$/A$
χ_6	-1	1	-1	1	1	1	1	.	.	.	$/A$	A	1	.	$/A$	A
χ_7	2	.	-2	.	.	.	.	.	.	.	-1	-1	.	.	1	1
χ_8	2	.	2	.	.	.	.	.	.	.	-1	-1	.	.	-1	-1
χ_9	-1	-1	1	1	-1	1	-1	1	-1	1	.	.	1	-1	.	.
χ_{10}	-1	-1	-1	-1	-1	-1	-1	1	1	1	.	.	-1	1	.	.
χ_{11}	-1	1	1	-1	1	-1	-1	1	-1	-1	.	.	1	1	.	.
χ_{12}	-1	1	-1	1	1	1	-1	1	1	-1	.	.	-1	-1	.	.
χ_{13}	-1	-1	1	1	-1	1	1	1	-1	-1	.	.	-1	1	.	.
χ_{14}	-1	-1	-1	-1	-1	-1	1	1	1	-1	.	.	1	-1	.	.
χ_{15}	.	.	.	.	.	.	.	-1	1	-1	1	1	.	1	-1	-1
χ_{16}	.	.	.	.	.	.	.	-1	-1	-1	1	1	.	-1	1	1
χ_{17}	-2	.	2	.	.	.	.	-1	1	1	.	.	.	-1	.	.
χ_{18}	-2	.	-2	.	.	.	.	-1	-1	1	.	.	.	1	.	.
χ_{19}	-2	-2	.	.	2	.	.	-2	.	.	.	.	.	.	.	.
χ_{20}	2	2	.	.	-2	.	.	-2	.	.	.	.	.	.	.	.
χ_{21}	1	-1	-1	1	-1	1	1	.	.	.	.	.	-1	.	.	.
χ_{22}	1	-1	1	-1	-1	-1	1	.	.	.	.	.	1	.	.	.
χ_{23}	1	1	-1	-1	1	-1	-1	.	.	.	.	.	1	.	.	.
χ_{24}	1	1	1	1	1	1	-1	.	.	.	.	.	-1	.	.	.
χ_{25}	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.
χ_{26}	-2	2	.	.	-2	.	.	.	.	.	.	.	.	.	.	.
χ_{27}	2	-2	.	.	2	.	.	.	.	.	.	.	.	.	.	.
χ_{28}	2	.	.	2	.	-2	.	.	.	.	.	.	.	.	.	.
χ_{29}	2	.	.	-2	.	2	.	.	.	.	.	.	.	.	.	.
χ_{30}	-2	.	.	-2	.	2	.	.	.	.	.	.	.	.	.	.
χ_{31}	-2	.	.	2	.	-2	.	.	.	.	.	.	.	.	.	.
χ_{32}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

TABLE V. Character Table of D^* (continued). Here $A = -1 - b7$.

2	10	10	9	9	7	2	3	7	7	6	6	5	4	.	3	2
3	2	1	1	.	1	2	2	1	.	.	.	.	.	1	1	1
5	1	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.
7	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	$1a$	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$5a$	$6a$	$6b$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	7	-1	-1	3	4	1	-1	3	-1	-1	-1	1	2	1	-1
χ_3	14	14	6	6	2	-1	2	6	2	2	2	2	.	-1	2	.
χ_4	15	-1	7	-1	3	.	3	-1	-1	3	-1	-1	1	.	-1	1
χ_5	20	20	4	4	4	5	-1	4	4	.	.	.	.	.	-1	1
χ_6	21	21	-3	-3	1	6	.	-3	1	1	1	1	-1	1	.	.
χ_7	21	21	-3	-3	1	-3	.	-3	1	1	1	1	-1	1	.	.
χ_8	21	21	-3	-3	1	-3	.	-3	1	1	1	1	-1	1	.	.
χ_9	28	28	-4	-4	4	1	1	-4	4	.	.	.	.	-2	1	-1
χ_{10}	35	35	3	3	-5	5	2	3	-5	-1	-1	-1	-1	.	2	.
χ_{11}	45	45	-3	-3	-3	.	.	-3	-3	1	1	1	1	.	.	.
χ_{12}	45	45	-3	-3	-3	.	.	-3	-3	1	1	1	1	.	.	.
χ_{13}	45	-3	-3	5	-3	.	.	-3	1	1	-3	1	1	.	.	.
χ_{14}	45	-3	-3	5	-3	.	.	-3	1	1	-3	1	1	.	.	.
χ_{15}	56	56	8	8	.	-4	-1	8	.	.	.	.	.	1	-1	-1
χ_{16}	64	64	.	.	.	4	-2	.	.	.	.	.	.	-1	-2	.
χ_{17}	70	70	-2	-2	2	-5	1	-2	2	-2	-2	-2	.	.	1	1
χ_{18}	90	-6	18	2	6	.	.	-6	-2	2	2	-2	.	.	.	.
χ_{19}	105	-7	17	-7	-3	.	3	1	1	1	-3	1	-1	.	-1	-1
χ_{20}	105	-7	1	9	-3	.	3	-7	1	-3	1	1	-1	.	-1	1
χ_{21}	105	-7	-7	1	9	.	3	1	-3	-3	1	1	1	.	-1	-1
χ_{22}	120	-8	8	8	.	.	-3	-8	.	.	.	.	.	.	1	-1
χ_{23}	210	-14	10	-6	6	.	-3	2	-2	-2	-2	2	.	.	1	1
χ_{24}	315	-21	3	-5	-9	.	.	3	3	-1	3	-1	1	.	.	.
χ_{25}	315	-21	-21	3	3	.	.	3	-1	3	-1	-1	-1	.	.	.

TABLE VI. Character Table of H^*. Here $A = -1 - b7$ and $B = -1 - b15$.

	2	1	1	4	2	1	1	.	.
	1	.	.	.	1	.	.	1	1
	.	.	.	.	.	.	.	1	1
	.	1	1	.	.	1	1	.	.
	$6c$	$7a$	$7b$	$8a$	$12a$	$14a$	$14b$	$15a$	$15b$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	.	.	.	1	-1	.	.	-1	-1
χ_3	-1	.	.	.	.	.	.	-1	-1
χ_4	.	1	1	-1	-1	-1	-1	.	.
χ_5	1	-1	-1	.	1	-1	-1	.	.
χ_6	-2	.	.	-1	.	.	.	1	1
χ_7	1	.	.	-1	.	.	.	B	$/B$
χ_8	1	.	.	-1	.	.	.	$/B$	B
χ_9	1	.	.	.	-1	.	.	1	1
χ_{10}	1	.	.	-1	.	.	.	.	.
χ_{11}	.	A	$/A$	1	.	A	$/A$	.	.
χ_{12}	.	$/A$	A	1	.	$/A$	A	.	.
χ_{13}	.	A	$/A$	-1	.	$-A$	$-/A$	.	.
χ_{14}	.	$/A$	A	-1	.	$-/A$	$-A$	.	.
χ_{15}	.	.	.	.	-1	.	.	1	1
χ_{16}	.	1	1	.	.	1	1	-1	-1
χ_{17}	-1	.	.	.	1	.	.	.	.
χ_{18}	.	-1	-1	.	.	1	1	.	.
χ_{19}	.	.	.	1	1	.	.	.	.
χ_{20}	.	.	.	1	-1	.	.	.	.
χ_{21}	.	.	.	-1	1	.	.	.	.
χ_{22}	.	1	1	.	1	-1	-1	.	.
χ_{23}	.	.	.	.	-1	.	.	.	.
χ_{24}	.	.	.	-1	.	.	.	.	.
χ_{25}	.	.	.	1	.	.	.	.	.

TABLE VI. Character Table of H^* (continued). Here $A = -1 - b7$ and $B = -1 - b15$

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