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ESTIMATES FOR THE SZEGÖ KERNEL ON A MODEL NON-PSEUDOCONVEX DOMAIN

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ABSTRACT. The Szegö kernel $S(z,\zeta)$ on the boundary of strictly pseudoconvex domains has been studied extensively. We can consider model domains $\Omega = \{(z_1,z_2) \in \mathbb{C}^2 \mid -\operatorname{Im} z_2 > b(\operatorname{Re} z_1)\}$. If b is convex, one has $|S(z,\zeta)| \leq c|B(z,\delta)|^{-1}$, where $B(z,\delta)$ is the nonisotropic ball with center z and radius δ, and δ is the nonisotropic distance from z to ζ. The only singularities are on the diagonal $z=\zeta$. In this paper, we obtain estimates for $|S|$ when the function b is a certain non-convex function. We show that near certain points, there are singularities off the diagonal.

Introduction

0.1. Background. Associated to any domain Ω in $\mathbb{C}^n$, there exist certain natural operators, the Bergman and Szegö projections. The Bergman projection, the easier one to define, is the projection operator from $L^2(\Omega)$ to $H^2(\Omega)$. The Szegö projection is the analog of the Bergman projection on the boundary $\partial\Omega$. We consider domains of the form $\Omega = \{x \in \mathbb{C}^n \mid \rho(x) < 0\}$, where the defining function $\rho \in C^\infty(\mathbb{C}^n)$ satisfies $\nabla\rho \neq 0$ when $\rho = 0$. An antiholomorphic vector field $L = \sum_{j=1}^n a_j \frac{\partial}{\partial \bar{z}_j}$ is tangential if $L\rho = 0$. Then

$$\begin{aligned} H^2(\partial\Omega) = \{f \in L^2(\partial\Omega) \mid Lf = 0 & \text{ as a distribution,} \\ & \text{for all tangential antiholomorphic vector fields L}\}. \end{aligned}$$

Note that by requiring the vector fields to be tangential, we force a linear relation on the coefficients a_j, and so locally we can find a basis consisting of $n-1$ vector fields, and we could simply require that $Lf = 0$ for these vector fields. The Szegö projection is then the orthogonal projection from $L^2(\partial\Omega)$ to the closed subspace $H^2(\partial\Omega)$.

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These operators can be written formally as integral operators:

$$Bf(z) = \int_\Omega K(z,\zeta)f(\zeta)dV(\zeta) \quad \text{and} \quad Sf(\zeta) = \int_{\partial\Omega} S(z,\zeta)f(\zeta)d\sigma(\zeta).$$

$K(z,\zeta)$ is the Bergman kernel and $S(z,\zeta)$ is the Szegö kernel. These kernels have been studied extensively for a variety of domains. If Ω is the unit ball, we have explicit formulas (see [5]). For more general domains, such explicit formulas do not often exist.

The Bergman and Szegö kernels on pseudoconvex domains have been studied extensively. J.J. Kohn introduced "Kohn's formula" for the Bergman kernel, $B = I - \bar\partial^* N \bar\partial$, where N is the $\bar\partial$-Neumann operator. This formula gives a connection between the $\bar\partial$-Neumann problem and the Bergman kernel B. This formula appears in [3] and [4]. N. Kerzman showed in [2] that on strictly pseudoconvex domains Ω with smooth boundary, the Bergman kernel extends smoothly to $\bar\Omega \times \bar\Omega - \{\text{the boundary diagonal}\}$. His work used the then-known results on the solution of the $\bar\partial$-Neumann problem. Using his techniques, the same result follows for certain classes of weakly pseudoconvex domains. Estimates for the Bergman and Szegö kernels in pseudoconvex domains of finite type in $\mathbb{C}^2$ were obtained by Nagel, Rosay, Stein, and Wainger in [12] and [13], and by J. McNeal in [6]. McNeal obtained estimates for the size of the Bergman and Szegö kernels in convex domains of finite type in $\mathbb{C}^n$ in [8] and [7]. McNeal and Stein obtained regularity theorems for the Bergman and Szegö projections on convex domains in [9] and [10].

We would like to understand Szegö kernel for non-pseudoconvex domains. However, the problem is in general complicated; this article gives estimates for a specific non-pseudoconvex domain.

0.2. The Szegö kernel as an integral. We consider certain model domains

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid -\operatorname{Im} z_2 > b(\operatorname{Re} z_1)\}, \tag{0.1}$$

where b is a polynomial of degree m. We put $z_1 = x + iy$ and $z_2 = t + i \operatorname{Im} z_2$, and on the boundary $\operatorname{Im} z_2 = -b(x)$. So we can identify $\partial\Omega$ with $\mathbb{C} \times \mathbb{R}$. Then

$$L = \frac{\partial}{\partial x} + i\left[\frac{\partial}{\partial y} + b'(x)\frac{\partial}{\partial t}\right] \tag{0.2}$$

is a vector field on the boundary which is identified with a global tangential antiholomorphic vector field in $\mathbb{C}^2$, and $H^2(\partial\Omega) = \{f \in L^2(\partial\Omega) \mid Lf = 0 \text{ as a distribution}\}$.

The pseudoconvexity of the model domain Ω is directly related to the function b. The domain is then pseudoconvex if $b''(x) \geq 0$, that is, if b is convex. If b is not convex, the domain is not pseudoconvex.

Formally, we can write the Szegö projection, which we now denote by P, defined on $f \in L^2(\mathbb{R}^3)$, as a singular integral operator:

$$Pf(x,y,t) = \iiint f(r,s,u)S((x,y,t);(r,s,u))drdsdu, \tag{0.3}$$

where $S((x,y,z);(r,s,u))$ is the Szegö kernel.

We look at the Szegö kernel $S((x,y,t);(r,s,u))$. If $b(x)$ grows quadratically at infinity, we can write it as

$$\begin{aligned} &S((x,y,t);(r,s,u)) \\ &\quad = \frac{1}{4\pi}\int_0^\infty e^{-\tau(b(x)+b(r)+i(t-u))}\int_{-\infty}^\infty \frac{e^{\eta(x+r)}e^{i\eta(y-s)}}{\int_{-\infty}^\infty e^{-2\tau(b(\lambda)-\frac{\eta}{\tau}\lambda)}d\lambda}d\eta\, d\tau, \end{aligned} \tag{0.4}$$

where the formula is understood as a principal value integral. This is shown in [11]. It is not clear at first where this is a smooth function. It can be shown (see [11]) that for convex functions b, formula (0.4) gives a function which is smooth off the diagonal $(x,y,t) = (r,s,u)$. We will see later that for the non-convex functions b we consider, singularities occur off the diagonal, where we might not necessarily expect them.

Before studying the nature of the singularities of the Szegö kernel S in the non-pseudoconvex situation, it is helpful to look at the case when b is a convex polynomial. In this case, an estimate for this kernel is stated in [11]. For standard Calderón-Zygmund singular integrals, we expect a size estimate like $|K(p,q)| \le c|B|^{-1}$, where B is the smallest (Euclidean) ball containing p and q. In the estimate for S in equation (0.4), the ball must be understood in terms of a nonisotropic metric, derived from appropriate vector fields.

0.2.1. *For reference: The convex case.* When b is a convex polynomial,

$$|S((x,y,t);(r,s,u))| \le c\frac{1}{|B((x,y,t),\delta)|}, \tag{0.5}$$

where B is a nonisotropic ball derived from the vector fields $X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y} + b'(x)$, and $T = b''(x)\frac{\partial}{\partial t}$, and δ is the nonisotropic distance from (x,y,t) to (r,s,u) (see [11]). The only possible singularities are on the diagonal $x = r, y = s, t = u$. For an explanation of nonisotropic balls and metrics, see [11] and [15].

It will be helpful later to compare our estimates to those for the convex functions $b(\lambda) = (\lambda-1)^2$ and $b(\lambda) = (\lambda+1)^2$. If $b(\lambda) = (\lambda \pm 1)^2$, then the domain Ω is convex and in fact biholomorphic to the Siegel upper half space, which is well understood. Since (0.4) is translation invariant in y and t, we assume $s = u = 0$. Then we can easily calculate that the singularities are when $(x-r) = t = y = 0$.

Indeed, one can calculate S exactly and take the norm:

$$|S((x,y,t);(r,0,0))| = c\frac{1}{((x-r)^2+y^2)^2+(2(t-y(x+r\pm 2)))^2}, \tag{0.6}$$

which agrees with (0.5).

0.3. Results. In this paper, I obtain results for S in (0.4) with a certain non-convex b, namely the function

$$b(x) = \begin{cases} (x-1)^2 & \text{if } x \geq 1/2, \\ -x^2+\frac{1}{2} & \text{if } -1/2 < x < 1/2, \\ (x+1)^2 & \text{if } x \leq -1/2. \end{cases} \tag{0.7}$$

0.3.1. *Position of singularities.* As $b(x)$ is convex in some places and concave in others, and S depends on $b(x)+b(r)$, the behavior of S varies depending on the values of x and r.

The main result concerns the points $(x,r) = (1,1)$ and $(-1,-1)$. It is here where the lack of convexity of b comes through. One can expect that for $|t|, |y| = 0$, S will be singular at $(x,r) = (1,1)$ and $(-1,-1)$, since this matches the result for the convex function. However, what is new here is that the singularity persists even if $|y| > 0$. At $(1,-1)$ and $(-1,1)$, an upper bound obtained indicates a similar situation, though the singularities are weaker.

For $x+r \geq c > 2$ and $x, r \geq 1/2$ (or $x+r \leq c < -2$, and $x, r \leq -1/2$), we have that $b(x), b(r), b(\frac{x+r}{2})$ are convex. Based on this, we expect S to behave like in the convex case. Indeed it does: if $t, y = 0$, the singularities are on the diagonal $x = r$. These singularities disappear if $t \neq 0$ or $y \neq 0$.

For $|x+r| \leq c < 2$, if we stay away from $(\pm 1, \pm 1)$, then S is bounded.

In the following subsection, we describe the singularities. We define $\epsilon = b(x)+b(r)$; this is in the important cases the square of the distance to the nearest singular point in $\{(\pm 1, \pm 1)\}$ and is always positive. For example, near $x = 1$ and $r = -1$, $\epsilon = (x-1)^2 + (r+1)^2$. We will use ϵ in the various cases, and its expression each time will be made clear.

0.3.2. *Description of some singularities.* We describe the situation near the singularity $x = r = 1$, where we define $\epsilon = (x-1)^2 + (r-1)^2$. There are two main cases: $\epsilon \leq 2|t|$ and $\epsilon \geq 2|t|$. In the first case, the methods used and the results are similar to those of the convex case. The other case is where things differ greatly. The methods applied to convex functions would not work here. The estimates are obtained using a contour integral. As long as $|t| = 0$, there is a singularity at $\epsilon = 0$ even if $|y| > 0$. The result is detailed in the following theorem.

First we define the terms D and D_e for $x, r \geq 1/2$:

$$D = \sqrt{2\epsilon} - (x+r-2) \qquad \text{and} \qquad D_e = \mathrm{Re}((2(\epsilon+it))^{1/2} - \sqrt{2\epsilon}.$$

These are positive quantities as will be shown later (see Claim 4.3). D is equivalent to the distance from (x, r) to the diagonal $x = r$ if $x + r \geq 2$ and to the point $(1, 1)$ if $x + r < 2$, and D_e is 0 if $t = 0$. We will also show in Claim 4.3 that $D, D_e \leq c\sqrt{\epsilon}$.

We explain the region in the (x, r)-plane where the theorem holds: we require $\epsilon \leq 1$ so we are in a circle around the point at $(1, 1)$, though we cannot have $(x, r) = (1, 1)$. We also require that (x, r) is either below the line $x + r = 2$, which goes through $(1, 1)$, or that it is off the diagonal $x = r$. This is because if we are above the line $x + r = 2$, the situation is similar to the convex case, so we must stay off the diagonal.

Define

$$|\operatorname{Im}_a| = \left|5t - y(x+r) + 4\operatorname{Im}((2(\epsilon + it))^{1/2}) + \frac{1}{2}\operatorname{Im}((2(\epsilon + it))^{3/2})\right|,$$

$$|\operatorname{Im}_b| = \left|t - y(x+r) + 2\operatorname{Im}((2(\epsilon + it))^{1/2})\right|.$$

Then we have:

THEOREM 0.1. *If $x, r \geq 1/2, 0 < \epsilon = (x-1)^2 + (r-1)^2, \epsilon, |t| \leq 1$, and $x + r - 2 < 0$ or $|x - r| > 0$, then $S = S_{M_1} + E_1 + E$, where if*

$$Q = \frac{\sqrt{\epsilon} + \sqrt{|t|} + y^2 + |\operatorname{Im}_a|}{(\epsilon + |t|)^{3/2}\left((y^2 + D + D_{\mathrm{e}}) + |\operatorname{Im}_b|\right)^2}, \tag{0.8}$$

then for some positive constant c,

$$\frac{1}{c}Q \leq |S_{M_1}| \leq cQ,$$

and for c_1 and c_2 positive constants,

$$|E_1| \leq c_1 \frac{1}{(\epsilon + |t|)^{3/2}} \qquad \text{and} \qquad |E| \leq c_2 < \infty.$$

The quantity Q is acquired by taking the norm of a fraction with complex expressions in the numerator and denominator. The expressions $|\operatorname{Im}_a|$ and $|\operatorname{Im}_b|$ are so named because they are the imaginary pieces. We will refer to these many times.

We understand Q in (0.8) by looking at different cases. One such case shows how our result differs from what might be expected. If $t = 0$ and $0 < \sqrt{\epsilon} \leq |y| \leq 1$, we can simplify Q as follows: If $t = 0$, then $\sqrt{|t|} = D_e = 0$ and $|\operatorname{Im}_a|, |\operatorname{Im}_b| = |y(x + r)|$, which is equivalent to $|y|$ since $x, r \geq 1/2$ and $\epsilon \leq 1$. Then $\sqrt{\epsilon} \leq |y| \leq 1$, so the numerator in Q is

$$\sqrt{\epsilon} + y^2 + |y(x+r)| \approx \sqrt{\epsilon} + y^2 + |y| \approx |y|,$$

and since $D \leq c\sqrt{\epsilon} \leq c|y|$ the denominator in Q is

$$\epsilon^{3/2}(y^2 + D + |y(x+r)|)^2 \approx \epsilon^{3/2}(y^2 + D + |y|)^2 \approx \epsilon^{3/2}|y|^2,$$

so we have

$$|S_{M_1}| \approx \frac{1}{\epsilon^{3/2}|y|}.$$

If $|y|$ is small enough, $|E_1|$ will not cancel this. So we see that S has singularities along the line $\epsilon = 0$, that is, even if $|y| > 0$, so there are singularities off the diagonal. This result is unlike those for standard singular integrals.

In order to better understand S, I combine Theorem 0.1 and other theorems I have obtained into the following estimate in Theorem 0.2. As it is only an upper estimate, it is weaker than Theorem 0.1; however, the estimate is simpler and closer to the kind of estimate one would expect.

Theorem 0.2. *If $x, r \geq 1/2$, $\epsilon = (x-1)^2 + (r-1)^2 \leq 1$, and either*

(1) *$\epsilon > 0$, $\{x + r < 2$ or $|x - r| > 0\}$, and $|y| \leq 1$, or*
(2) *$|t| \geq \epsilon/k$ for some positive constant k,*

then for some positive constants c (depending on k) and c_1,

$$|S((x,y,t);(r,0,0))| \leq c\frac{1}{\left[(\epsilon D^2)^{1/4} + \sqrt{\sqrt{\epsilon^2+t^2}-\epsilon}\right]^3 |\mathrm{Im}_b|} + E,$$

with $|E| \leq c_1 < \infty$.

This estimate is similar to those for what are called product singular integrals. Simply put, product singular integrals satisfy estimates involving not one volume of one ball, but the product of the volumes of two (or more) balls as understood in two (or more) spaces. These estimates were discussed in [1], where the authors obtained results for product singular integrals with convolution kernels. In a recent paper [14], Nagel and Stein obtained results for non-convolution operators and nonisotropic balls where the operator satisfies some size and derivative estimates similar to those for standard Calderón-Zygmund operators.

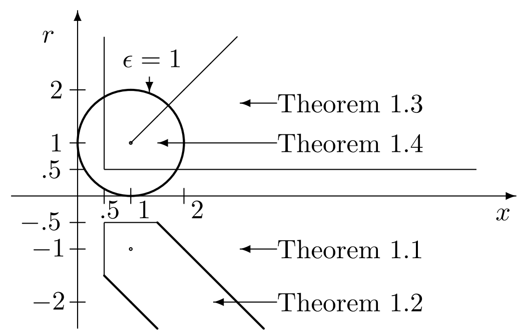
0.4. Notation. We use $a \lesssim b$ to mean that $a \leq cb$, for some positive constant c. We use $a \approx b$ to mean $1/ca \leq b \leq ca$ for some positive constant c, in which case we say that a is equivalent to b. We use E for a generic positive error which satisfies $E \leq$ constant. In certain cases, as will be pointed out, we will add to this E another bounded term, and call the result E. We use c for a generic constant which changes from line to line. Sometimes to make this difference clear we also use $c_1, c_2, \ldots$. We use $a \ll 1$ to mean a is less than the constant required in the argument. We continue to use the letter Q for relevant quantities in the various theorems, though its meaning changes with each use, and we redefine it every time. The same is true of ϵ. $|S|$ refers to $|S((x,y,t);(r,0,0))|$.

1. Statement of the main theorems

We obtain estimates for S in (0.4) with b in (0.7). The main theorems are stated here. Explanations of the proofs will be given in the following sections. As in the Introduction, we look at the behavior of S in the various regions of the (x, r)-plane.

Symmetry in x and r: The following theorems are stated for the right half of the xr-plane, that is, for $x \geq 0$. Since b is even, we can transfer the results to the other half. So all of our theorems hold for $(-x, -r)$ if, in the result, we replace (x, r) with $(-x, -r)$ and y with $-y$. From now on we assume $x \geq 0$.

The following theorems each concern a separate region. In the following diagram, the darker lines and the circle show the boundaries of the regions. If $x, r \geq 1/2$, then we define $\epsilon = (x-1)^2 + (r-1)^2$, and $\epsilon = 1$ is the circle shown. The lighter line on the diagonal $x = r, x + r \geq 2$, and the dot at $(1, -1)$ show the singular points.



Theorem 1.1 covers large areas where S is bounded: the areas outside of $\{x, r \geq 1/2\}$, which is in the first quadrant, and outside of the strip $\{|x+r| < 1, x \geq 1/2, r \leq -1/2\}$, which is in the fourth quadrant. Theorem 1.2 covers those points in the strip $\{|x+r| < 1, x \geq 1/2, r \leq -1/2\}$. Theorem 1.3 covers the area in the first quadrant where $x, r \geq 1/2$ and x and r are outside the circle $\epsilon = (x-1)^2 + (r-1)^2 = 1$. Theorem 1.4 and Corollary 1.5 cover the area in the first quadrant where $x, r \geq 1/2$ and x and r are inside the circle $\epsilon = (x-1)^2 + (r-1)^2 = 1$. This is the most important region as it contains the point $(1, 1)$.

Theorem 1.1 ((x, r) far away from singularities). *If $x \geq 0$ and $(x, r) \notin \{x, r \geq 1/2\} \cup \{|x + r| < 1, x \geq 1/2, r \leq -1/2\}$, then*

$$|S((x, y, t); (r, 0, 0))| \leq c < \infty.$$

This theorem specifies a region where $|S|$ is bounded. However, the region is actually larger than claimed in this theorem; $|S|$ is bounded as long as we stay away from the ray $|x-r|=0, x+r-2\geq 0$, and the point $(x,r)=(1,-1)$. We discuss the regions near these points in the following theorems.

THEOREM 1.2 (Near $(1,-1)$). *If $x\geq 1/2, r\leq -1/2$, and $|x+r|<1$, with $\epsilon=(x-1)^2+(r+1)^2$, then we have for some positive constant c*

$$|S|\leq c\frac{1}{(|t|+\epsilon)^{3/2}(|y|+1)}+|E|, \tag{1.1}$$

where $|E|\leq c_1<\infty$.

If $|y|\leq 1$ and $\epsilon\leq|t|$, at worst we have a singularity that looks like $1/|t|^{3/2}$. This is better than the $1/t^2$ that we have later (see (1.7) and the statement after it). This is why we say that the singularity is weaker here. Also note that $|t|$ and ϵ play the same role; we have

$$|S|\leq c\begin{cases}\frac{1}{|t|^{3/2}(|y|+1)} & \text{if } |t|\geq\epsilon,\\ \frac{1}{\epsilon^{3/2}(|y|+1)} & \text{if } \epsilon\geq|t|.\end{cases}$$

We will see this $|t|$ and ϵ playing the same role later also. Also, if either ϵ or $|t|$ are away from 0 in the region of this theorem, then $|S|$ is bounded. Specifically, if $|x+r|<1, x\geq 1/2, r\leq -1/2$, but we stay away from $(1,-1)$, then $|S|$ is bounded. This matches the result in Theorem 1.1 above.

THEOREM 1.3 ($|x|,|r|$ large and in convex part). *If $x,r\geq 1/2$ and $\epsilon=(x-1)^2+(r-1)^2\geq 1$, then for some positive constant c we have*

$$S=S_{M_1}+E,$$

where

$$|S_{M_1}|=c\frac{1}{((x-r)^2+y^2)^2+(2(t-y(x+r-2)))^2}, \tag{1.2}$$

with $|E|\leq c_1<\infty$.

This matches the result for the convex function $b(\lambda)=(\lambda-1)^2$ above in (0.6). This is expected, because in the region where $x,r\geq 1/2$ and $\epsilon\geq 1$, our b is convex and we are away from the transition point $(x,r)=(1,1)$.

For the following, we recall the definitions of the positive terms D and D_e for $x,r\geq 1/2$:

$$D=\sqrt{2\epsilon}-(x+r-2)\qquad\text{and}\qquad D_e=\mathrm{Re}((2(\epsilon+it))^{1/2})-\sqrt{2\epsilon}.$$

So

$$\begin{aligned}&\mathrm{Re}((2(\epsilon+it))^{1/2})-(x+r-2)\\ &\quad=\Big(\sqrt{2\epsilon}-(x+r-2)\Big)+\Big(\mathrm{Re}((2(\epsilon+it))^{1/2})-\sqrt{2\epsilon}\Big)\\ &\quad=D+D_e.\end{aligned}$$

Also recall

$$|\operatorname{Im}_a| = \left|5t - y(x+r) + 4\operatorname{Im}((2(\epsilon+it))^{1/2}) + \frac{1}{2}\operatorname{Im}((2(\epsilon+it))^{3/2})\right|,$$
$$|\operatorname{Im}_b| = \left|t - y(x+r) + 2\operatorname{Im}((2(\epsilon+it))^{1/2})\right|.$$

In the following theorem, we are in the region $x, r \geq 1/2$, inside the circle $\epsilon = (x-1)^2 + (r-1)^2 \leq 1$. Unless $t \geq \epsilon/k$ for some constant k as in part (c), we must also specify that (x, r) is either below the line $x + r < 2$, or off the diagonal $x = r$. So whenever we have the condition $\epsilon > 0$, we also have the condition $\{x + r < 2 \text{ or } |x - r| > 0\}$.

Theorem 1.4 (Near $(1,1)$). *For $x, r \geq 1/2$ and $\epsilon = (x-1)^2+(r-1)^2 \leq 1$:*

(a) *If $\epsilon > 0$, $\{x + r < 2$ or $|x - r| > 0\}$, and $\epsilon, |t| \leq 1$, then there is an equivalence*

$$S = S_{M_1} + E_1 + E,$$

where if

$$Q = \frac{\sqrt{\epsilon} + \sqrt{|t|} + y^2 + |\operatorname{Im}_a|}{(\epsilon+|t|)^{3/2}\left(y^2 + D + D_{\mathrm{e}} + |\operatorname{Im}_b|\right)^2}, \tag{1.3}$$

then for positive constants c_1, c_2, c_3, we have

$$\frac{1}{c_1} Q \leq |S_{M_1}| \leq c_1 Q$$

and

$$|E_1| \leq c_2 \frac{1}{(\epsilon+|t|)^{3/2}(y^2+1)} \qquad \text{and} \qquad |E| \leq c_3 < \infty.$$

(b) *Assume $\epsilon > 0$ and $\{x + r < 2$ or $|x - r| > 0\}$. There are positive constants $c_4 < 1, c_5, c_6$, such that if $\epsilon, |t|, |y| \leq c_4$, then*

$$\frac{1}{c_5}|S_{M_1}| \leq |S_{M_1} + E_1| \leq c_5 |S_{M_1}|.$$

If $\epsilon, |t| \leq 1$, then

$$|S_{M_1} + E_1| \leq c_6 |S_{M_1}|.$$

(c) *Assume $\epsilon \leq 1$. If either*
 (1) *$\epsilon > 0$, $\{x + r < 2$ or $|x - r| > 0\}$, and $|y| \leq 1$, or*
 (2) *$|t| \geq \epsilon/k$ for any positive constant k,*

then for positive constants c_7 (depending on k) and c_8, we have

$$|S| \leq c_7 \frac{\sqrt{\epsilon} + \sqrt{|t|} + |\operatorname{Im}_a|}{(\epsilon+|t|)^{3/2}(D + D_e + |\operatorname{Im}_b|)^2} + |E|,$$

with $|E| \leq c_8 < \infty$.

(d) *If* $\epsilon > 0$, $\{x + r < 2$ *or* $|x - r| > 0\}$, $\epsilon \le 1$, $|t| \le 1$ *and* $|y| > 1$, *then for positive constants* c_9, c_{10}, *we have*

$$|S| \le c_9 \frac{1}{(\epsilon + |t|)^{3/2} y^2} + |E|,$$

where $|E| \le c_{10} < \infty$.

Using estimates on the various terms involved (see Claim 4.3), we can then obtain the following:

1. By (a), if $t = 0$, $1 \ge |y| \ge \sqrt{\epsilon} > 0$, and $\{x + r < 2$ or $|x - r| > 0\}$ (so if $t = 0$, $|y|$ is dominant, and we are off the diagonal), then $|\operatorname{Im}_a| = |\operatorname{Im}_b| = |y(x + r)| \approx |y| \gtrsim y^2 + \sqrt{\epsilon}$ and $D_e = 0$, so

$$|S_{M_1}| \approx \frac{1}{\epsilon^{3/2}|y|}, \tag{1.4}$$

and so for $\epsilon, |y| \ll 1$, the error E_1 cannot cancel this, so we have singularities along the line $\epsilon = 0$, even if $|y| > 0$.

2. By (a), if $t = 0, 1 \ge \sqrt{\epsilon} \ge cD \ge |y|, \epsilon > 0$, and $\{x + r < 2$ or $|x - r| > 0\}$ (so if $t = 0$, ϵ is dominant, and we are off the diagonal), then $|\operatorname{Im}_a| = |\operatorname{Im}_b| = |y(x + r)| \approx |y|$. Also, $\sqrt{\epsilon} + y^2 + |y| \approx \sqrt{\epsilon}$ and $y^2 + D + D_e \approx D$, so

$$|S_{M_1}| \approx \frac{1}{\epsilon D^2} \approx \begin{cases} \frac{1}{(x-r)^4} & \text{if } x + r - 2 \ge 0, \\ \frac{1}{\epsilon^2} & \text{if } x + r - 2 < 0. \end{cases} \tag{1.5}$$

If $x + r - 2 \ge 0$, there are singularities all along the diagonal $x = r$. If $x + r - 2 < 0$, there is no singularity until one reaches $x = r = 1$. This matches the results of Theorems 1.1 and 1.3.

3. In fact, if $|t| \le 2\epsilon \le 1$, $\epsilon > 0$, $\{x + r < 2$ or $|x - r| > 0\}$, and $|y| \le 1$, we will show in the proof of part (b) (see (4.12)) that

$$|S_{M_1}| \approx \frac{\sqrt{\epsilon} + |y|}{\epsilon^{3/2}(D + \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} + |\operatorname{Im}_b|)^2}.$$

Then if $t = 0$, this can be reduced to (1.4) or (1.5) given the conditions for those equations.

4. Part (b) shows in fact that the error E_1 will not cancel the main term S_{M_1} as long as $\epsilon > 0$, $\{x + r < 2$ or $|x - r| > 0\}$, and $\epsilon, |t|, |y| \ll 1$. We prove this by showing that Q is large compared to E_1, so any singularities in Q are singularities of S. The second statement shows that if we just have $\epsilon, |t| \le 1$, we can still write

$$|S| = |S_{M_1} + E_1 + E| \le |S_{M_1} + E_1| + |E| \lesssim |S_{M_1}| + |E|.$$

5. By (c), if $\epsilon = 0$ and $|y(x + r)| > 30\sqrt{|t|}$, then we can show $|\operatorname{Im}_a| \approx |\operatorname{Im}_b| \approx |y(x + r)| \approx |y|$ (see the end of Section 4.1). So

$$|S| \lesssim \frac{1}{|t|^{3/2}|y|} + |E|. \tag{1.6}$$

6. By (c), if $\epsilon = y = 0$, then $D = 0$. Also, using $D_e \approx \operatorname{Im}((2(\epsilon + it))^{1/2}) \approx \sqrt{|t|}$ and $\operatorname{Im}((2(\epsilon+it))^{3/2}) \approx |t|^{3/2}$, we see $|\operatorname{Im}_a| \approx \sqrt{|t|} + |t|^{3/2}$ and $|\operatorname{Im}_b| \approx \sqrt{|t|} + |t|$. Splitting this up into the cases $|t| \geq 1$ and $|t| < 1$, we see in each case that

$$|S| \lesssim \frac{1}{t^2} + |E|. \tag{1.7}$$

7. In fact, if $|t| \geq \epsilon/k$ for any positive constant k, we will show in the proof of part (c) (see (4.18)) that

$$|S| \lesssim \frac{1}{|t|^{3/2}(\sqrt{|t|} + |\operatorname{Im}_b|)^2} + |E|.$$

This can be reduced to (1.6) if $\sqrt{|t|} \leq |y|$ or (1.7) if $\sqrt{|t|} \geq |y|$.

Corollary 1.5 (Near $(1,1)$). *If $x, r \geq 1/2$, $\epsilon = (x-1)^2 + (r-1)^2 \leq 1$, and either*

(1) *$\epsilon > 0$, $\{x + r < 2$ or $|x - r| > 0\}$, and $|y| \leq 1$, or*
(2) *$|t| \geq \epsilon/k$ for some positive constant k,*

then for positive constants c (depending on k) and c_1, we have

$$|S((x,y,t);(r,0,0))| \leq c\frac{1}{\left[(\epsilon D^2)^{1/4} + \sqrt{\sqrt{\epsilon^2 + t^2} - \epsilon}\right]^3 |\operatorname{Im}_b|} + |E|,$$

where $|E| \leq c_1 < \infty$.

This result will be shown to be a corollary of Theorem 1.4(c). Though it is weaker than those in Theorem 1.4, we include it because it is a unified estimate for all $\epsilon, |y| \leq 1$ and all t which is simpler than the previous one. It is closer to the kind of estimate we are ultimately looking for, that is, where the denominator looks like the volume of a ball or possibly the product of the volumes of two balls.

2. The integral in λ

Before we discuss the estimates for S in more detail, I want to state a lemma describing the main term. We can immediately make an observation about the integral in (0.4). The only place where S depends on the global behavior of b is in the λ-integral in the denominator. So if we compare our S to S when b is the convex function $b(\lambda) = (\lambda - 1)^2$, for example, and look at the case where $x, r \geq 1/2$, the only difference between the two is in the λ-integral. This integral was evaluated simply for $b(\lambda) = (\lambda-1)^2$ (see Section 0.2.1), but it is complicated in our case. In the following lemma, I analyze the λ-integral, and explain what I have found to be the main term of the integral S.

If we define

$$\mathcal{I}(\tau,\eta)=\int_{-\infty}^{\infty} e^{-2\tau(b(\lambda)-\frac{\eta}{\tau}\lambda)}\,d\lambda, \tag{2.1}$$

then we have:

Lemma 2.1.

$$\frac{1}{\mathcal{I}(\tau,\eta)}=Mc_0+E_0,$$

where

$$M=\frac{e^{-\frac{\eta^2}{2\tau}}\sqrt{2\tau}}{e^{2\eta}+e^{-2\eta}} \qquad \textit{and} \qquad c_0=\left(\int_{-\infty}^{\infty} e^{-\lambda^2}d\lambda\right)^{-1}=\frac{1}{\sqrt{\pi}} \tag{2.2}$$

and

$$\left|\int_0^{\infty} e^{-\tau(b(x)+b(r))}e^{-it\tau}\int_{-\infty}^{\infty} e^{\eta(x+r)}e^{i\eta y}E_0\,d\eta\,d\tau\right|\le c<\infty. \tag{2.3}$$

So then the main term of S is a constant times

$$S_M=\int_0^{\infty} e^{-\tau(b(x)+b(r)+it)}\int_{-\infty}^{\infty}\frac{e^{-\eta^2/2\tau}e^{\eta(x+r)}e^{i\eta y}}{e^{2\eta}+e^{-2\eta}}\,d\eta\sqrt{\tau}\,d\tau. \tag{2.4}$$

For all of the theorems, we use this lemma.

Proof. We will need the following facts, which can be proven by a standard argument: If $a\ge 1$, then
(2.5)

$$\frac{1}{4a}e^{-a^2}\le\int_a^{\infty} e^{-\lambda^2}d\lambda\le\frac{1}{2a}e^{-a^2} \quad \text{and} \quad \frac{1}{4a}e^{-a^2}\le\int_{-\infty}^{-a} e^{-\lambda^2}d\lambda\le\frac{1}{2a}e^{-a^2}.$$

Now we prove Lemma 2.1. First we break the integral $\mathcal{I}$ in (2.1) into three pieces, I, II and III:

$$\begin{aligned}
\mathrm{I}&=\int_{-\infty}^{-\frac{1}{2}} e^{-2\tau\left((\lambda+1)^2-\frac{\eta}{\tau}\lambda\right)}\,d\lambda=\frac{1}{\sqrt{2\tau}}\int_{-\infty}^{\frac{1}{2}\left(1-\frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{-\lambda^2}\,d\lambda\,e^{-2\eta}e^{\frac{\eta^2}{2\tau}},\\
\mathrm{III}&=\int_{\frac{1}{2}}^{\infty} e^{-2\tau\left((\lambda-1)^2-\frac{\eta}{\tau}\lambda\right)}\,d\lambda=\frac{1}{\sqrt{2\tau}}\int_{\frac{1}{2}\left(-1-\frac{\eta}{\tau}\right)\sqrt{2\tau}}^{\infty} e^{-\lambda^2}\,d\lambda\,e^{2\eta}e^{\frac{\eta^2}{2\tau}},\\
\mathrm{II}&=\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\tau\left(-\lambda^2+\frac{1}{2}-\frac{\eta}{\tau}\lambda\right)}\,d\lambda=\frac{1}{\sqrt{2\tau}}\int_{\frac{1}{2}\left(-1+\frac{\eta}{\tau}\right)\sqrt{2\tau}}^{\frac{1}{2}\left(1+\frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{\lambda^2}\,d\lambda\,e^{-\tau}e^{-\frac{\eta^2}{2\tau}}.
\end{aligned}$$

Write

(2.6)

$$S = \int_0^\infty e^{-\tau(b(x)+b(r)+it)} \int_{-\infty}^\infty e^{\eta(x+r)} e^{i\eta y} M c_0 \, d\eta \, d\tau \\ - \int_0^\infty e^{-\tau(b(x)+b(r)+it)} \int_{-\infty}^\infty e^{\eta(x+r)} e^{i\eta y} \underbrace{\left(M c_0 - \frac{1}{\mathrm{I} + \mathrm{II} + \mathrm{III}} \right)}_{E_0} \, d\eta \, d\tau$$

Now we will show that the second term is bounded in absolute value by a constant. This term is in absolute value $\leq$

(2.7)

$$\int_0^\infty e^{-\tau\epsilon} \int_{-\infty}^\infty e^{\eta(x+r)} \underbrace{\left| \left(\frac{e^{-\frac{\eta^2}{2\tau}} \sqrt{2\tau} c_0}{e^{2\eta} + e^{-2\eta}} \right) \left[\frac{\int_{-\infty}^\infty e^{-\lambda^2} \, d\lambda - \frac{\mathrm{I}_1 + \mathrm{II}_1 + \mathrm{III}_1}{e^{2\eta} + e^{-2\eta}}}{\frac{\mathrm{I}_1 + \mathrm{II}_1 + \mathrm{III}_1}{e^{2\eta} + e^{-2\eta}}} \right] \right|}_{|E_0|} d\eta d\tau,$$

where $\epsilon = b(x) + b(r)$ and

$$\mathrm{I}_1 = e^{-2\eta} \int_{-\infty}^{\frac{1}{2}\left(1 - \frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{-\lambda^2} \, d\lambda,$$

$$\mathrm{III}_1 = e^{2\eta} \int_{\frac{1}{2}\left(-1 - \frac{\eta}{\tau}\right)\sqrt{2\tau}}^\infty e^{-\lambda^2} \, d\lambda,$$

$$\mathrm{II}_1 = e^{-\frac{2\eta^2}{2\tau}} e^{-\tau} \int_{\frac{1}{2}\left(-1 + \frac{\eta}{\tau}\right)\sqrt{2\tau}}^{\frac{1}{2}\left(1 + \frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{\lambda^2} \, d\lambda.$$

We show that (2.7) is $\leq c$.

Step 1: Show that

$$\frac{\mathrm{I}_1 + \mathrm{II}_1 + \mathrm{III}_1}{e^{2\eta} + e^{-2\eta}} \approx c.$$

The idea is to consider the cases $\eta \geq 0$ and $\eta \leq 0$ separately. If $\eta \geq 0$, then

$$\frac{\mathrm{III}_1}{e^{-2\eta} + e^{2\eta}} \approx \frac{e^{2\eta}}{e^{2\eta}} \int_{\frac{1}{2}\left(-1 - \frac{\eta}{\tau}\right)\sqrt{2\tau}}^\infty e^{-\lambda^2} \, d\lambda \approx c,$$

since $-\infty < \frac{1}{2}\left(-1 - \frac{\eta}{\tau}\right)\sqrt{2\tau} \leq 0$. This is the dominant term. The others are positive and bounded by a constant. If $\eta \leq 0$, the situation is similar, but now the term with I_1 is dominant.

Now we have

$$|(2.7)| \lesssim \int_0^\infty \int_{-\infty}^\infty \frac{e^{-\tau\epsilon} e^{\eta(x+r)} e^{-\frac{\eta^2}{2\tau}} \sqrt{2\tau}}{e^{2\eta} + e^{-2\eta}} \left| \int_{-\infty}^\infty e^{-\lambda^2} d\lambda - \frac{\mathrm{I}_1 + \mathrm{II}_1 + \mathrm{III}_1}{e^{2\eta} + e^{-2\eta}} \right| d\eta d\tau.$$

Step 2: Show that

$$\frac{e^{-\tau\epsilon}e^{\eta(x+r)}e^{-\frac{\eta^2}{2\tau}}}{e^{2\eta}+e^{-2\eta}} \tag{2.8}$$

is bounded by a constant. Recall that we are assuming $x \geq 0$.

Case 1: $x, r \geq 1/2$. Here $\epsilon = b(x)+b(r) = (x-1)^2+(r-1)^2$. We complete the square and use $\epsilon - \frac{1}{2}(x+r-2)^2 = \frac{1}{2}(x-r)^2$.

Case 2: $|x+r| < 2$. Then

$$\frac{e^{-\tau\epsilon}e^{\eta(x+r)}e^{-\frac{\eta^2}{2\tau}}}{e^{2\eta}+e^{-2\eta}} \leq \begin{cases} e^{-\tau\epsilon}e^{-\frac{\eta^2}{2\tau}}e^{\eta(x+r-2)} \leq c & \text{if } \eta \geq 0 \quad (\text{use } x+r<2), \\ e^{-\tau\epsilon}e^{-\frac{\eta^2}{2\tau}}e^{\eta(x+r+2)} \leq c & \text{if } \eta < 0 \quad (\text{use } x+r>-2). \end{cases}$$

Other cases: In proving the other cases, we make and prove a claim, which we will use later.

Claim 2.2. If $x \geq 0$ and $(x, r) \notin \{x, r \geq 1/2\} \cup \{|x+r| < 2\}$, then for some $c > 0$,

$$\frac{e^{-\tau\epsilon}e^{\eta(x+r)}e^{-\frac{\eta^2}{2\tau}}}{e^{2\eta}+e^{-2\eta}} \leq e^{-c\tau}e^{-\frac{1}{2\tau}(\eta+(x+r\pm 2)\tau)^2},$$

where the plus or minus sign depends on whether (x, r) is above or below the strip $|x+r| < 2$.

Proof. For these cases, we will use the inequality

$$\frac{1}{e^{2\eta}+e^{-2\eta}} \leq e^{\pm 2\eta},$$

where we chose the plus or minus depending on whether (x, r) is above or below the strip $|x+r| < 2$, and then we complete the square:

$$\frac{e^{-\tau\epsilon}e^{\eta(x+r)}e^{-\frac{\eta^2}{2\tau}}}{e^{2\eta}+e^{-2\eta}} \leq e^{-(b(x)+b(r)-\frac{1}{2}(x+r\pm 2)^2)\tau}e^{-\frac{1}{2\tau}(\eta+(x+r\pm 2)\tau)^2}.$$

There are four cases. In each case we have $b(x)+b(r)-\frac{1}{2}(x+r\pm 2)^2 \geq c > 0$.

Case $x \geq 1/2, |r| \leq 1/2, x+r \geq 2$: Note if $r \geq 1/2, |x| \leq 1/2, x+r \geq 2$, the same argument will hold with x, r switched. Note here $x \geq 3/2$. We have

$$b(x)+b(r)-\frac{1}{2}(x+r-2)^2 = (x-1)^2 - r^2 + \frac{1}{2} - \frac{1}{2}(x+r-2)^2 \geq \frac{1}{4}.$$

There are three other cases: where $r \leq -1/2, |x| \leq 1/2, x+r \leq -2$, where $x \geq 1/2, r \leq -1/2, x+r \geq 2$, and where $x \geq 1/2, r \leq -1/2, x+r \leq -2$. The arguments for these are similar to the one shown above. □

Now we have

$$|(2.7)| \lesssim \int_0^\infty \int_{-\infty}^\infty \sqrt{2\tau}\left|\int_{-\infty}^\infty e^{-\lambda^2}\,d\lambda - \frac{\mathrm{I}_1+\mathrm{II}_1+\mathrm{III}_1}{e^{2\eta}+e^{-2\eta}}\right|\,d\eta\,d\tau.$$

Step 3: We show that this is bounded by a constant. The term $|.|$ is

$$= \frac{1}{e^{2\eta}+e^{-2\eta}} \times \left| \left(e^{-2\eta} \int_{-\infty}^{\infty} e^{-\lambda^2}\, d\lambda - \mathrm{I}_1 \right) + \left(e^{2\eta} \int_{-\infty}^{\infty} e^{-\lambda^2}\, d\lambda - \mathrm{III}_1 \right) - \mathrm{II}_1 \right|$$

$$= \frac{1}{e^{2\eta}+e^{-2\eta}} \times \left| e^{-2\eta} \left(\int_{\frac{1}{2}\left(1-\frac{\eta}{\tau}\right)\sqrt{2\tau}}^{\infty} e^{-\lambda^2} d\lambda \right) + e^{2\eta} \left(\int_{-\infty}^{\frac{1}{2}\left(-1-\frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{-\lambda^2}\, d\lambda \right) - \mathrm{II}_1 \right|$$

$$\leq \left| \frac{e^{-2\eta}}{e^{2\eta}+e^{-2\eta}} \left(\int_{\frac{1}{2}\left(1-\frac{\eta}{\tau}\right)\sqrt{2\tau}}^{\infty} e^{-\lambda^2} d\lambda \right) \right| + \left| \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}} \left(\int_{-\infty}^{\frac{1}{2}\left(-1-\frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{-\lambda^2}\, d\lambda \right) \right| + \left| \frac{1}{e^{2\eta}+e^{-2\eta}}\, \mathrm{II}_1 \right| .$$

So now

$$\begin{aligned} |(2.7)| \quad &\leq \int_0^{\infty} \sqrt{2\tau} \int_{-\infty}^{\infty} \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}} \int_{-\infty}^{\frac{1}{2}\left(-1-\frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{-\lambda^2} d\lambda d\eta d\tau \\ &\quad + \int_0^{\infty} \sqrt{2\tau} \int_{-\infty}^{\infty} \frac{e^{-2\eta}}{e^{2\eta}+e^{-2\eta}} \int_{\frac{1}{2}\left(1-\frac{\eta}{\tau}\right)\sqrt{2\tau}}^{\infty} e^{-\lambda^2} d\lambda d\eta d\tau \\ &\quad + \int_0^{\infty} \sqrt{2\tau} \int_{-\infty}^{\infty} \frac{\mathrm{II}_1}{e^{2\eta}+e^{-2\eta}} d\eta d\tau, \end{aligned}$$

which we denote as

$$\int \mathrm{III} + \int \mathrm{I} + \int \mathrm{II}\,. \tag{2.9}$$

We show that each of these is bounded by a constant.

$\int \mathrm{III}$ *in* (2.9)*:*

$$\int \mathrm{III} = \int_0^{\infty} \sqrt{2\tau} \int_{-\infty}^{\infty} \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}} \int_{-\infty}^{\frac{1}{2}\left(-1-\frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{-\lambda^2}\, d\lambda\, d\eta\, d\tau. \tag{2.10}$$

Case 1: $\frac{1}{2}\left(-1-\frac{\eta}{\tau}\right)\sqrt{2\tau} \leq -1$. Using (2.5), we see that

$$\begin{aligned} (2.10) &\leq \int_0^{\infty} \int_{-\infty}^{\infty} \sqrt{2\tau} \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}} e^{-\frac{\tau}{2}\left(1+2\frac{\eta}{\tau}+\left(\frac{\eta}{\tau}\right)^2\right)}\, d\eta\, d\tau \\ &\leq c \int_0^{\infty} \sqrt{\tau} e^{-\frac{\tau}{2}}\, d\tau \leq c. \end{aligned}$$

Case 2: $\frac{1}{2}\left(-1-\frac{\eta}{\tau}\right)\sqrt{2\tau} \geq -1$. Here we have $\eta \leq \sqrt{2\tau}-\tau$ and

$$\int_{-\infty}^{\frac{1}{2}\left(-1-\frac{\eta}{\tau}\right)\sqrt{2\tau}} e^{-\lambda^2}\, d\lambda \approx c.$$

So

$$\begin{aligned}
(2.10) &\lesssim \int_0^\infty \sqrt{\tau}\int_{-\infty}^{\sqrt{2\tau}-\tau} \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}} d\eta d\tau \\
&\leq \int_0^8 \sqrt{\tau}\int_{-\infty}^{\sqrt{2\tau}-\tau} \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}} d\eta d\tau + \int_8^\infty \sqrt{\tau}\int_{-\infty}^{\sqrt{2\tau}-\tau} \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}} d\eta d\tau \\
&\leq c + \int_8^\infty \sqrt{\tau}\int_{-\infty}^{\sqrt{2\tau}-\tau} \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}} d\eta d\tau.
\end{aligned}$$

For the second term, we use $\tau \geq 8$ to show that $-\tau/2 \geq \sqrt{2\tau}-\tau$. So we see $\eta \leq \sqrt{2\tau}-\tau \leq -\tau/2$, and so the second term is $\leq$

$$\int_8^\infty \sqrt{\tau}\int_{-\infty}^{-\tau/2} \frac{e^{2\eta}}{e^{2\eta}+e^{-2\eta}}\, d\eta\, d\tau \leq c.$$

$\int$ I *in* (2.9)*:* The argument is similar to that for $\int$ III. Instead of the cases there, we have $\frac{1}{2}\left(1-\frac{\eta}{\tau}\right)\sqrt{2\tau} \geq 1$ and $\frac{1}{2}\left(1-\frac{\eta}{\tau}\right)\sqrt{2\tau} \leq 1$.

$\int$ II *in* (2.9)*:* We have

$$\int \mathrm{II} \leq c\int_0^\infty \sqrt{\tau}\int_{-\infty}^\infty \frac{\mathrm{II}_1}{e^{2\eta}+e^{-2\eta}} d\eta\, d\tau. \tag{2.11}$$

As in Step 1, an easy calculation shows

$$\frac{\mathrm{II}_1}{e^{2\eta}+e^{-2\eta}} \lesssim \sqrt{\tau}e^{-\frac{\tau}{2}}e^{-\frac{\eta^2}{2\tau}} \times \begin{cases} e^{-\eta} & \text{if } \eta \geq 0, \\ e^{\eta} & \text{if } \eta < 0. \end{cases}$$

From this one obtains $\left|\int \mathrm{II}\right| \leq c$. This completes the proof of (2.3) and hence the proof of Lemma 2.1. □

3. Proofs of Theorems 1.1–1.3

3.1. Proof of Theorem 1.1. Here we assume that (x, r) is not in $\{x, r \geq 1/2\}$. We also assume that if $x \geq 1/2$ and $r \leq -1/2$, then $\epsilon = b(x) + b(r) = (x-1)^2 + (r+1)^2 \geq c > 0$. We will show that under these assumptions $|S|$ is bounded.

Proof. By Lemma 2.1, we need only look at

$$S_M = \int_0^\infty e^{-\tau(\epsilon+it)} \int_{-\infty}^\infty \frac{e^{\frac{-\eta^2}{2\tau}} e^{\eta(x+r)} e^{i\eta y}}{e^{-2\eta}+e^{2\eta}}\, d\eta \sqrt{\tau}\, d\tau.$$

Case 1: $|x+r| \geq 2$. The proof follows from Claim 2.2.

Case 2: $|x+r| < 2$. We have that $|S_M|$ is

$$\leq \int_0^\infty e^{-\tau(b(x)+b(r))}\sqrt{\tau}\left(\int_{-\infty}^0 e^{-\frac{\eta^2}{2\tau}} e^{\eta(x+r+2)} d\eta + \int_0^\infty e^{-\frac{\eta^2}{2\tau}} e^{\eta(x+r-2)} d\eta\right) d\tau$$
$$\leq c \int_0^\infty e^{-\tau(b(x)+b(r))}\tau\, d\tau.$$

This integral is bounded as long as $b(x)+b(r) \geq c > 0$, which one can show to be true by considering separately the cases $x \geq 1/2$, $|r| \leq 1/2$, $r \leq -1/2, |x| \leq 1/2$, and $x \geq 1/2, r \leq -1/2$. □

3.2. Proof of Theorem 1.2. We covered part of the region in this theorem in the above section; that is, we showed that if $x \geq 1/2, r \leq -1/2$ and $\epsilon \geq c > 0$, then $|S|$ is bounded. So we assume $\epsilon \leq c$. We can also assume $|t| \leq c$ since otherwise $|S| \lesssim E$ as we will see later in Lemma 4.2.

Proof. As above, we need only consider

$$S_M = \lim_{\delta\to 0} \int_0^\infty e^{-\tau(\epsilon+it)} \int_{-\infty}^\infty \frac{e^{\frac{-\eta^2}{2\tau}} e^{\eta(x+r)} e^{i\eta y}}{e^{-2\eta}+e^{2\eta}}\, d\eta \sqrt{\tau}\chi(\delta\tau)\, d\tau,$$

where $\chi(\tau)$ is a Schwartz cutoff function which is 1 for $\tau \leq 1$ and 0 for $\tau \geq 2$; it is easily seen that this limit exists by doing the integration by parts below. We consider two cases.

Case 1: $|y| \geq c$. We split up the integral at $\frac{1}{\epsilon+|t|}$. We merely note that for the second piece, we do integration by parts with

$$u = \int_{-\infty}^\infty \frac{e^{\frac{-\eta^2}{2\tau}} e^{\eta(x+r)} e^{i\eta y}}{e^{-2\eta}+e^{2\eta}}\, d\eta \sqrt{\tau}\chi(\delta\tau),$$

and then one can show that

$$\left|\frac{d^p u}{d\tau^p}\right| \leq \frac{c}{\tau^p}\sqrt{\tau}\chi.$$

We then do integration by parts two times and the result follows.

Case 2: $|y| \le c$. We first do integration by parts in the η integral to bring down a power of $|y|$. We obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-\frac{\eta^2}{2\tau}} e^{\eta(x+r)}}{e^{-2\tau}+e^{2\tau}} e^{i\eta y} d\eta \\ = -\frac{1}{iy}\int_{-\infty}^{\infty} \left(\frac{-\frac{2\eta}{2\tau}+(x+r)}{e^{-2\eta}+e^{-2\eta}} + \frac{2e^{-2\eta}-2e^{2\eta}}{(e^{-2\eta}+e^{2\eta})^2}\right) e^{-\frac{\eta^2}{2\tau}} e^{\eta(x+r)} e^{i\eta y} d\eta.\end{aligned}$$

Then with this as our u in the integration by parts in τ, one can show that

$$\left|\frac{d^p u}{d\tau^p}\right| \lesssim \frac{1}{|y|}\frac{1}{\tau^p}\sqrt{\tau}\chi,$$

and so we proceed as above. □

3.3. Proof of Theorem 1.3. If $\epsilon \ge 1$, S behaves as if we had replaced our $b(x)$ with the convex function $b(x) = (x-1)^2$. The following lemma describes how the main term of S exhibits this behavior.

LEMMA 3.1. *If $x, r \ge 1/2$ and $x + r - 2 \ge c > 0$, then for $\epsilon = (x-1)^2 + (r-1)^2$ and M as in Lemma 2.1 we have*

$$M = M_1 + E_{01},$$

where

$$M_1 = e^{-\frac{\eta^2}{2\tau}} \sqrt{2\tau} e^{-2\eta}$$

and

$$\left|\int E_{01}\right| = \left|\int_0^{\infty} e^{-\tau(\epsilon+it)} \int_{-\infty}^{\infty} e^{i\eta y} e^{\eta(x+r)} E_{01} d\eta d\tau\right| \le c.$$

Proof. Write

$$\frac{1}{e^{2\eta}+e^{-2\eta}} = \frac{e^{-2\eta}}{1+e^{-4\eta}} = e^{-2\eta} - \frac{e^{-6\eta}}{1+e^{-4\eta}}.$$

We use the first piece for M_1 and the second for E_{01}. So

$$M_1 = e^{-\frac{\eta^2}{2\tau}} \sqrt{2\tau} e^{-2\eta} \quad \text{and} \quad E_{01} = e^{-\frac{\eta^2}{2\tau}} \sqrt{2\tau} \frac{e^{-6\eta}}{1+e^{-4\eta}}.$$

For E_{01} we have

$$\left|\int E_{01}\right| \lesssim \int_0^{\infty} e^{-\epsilon\tau} \left(\int_{-\infty}^{0} e^{-\frac{\eta^2}{2\tau}} e^{\eta(x+r-2)} d\eta + \int_0^{\infty} e^{-\frac{\eta^2}{2\tau}} e^{\eta(x+r-6)} d\eta \right) \sqrt{\tau} d\tau.$$

We then show that this is bounded by considering each term separately. We complete the square and use (2.5). □

CLAIM 3.2. Upon applying this lemma, one can obtain Theorem 1.3.

Proof. First note that $x, r \geq 1/2$ and $\epsilon \geq 1$ imply $x + r - 2 \geq 1/3$. One can see this by considering the situation in the xr-plane. So we have

$$S_{M_1} = c \int_0^\infty e^{-(\epsilon + it)\tau} \int_{-\infty}^\infty e^{\eta(x+r-2)} e^{i\eta y} e^{-\frac{\eta^2}{2\tau}} d\eta \sqrt{\tau} d\tau.$$

Then, by completing the square, using $\epsilon - \frac{(x+r-2)^2}{2} = \frac{(x-r)^2}{2}$, and doing a change of variables, we have that this

$$= c \int_0^\infty e^{-\left(\frac{(x-r)^2}{2} + \frac{y^2}{2} + i(t - y(x+r-2))\right)\tau} \int_{-\infty}^\infty e^{-\eta^2} d\eta \tau d\tau.$$

Since the real part of the exponent is positive, we can use the Identity Theorem to get that this

$$= c \frac{1}{[(x-r)^2 + y^2 + 2i(t - y(x+r-2))]^2}.$$

We obtain (1.2) by taking the norm. This finishes the proof of Theorem 1.3. □

4. Proofs of Theorem 1.4 and Corollary 1.5

Both theorems rely on Lemma 2.1. For Theorem 1.4, there are some additional main lemmas used, and these are stated here; they will be proven in later sections as the proofs are long and detailed. Following the statements of these lemmas, we will show how to use the lemmas to prove the theorems. Corollary 1.5 will be obtained as a corollary of Theorem 1.4(c).

In Theorem 1.4, there are a number of constants. The proof does not refer to each one, but rather they are implicit in the $a \approx b$ and $a \lesssim b$ statements.

4.1. Proof of Theorem 1.4.

4.1.1. *Main estimates used for Theorem 1.4.* We have two main lemmas, which correspond to ϵ or $|t|$ being dominant, though the ranges where they apply actually overlap.

Again recall that D and D_e are defined as follows if $x, r \geq 1/2$:

$$D = \sqrt{2\epsilon} - (x + r - 2) \qquad \text{and} \qquad D_e = \mathrm{Re}((\epsilon + it)^{1/2}) - \sqrt{2\epsilon}.$$

Also recall that

$$|\mathrm{Im}_a| = \left| 5t - y(x+r) + 4\,\mathrm{Im}((2(\epsilon+it))^{1/2}) + \frac{1}{2}\mathrm{Im}((2(\epsilon+it))^{3/2}) \right|,$$
$$|\mathrm{Im}_b| = \left| t - y(x+r) + 2\,\mathrm{Im}((2(\epsilon+it))^{1/2}) \right|.$$

Lemma 4.1 (ϵ dominant). *If $\epsilon > 0, x, r \geq 1/2$ and $\{x + r - 2 < 0$ or $|x - r| > 0)\}$, then*

$$S = S_{M_1} + E_1 + E,$$

where if

(4.1)
$$Q = \frac{\left[\left(\mathrm{Re}((\epsilon+it)^{3/2}) + \mathrm{Re}((\epsilon+it)^{1/2}) + D + D_{\mathrm{e}} + y^2\right)^2 + (\mathrm{Im}_a)^2\right]^{1/2}}{(\epsilon+|t|)^{3/2}\left[\left((x-r)^2 + y^2 + D + D_{\mathrm{e}}\right)^2 + (\mathrm{Im}_b)^2\right]},$$

then

$$\frac{1}{c}Q \le |S_{M_1}| \le cQ$$

and

$$|E_1| \le c_1 \begin{cases} \frac{1}{(\epsilon+|t|)^{3/2}(y^2+1)} & \text{if } |\epsilon+it| \le 2, \\ 1 & \text{if } |\epsilon+it| > 2, \end{cases} \qquad \text{and} \qquad |E| \le c_2 < \infty.$$

Proof. See Section 4.3. The idea is to interchange the order of integration and then use a contour integral. □

Lemma 4.2 ($|t|$ dominant). *If $|t| \ge c\epsilon$ for any positive c, then for c_1 depending on c,*

$$|S| \le c_1 \frac{1}{|t|^{3/2}(|t|^{1/2} + |y|)} + |E|. \tag{4.2}$$

Proof. See Section 4.4. We first show that $|S| \le c_1 \frac{1}{t^2} + |E|$. The idea is to use integration by parts in the τ-integral. Then we show that $|S| \le c_1 \frac{1}{|t|^{3/2}|y|} + |E|$. The idea is again to use integration by parts: We can integrate by parts once in η, bringing down one power of y. Then we integrate by parts in τ. □

These two lemmas overlap when $\epsilon > 0$, $|t| \ge c\epsilon$ for any positive c. We will show in the next section that they give the same estimate where they overlap if $\epsilon, |t|, |y| \le 1$.

4.1.2. *Proof of Theorem 1.4 using the main estimates.* We prove Theorem 1.4 based on the lemmas in Section 4.1.1 and the approximations in the following claim, which can be proven using Taylor series.

Claim 4.3. The following approximations hold:

1.

$$\begin{aligned} D &= \sqrt{2\epsilon} - (x + r - 2) \\ &\approx \left\{ \begin{array}{ll} \frac{(x-r)^2}{\sqrt{\epsilon}} & \text{if } x + r - 2 \ge 0 \\ \sqrt{\epsilon} & \text{if } x + r - 2 \le 0 \end{array} \right\} \lesssim \sqrt{\epsilon}, \end{aligned} \tag{4.3}$$

if $x, r \ge 1/2$.

2.

$$D_e = \left(\mathrm{Re}((2(\epsilon+it))^{1/2}) - \sqrt{2\epsilon}\right) \tag{4.4}$$
$$\approx \left\{\begin{array}{ll} \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} & \text{if } \epsilon \geq 4|t| \\ \sqrt{|t|} & \text{if } \epsilon \leq \frac{|t|}{4} \\ \sqrt{|t|} \approx \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} & \text{if } \frac{1}{4} \leq \frac{\epsilon}{|t|} \leq 4 \end{array}\right\}$$
$$\lesssim \left\{\begin{array}{ll} \sqrt{\epsilon} & \text{if } \epsilon \geq 4|t| \\ \sqrt{|t|} & \text{if } \epsilon \leq \frac{|t|}{4} \\ \sqrt{|t|} \approx \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} & \text{if } \frac{1}{4} \leq \frac{\epsilon}{|t|} \leq 4 \end{array}\right. .$$

3.

$$\mathrm{Re}((2(\epsilon+it))^{1/2}) = \sqrt{\sqrt{\epsilon^2+t^2}+\epsilon} \tag{4.5}$$
$$\approx \left\{\begin{array}{ll} \sqrt{\epsilon} & \text{if } \epsilon \geq |t| \\ \sqrt{|t|} & \text{if } \epsilon \leq |t| \end{array}\right. .$$

4.

$$\mathrm{Im}((2(\epsilon+it))^{1/2}) = \mathrm{sgn}(t)\sqrt{\sqrt{\epsilon^2+t^2}-\epsilon} \tag{4.6}$$
$$\approx \left\{\begin{array}{ll} \frac{t}{\sqrt{\epsilon}} & \text{if } \epsilon \geq 2|t| \\ \mathrm{sgn}(t)\sqrt{|t|} & \text{if } \epsilon \leq \frac{|t|}{2} \\ \frac{t}{\sqrt{\epsilon}} \approx \mathrm{sgn}(t)\sqrt{|t|} & \text{if } \frac{1}{2} \leq \frac{\epsilon}{|t|} \leq 2 \end{array}\right. .$$

5.

$$\mathrm{Re}((2(\epsilon+it))^{3/2}) = 2\sqrt{(\epsilon^2+t^2)^{3/2}+\epsilon^3-3\epsilon t^2} \tag{4.7}$$
$$\approx \left\{\begin{array}{ll} \epsilon^{3/2} & \text{if } \epsilon \geq |t|\frac{1}{.95} \\ |t|^{3/2} & \text{if } \epsilon < |t| \\ \epsilon^{3/2} \approx |t|^{3/2} & \text{if } .9 \leq \frac{\epsilon}{|t|} \leq \frac{1}{.95} \end{array}\right. .$$

6.

$$\mathrm{Im}((2(\epsilon+it))^{3/2}) = \mathrm{sgn}(t)2\sqrt{(\epsilon^2+t^2)^{3/2}-\epsilon^3+3\epsilon t^2} \tag{4.8}$$
$$\approx \left\{\begin{array}{ll} \sqrt{\epsilon}t & \text{if } \epsilon \geq \frac{3}{4}|t| \\ \mathrm{sgn}(t)|t|^{3/2} & \text{if } \epsilon \leq \frac{3}{4}|t| \end{array}\right. .$$

We also prove in some cases more specific results. Here k is any positive constant:

7.

$$\mathrm{Im}((2(\epsilon+it))^{1/2}) = \frac{t}{\sqrt{2\epsilon}} + err,$$

where

$$|err| \leq \frac{1}{4}\frac{|t|}{\sqrt{2\epsilon}}\left(\frac{t}{\epsilon}\right)^2 \leq \frac{1}{16}\frac{|t|}{\sqrt{2\epsilon}} \text{ if } |t| \leq \frac{\epsilon}{2},\ \epsilon > 0.$$

8.
$$\left|\operatorname{Im}((2(\epsilon+it))^{1/2})\right| \leq (k+1)^{1/4}\sqrt{|t|} \quad \text{if } |t| \geq \frac{\epsilon}{k}.$$
In particular, if $k=2$,
$$\left|\operatorname{Im}((2(\epsilon+it))^{1/2})\right| \leq 2\sqrt{|t|} \quad \text{if } |t| \geq \frac{\epsilon}{2}.$$

9.
$$\left|\operatorname{Im}((2(\epsilon+it))^{3/2})\right| \leq 2((k^2+1)^{3/2}+3k)^{1/2}|t|^{3/2} \quad \text{if } |t| \geq \frac{\epsilon}{k}.$$
In particular, if $k=2$,
$$\left|\operatorname{Im}((2(\epsilon+it))^{3/2})\right| \leq 12|t|^{3/2} \quad \text{if } |t| \geq \frac{\epsilon}{2}.$$

10.
$$\left|\operatorname{Im}((2(\epsilon+it))^{3/2})\right| \leq 6\sqrt{\epsilon}|t| \quad \text{if } |t| \leq \frac{\epsilon}{2}.$$

Proof of Theorem 1.4(a). We make some observations about Lemma 4.1:

- $\operatorname{Re}((\epsilon+it)^{1/2}) \approx \epsilon^{1/2}+|t|^{1/2}$ and $\operatorname{Re}((\epsilon+it)^{3/2}) \approx \epsilon^{3/2}+|t|^{3/2}$ by (4.5) and (4.7).
- If $\epsilon, |t| \leq 1$, then $\epsilon^{3/2}+|t|^{3/2} \leq \epsilon^{1/2}+|t|^{1/2}$.
- Also D is always $\lesssim \epsilon^{1/2}$, and $D_e \lesssim \sqrt{|t|}+\sqrt{\epsilon}$ (see (4.3) and (4.4)).
- If $\epsilon \leq 1$, then $D \gtrsim (x-r)^2$ (see (4.3)).

So we can reduce (4.1) to (1.3) in Theorem 1.4(a). For the error, note that if $\epsilon, |t| \leq 1$, then $|\epsilon+it| \leq 2$. ☐

Proof of Theorem 1.4(b),(d). The plan here is to show that $Q \geq |E_1|$.

First assume $|y| \leq 1$. We show that the piece in Q multiplying the term $\frac{1}{(\epsilon+|t|)^{3/2}}$ is greater than a constant if $\epsilon, |t|$ and $|y|$ are ≤ 1, and that it can be made as large as we want if $\epsilon, |t|$, and $|y|$ are made small enough. We start with the following claim:

Claim 4.4. If $|y| \leq 1$, then the term y^2 is always less than or equal to one of the other existing terms.

Proof. 1. For $|t| \leq \epsilon/2$, recall that
$$|\operatorname{Im}_b| = \left|t - y(x+r) + 2\operatorname{Im}((2(\epsilon+it))^{1/2})\right|,$$
$$|\operatorname{Im}_a| = \left|5t - y(x+r) + 4\operatorname{Im}((2(\epsilon+it))^{1/2}) + \frac{1}{2}\operatorname{Im}((2(\epsilon+it))^{3/2}\right|.$$
If $|y(x+r)| \geq 30|\frac{t}{\sqrt{2\epsilon}}|$, then using Claim 4.3 and $|x+r| \approx 1$ (since $x, r \geq 1/2$ and $\epsilon \leq 1$) we see that
$$\left|t + 2\operatorname{Im}((2(\epsilon+it))^{1/2})\right| < 5\left|\frac{t}{\sqrt{2\epsilon}}\right|$$

and

$$\left|5t + 4\,\mathrm{Im}((2(\epsilon+it))^{1/2}) + \frac{1}{2}\,\mathrm{Im}((2(\epsilon+it))^{3/2}\right| < 21\left|\frac{t}{\sqrt{2\epsilon}}\right|,$$

and so

$$y^2 \le |y| \approx |\mathrm{Im}_b| \approx |\mathrm{Im}_a|. \tag{4.9}$$

If $|y(x+r)| \le 30|\frac{t}{\sqrt{2\epsilon}}|$, we use

$$y^2 \le c\left(\frac{t}{\sqrt{\epsilon}}\right)^2\left(\frac{1}{\sqrt{\epsilon}}\right) \approx D_e \lesssim \sqrt{\epsilon}.$$

2. For $1 \ge |t| \ge \epsilon/2$, we use similar arguments to show that either $y^2 \lesssim |\mathrm{Im}_b| \approx |\mathrm{Im}_a|$ or $y^2 \lesssim |D_e|$. $\square$

Using this we can simplify (1.3) in Theorem 1.4(a) to get

$$|S_{M_1}| \approx \frac{1}{(\epsilon+|t|)^{3/2}}\frac{\sqrt{\epsilon}+\sqrt{|t|}+|\mathrm{Im}_a|}{(D+D_{\mathrm{e}}+|\mathrm{Im}_b|)^2}. \tag{4.10}$$

Then we define F by

$$\frac{1}{(\epsilon+|t|)^{3/2}}\frac{\sqrt{\epsilon}+\sqrt{|t|}+|\mathrm{Im}_a|}{(D+D_{\mathrm{e}}+|\mathrm{Im}_b|)^2} = \frac{1}{(\epsilon+|t|)^{3/2}}(F). \tag{4.11}$$

Now we will show that if $\epsilon, |t|, |y| \ll 1$, then $|S_{M_1}+E_1| \approx |S_{M_1}|$, and for $\epsilon, |t|, |y| \le 1$, we have $|S_{M_1}+E_1| \lesssim |S_{M_1}|$. To do this, we first prove the following claim. We label the constant c_4 to make it clear that it is the same as in the Theorem.

Claim 4.5. Given a constant d, there exists a constant $c_4 < 1$ such that if $\epsilon, |t|, |y| \le c_4$, then $1/F \le 1/d$.

Proof. Case 1a: We look first at the case $t = 0$. Then $|\mathrm{Im}_a| = |\mathrm{Im}_b| \approx |y|$. Here

$$\frac{1}{F} = \frac{(D+|y|)^2}{(\sqrt{\epsilon}+|y|)} \approx \left\{\begin{array}{ll} |y| & \text{if } |y| \ge \sqrt{\epsilon} \\ \frac{y^2}{\sqrt{\epsilon}} & \text{if } D \le |y| \le \sqrt{\epsilon} \\ \frac{D^2}{\sqrt{\epsilon}} & \text{if } |y| \le D \end{array}\right\} \lesssim \left\{\begin{array}{ll} |y| & \text{if } |y| \ge \sqrt{\epsilon} \\ \sqrt{\epsilon} & \text{if } D \le |y| \le \sqrt{\epsilon} \\ \sqrt{\epsilon} & \text{always} \end{array}\right. .$$

So we see that we can make $1/F$ small if $\epsilon, |y| \ll 1$.

Case 1b: Now we look at the more general case $\epsilon \ge 2|t|$. First we use Claim 4.3 to see that

$$|\mathrm{Im}_b| \lesssim |t| + |y| + \frac{|t|}{\sqrt{\epsilon}} + \sqrt{\epsilon}|t| \lesssim |y| + \frac{|t|}{\sqrt{\epsilon}} \lesssim |y| + \sqrt{\epsilon}.$$

Next we show that

$$\sqrt{\epsilon} + |\mathrm{Im}_a| \approx \sqrt{\epsilon} + |y|.$$

To do so, we first note that we have already shown (see (4.9)) that

$$|\operatorname{Im}_a| \approx |y(x+r)| \approx |y| \text{ if } |y(x+r)| \geq 30\left|\frac{t}{\sqrt{2\epsilon}}\right|,$$

which is true if $|y| \geq 30\left|\frac{t}{\sqrt{2\epsilon}}\right|$ since $|x+r| \geq 1$. If $|y| \leq 30\left|\frac{t}{\sqrt{2\epsilon}}\right|$, then $|y| \leq 15\sqrt{\epsilon}$, so then

$$|\operatorname{Im}_a| \lesssim |t| + |y| + \frac{|t|}{\sqrt{\epsilon}} + \sqrt{\epsilon}|t| \lesssim \sqrt{\epsilon}.$$

So $\sqrt{\epsilon} + |\operatorname{Im}_a| \approx \sqrt{\epsilon} \approx \sqrt{\epsilon} + |y|$. Using these approximations, we get

$$\frac{1}{F} \approx \frac{(D + \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} + |\operatorname{Im}_b|)^2}{(\sqrt{\epsilon} + |\operatorname{Im}_a|)} \approx \frac{(D + \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} + |\operatorname{Im}_b|)^2}{(\sqrt{\epsilon} + |y|)} \lesssim \sqrt{\epsilon} + |y| \ .$$

So we see that we can make this small if $\epsilon, |y|$ are small enough.

Note that the second approximation shows that if $1 \geq \epsilon \geq 2|t|$, $\epsilon > 0$, $\{x + r < 2 \text{ or } |x - r| > 0\}$, and $|y| \leq 1$, then

$$|S_{M_1}| \approx \frac{\sqrt{\epsilon} + |y|}{\epsilon^{3/2}(D + \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} + |\operatorname{Im}_b|)^2}. \tag{4.12}$$

This expression is referred to in the notes after Theorem 1.4.

Case 2: If $\epsilon \leq 2|t|$ and $|t| \leq 1$, the argument is similar to that in Case 1b.

To finish the proof of the claim, we choose $c_4 < 1$ so that $1/F \leq 1/d$ when $\epsilon, |t|, |y| \leq c_4$. □

Now we choose d such that given c_2 in Theorem 1.4(a), $\frac{c_2}{2}d = d' > 1$. Then if we assume $\epsilon, |t|, |y| \leq c_4 < 1$, we have

$$\frac{|S_{M_1}|}{|E_1|} \geq \frac{c_2}{2}F \geq \frac{c_2}{2}d = d' > 1,$$

so

$$\frac{1}{d'}|S_{M_1}| \geq |E_1|,$$

so

$$\left(1 - \frac{1}{d'}\right)|S_{M_1}| \leq |S_{M_1} + E_1| \leq \left(1 + \frac{1}{d'}\right)|S_{M_1}|.$$

Therefore, if we pick c_5 so that $1 + \frac{1}{d'} \leq c_5$ and $1 - \frac{1}{d} \geq \frac{1}{c_5}$, we get the first claim in Theorem 1.4(b).

To show that $|S_{M_1} + E_1| \lesssim |S_{M_1}|$, we simply note that $1/F \leq c$ if $\epsilon, |t|, |y| \leq 1$. This gives us the second claim in Theorem 1.4(b) if $|y| \leq 1$.

So we have for $\epsilon > 0$, $\{x + r < 2 \text{ or } |x - r| > 0\}$, and $\epsilon, |t|, |y| \leq 1$,

$$|S| \leq |S_{M_1} + E_1| + |E| \lesssim |S_{M_1}| + |E| \approx \frac{\sqrt{\epsilon} + \sqrt{|t|} + |\mathrm{Im}_a|}{(\epsilon^{3/2} + |t|^{3/2})\,[D + D_e + |\mathrm{Im}_b|]^2} + |E|. \tag{4.13}$$

Now assume $|y| > 1$. We look again at (1.3), which is true if $\epsilon > 0$, $\{x+r < 2 \text{ or } |x - r| > 0\}$, and $\epsilon, |t| \leq 1$:

$$Q = \frac{(\sqrt{\epsilon} + \sqrt{|t|} + y^2) + |\mathrm{Im}_a|}{(\epsilon + |t|)^{3/2}\,[(y^2 + D + D_{\mathrm{e}}) + |\mathrm{Im}_b|]^2}.$$

Then by Claim 4.3 and the fact that $\epsilon, |t| \leq 1 < |y|$, we have

$$|\,\mathrm{Im}_a\,| = \left|5t - y(x + r) + 4\,\mathrm{Im}((2(\epsilon + it))^{1/2}) + \frac{1}{2}\,\mathrm{Im}((2(\epsilon + it))^{3/2})\right| < y^2.$$

We also have

$$|\,\mathrm{Im}_b\,| = \left|t - y(x + r) + 2\,\mathrm{Im}((2(\epsilon + it))^{1/2})\right| < y^2.$$

So we now have

$$Q \approx \frac{y^2}{(\epsilon + |t|)^{3/2}[y^2]^2} = \frac{1}{(\epsilon + |t|)^{3/2}y^2}.$$

Since $|S_{M_1}| \approx Q$, and we have

$$|E_1| \lesssim \frac{1}{(\epsilon + |t|)^{3/2}y^2},$$

we have now finished the proof of Theorem 1.4(d).

Furthermore, we have for $|y| > 1$,

$$|S_{M_1} + E_1| \leq c|S_{M_1}|.$$

This finishes the proof of Theorem 1.4(b). □

Proof of Theorem 1.4(c). We will obtain an upper bound for $|S|$ covering the more general case where $\epsilon \leq 1$ and either (1) $\epsilon > 0$, $\{x+r < 2 \text{ or } |x-r| > 0\}$, and $|y| \leq 1$ or (2) $|t| \geq \epsilon/k$ for some positive constant k. Although Lemma 4.1 is only known to be true for $\epsilon > 0$, we will use it to suggest an estimate and then prove this estimate. Here is the sequence of ideas:

1. We showed above (see (4.13)) that for $\epsilon > 0$, $\{x + r < 2 \text{ or } |x - r| > 0\}$, and $\epsilon, |t|, |y| \leq 1$,

$$|S| \lesssim \frac{\sqrt{\epsilon} + \sqrt{|t|} + |\mathrm{Im}_a|}{(\epsilon^{3/2} + |t|^{3/2})\,[D + D_e + |\mathrm{Im}_b|]^2} + |E|. \tag{4.14}$$

2. If $|t| > 1$, then $|t| \geq \epsilon$, so Lemma 4.2 shows that $|S| \leq c + |E|$, and the constant can be included in $|E|$, so we are done.

3. We must now allow for $\epsilon = 0$. In fact, we can show that the result is true as long as $1 \geq |t| \geq \epsilon/k$ for any positive constant k. Here we use k instead of c because it will be important to keep track of the constant in the approximations in Claim 4.3.

Define $(*)$ by writing (4.14) as

$$|S| \lesssim (*) + |E|.$$

We must show that this is true for $1 \geq |t| \geq \epsilon/k$. Here $D_e \approx \sqrt{|t|} \gtrsim D$ (see Claim 4.3), so we can simplify $(*)$ to get

$$(*) \approx \frac{\sqrt{|t|} + |\operatorname{Im}_a|}{|t|^{3/2}(\sqrt{|t|} + |\operatorname{Im}_b|)^2}. \tag{4.15}$$

Now we show that $|S| \lesssim (*) + |E|$ if $\epsilon \leq k|t| \leq k$. First we choose f_k so that $f_k \geq 5 + 4(k+1)^{1/4} + ((k^2+1)^{3/2} + 3k)^{1/2} + 1$. We will see shortly the importance of this expression. Then we have two cases:

Case a: $\sqrt{|t|} \geq \frac{1}{f_k}|y(x+r)|$. Here $|\operatorname{Im}_a|, |\operatorname{Im}_b| \lesssim \sqrt{|t|}$, so $(*) + |E| \approx \frac{1}{t^2} + |E| \gtrsim |S|$ by Lemma 4.2.

Case b: $\sqrt{|t|} \leq \frac{1}{f_k}|y(x+r)|$. Here $|\operatorname{Im}_a| \approx |\operatorname{Im}_b| \approx |y| \gtrsim \sqrt{|t|}$. Using Claim 4.3, we have

$$\begin{aligned}
|\operatorname{Im}_a| &= \left|5t + 4\operatorname{Im}((2(\epsilon+it))^{1/2}) + \frac{1}{2}\operatorname{Im}((2(\epsilon+it))^{3/2})\right| \\
&\leq 5|t| + 4(k+1)^{1/4}\sqrt{|t|} + \frac{1}{2}2((k^2+1)^{3/2} + 3k)^{1/2}|t|^{3/2} \\
&\leq \left(5 + 4(k+1)^{1/4} + ((k^2+1)^{3/2} + 3k)^{1/2}\right)\sqrt{|t|} \\
&\leq \frac{5 + 4(k+1)^{1/4} + ((k^2+1)^{3/2} + 3k)^{1/2}}{f_k}|y(x+r)|
\end{aligned}$$

and

$$\begin{aligned}
|\operatorname{Im}_b| &= \left|t + 2\operatorname{Im}((2(\epsilon+it))^{1/2})\right| \\
&\leq |t| + 2(k+1)^{1/4}\sqrt{|t|} \\
&\leq (1 + 2(k+1)^{1/4})\sqrt{|t|} \\
&\leq \frac{(1 + 2(k+1)^{1/4})}{f_k}|y(x+r)|.
\end{aligned}$$

Since the quantity in front of the term $|y(x+r)|$ is strictly less than 1, we have $|\operatorname{Im}_a| \approx |\operatorname{Im}_b| \approx |y(x+r)|$. Then we use the fact that $(x+r) \approx 1$ since $\epsilon \leq 1$ and $x, r \geq 1/2$ to see that $|\operatorname{Im}_a| \approx |\operatorname{Im}_b| \approx |y| \gtrsim \sqrt{|t|}$.

So

$$(*) + |E| \approx \frac{|y|}{|t|^{3/2}|y|^2} + |E| \approx \frac{1}{|t|^{3/2}|y|} + |E| \gtrsim |S|$$

by Lemma 4.2.

This shows that $|S| \lesssim (*) + |E|$ and finishes the proof of Theorem 1.4(c), which we state here again for reference: As long as $\epsilon \leq 1$ and either (1) $\epsilon > 0$, $\{x + r < 2 \text{ or } |x - r| > 0\}$, and $|y| \leq 1$ or (2) $|t| \geq \epsilon/k$ for some constant k, we have

$$|S| \lesssim \frac{\sqrt{\epsilon} + \sqrt{|t|} + |\mathrm{Im}_a|}{(\epsilon^{3/2} + |t|^{3/2})\,[D + D_e + |\mathrm{Im}_b|]^2} + |E|. \tag{4.16}$$

In fact, the previous discussion shows that if $1 \geq |t| \geq \epsilon/k$ for any positive constant k, then

$$(*) \approx \frac{1}{|t|^{3/2}(\sqrt{|t|} + |y|)}, \tag{4.17}$$

and we have shown $|S| \lesssim (*) + |E|$, so

$$|S| \lesssim \frac{1}{|t|^{3/2}(\sqrt{|t|} + |y|)} + |E|, \tag{4.18}$$

which is exactly the bound in Lemma 4.2. Since $(*)$ was obtained from (4.14), which was obtained from Lemma 4.1, we see that the two lemmas agree where they overlap if $\epsilon, |t|, |y| \leq 1$. This finishes the proof of Theorem 1.4. □

4.2. Proof of Corollary 1.5. In this section we will prove Corollary 1.5 as a corollary of Theorem 1.4(c).

Proof. Case 1: $\epsilon \geq 2|t|$. Start from (4.16). If $\epsilon \geq 2|t|$, we can reduce this to

$$|S| \lesssim \frac{\sqrt{\epsilon} + |\,\mathrm{Im}_a\,|}{\epsilon^{3/2}\left(D + \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} + |\,\mathrm{Im}_b\,|\right)^2} + |E|.$$

Case 1a: $|\,\mathrm{Im}_a\,| \geq c\sqrt{\epsilon}$. Here

$$|S| \lesssim \frac{|\,\mathrm{Im}_a\,|}{\epsilon^{3/2}|\,\mathrm{Im}_b\,|^2} + |E|. \tag{4.19}$$

We will show that either $|\,\mathrm{Im}_a\,| \approx |\,\mathrm{Im}_b\,|$ or $|\,\mathrm{Im}_a\,| \lesssim \sqrt{\epsilon}$, in which case we can use Case 1b below. Write

$$\mathrm{Im}_a = (\mathrm{Im}_b) + \left(4t + 2\,\mathrm{Im}((2(\epsilon + it))^{1/2}) + \frac{1}{2}\,\mathrm{Im}((2(\epsilon + it))^{3/2})\right) = A + B.$$

Case $|B| \leq \frac{1}{2}|A|$*:* Then $\mathrm{Im}_a \approx \mathrm{Im}_b$. So we can reduce (4.19) and, using $\epsilon \gtrsim D^2$, obtain the statement in Corollary 1.5:

$$\begin{aligned}|S| &\lesssim \frac{1}{\left[\epsilon^{1/2} + \sqrt{\sqrt{\epsilon^2+t^2}-\epsilon}\right]^3 |\,\mathrm{Im}_b\,|} + |E| \\ &\lesssim \frac{1}{\left[(\epsilon D^2)^{1/4} + \sqrt{\sqrt{\epsilon^2+t^2}-\epsilon}\right]^3 |\,\mathrm{Im}_b\,|} + |E|.\end{aligned}$$

Case $|B| > \frac{1}{2}|A|$*:* Then

$$\begin{aligned}|\,\mathrm{Im}_a\,| &\leq |\,\mathrm{Im}_b\,| + |B| \\ &\leq c|B| \\ &= c\left|4t + 2\,\mathrm{Im}((2(\epsilon+it))^{1/2}) + \frac{1}{2}\,\mathrm{Im}((2(\epsilon+it))^{3/2})\right| \\ &\lesssim \sqrt{\epsilon}.\end{aligned}$$

Case 1b: $|\,\mathrm{Im}_a\,| \leq c\sqrt{\epsilon}$. Here

$$\begin{aligned}|S| &\lesssim \frac{1}{\epsilon\left(D + \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon} + |\,\mathrm{Im}_b\,|\right)^2} + |E| \\ &\lesssim \frac{1}{\epsilon^{3/4}\epsilon^{1/4}\left(D + \frac{t}{\sqrt{\epsilon}}\frac{t}{\epsilon}\right)|\,\mathrm{Im}_b\,|} + |E| \\ &\lesssim \frac{1}{\left(\epsilon^{3/4}D^{3/2} + \frac{t}{\sqrt{\epsilon}}t\right)|\,\mathrm{Im}_b\,|} + |E|.\end{aligned}$$

Then, using the fact that $\sqrt{|t|} \gtrsim \frac{|t|}{\sqrt{\epsilon}}$, this is

$$\begin{aligned}&\lesssim \frac{1}{\left(\epsilon^{3/4}D^{3/2} + \left|\frac{t}{\sqrt{\epsilon}}\right|^3\right)|\,\mathrm{Im}_b\,|} + |E| \\ &\approx \frac{1}{\left(\epsilon^{1/4}D^{1/2} + \left|\frac{t}{\sqrt{\epsilon}}\right|\right)^3 |\,\mathrm{Im}_b\,|} + |E| \\ &\approx \frac{1}{\left(\epsilon^{1/4}D^{1/2} + \sqrt{\sqrt{\epsilon^2+t^2}-\epsilon}\right)^3 |\,\mathrm{Im}_b\,|} + |E|.\end{aligned}$$

For the last line, we use that fact that in this case $\sqrt{\sqrt{\epsilon^2+t^2}-\epsilon} \approx |t|/\sqrt{\epsilon}$, which is part of Claim 4.3.

Case 2: $\epsilon \le 2|t|$. Now we use (4.17). Using the cases which precede that equation, we see that $\sqrt{|t|}+|y| \approx \sqrt{|t|}+|\operatorname{Im}_b|$. So

$$|S| \lesssim \frac{1}{(\epsilon+|t|)^{3/2}\left(\sqrt{|t|}+|\operatorname{Im}_b|\right)} + |E|$$
$$\approx \frac{1}{\left(\epsilon^{3/2}+|t|^{3/2}\right)\left(\sqrt{|t|}+|\operatorname{Im}_b|\right)} + |E|$$
$$\lesssim \frac{1}{\left((\epsilon D^2)^{3/4}+\left(\sqrt{\sqrt{\epsilon^2+t^2}-\epsilon}\right)^3\right)|\operatorname{Im}_b|} + |E|.$$

Here we used the fact that in this case $\sqrt{\sqrt{\epsilon^2+t^2}-\epsilon} \approx \sqrt{|t|}$. This completes the proof of Corollary 1.5. □

4.3. Proof of Lemma 4.1.

Proof. By Lemma 2.1 we need only look at

$$S_M = \int_0^\infty e^{-\tau(\epsilon+it)} \int_{-\infty}^\infty \frac{e^{\frac{-\eta^2}{2\tau}} e^{\eta(x+r)} e^{i\eta y}}{e^{-2\eta}+e^{2\eta}}\, d\eta \sqrt{\tau}\, d\tau.$$

First assume $t=0$. By changing the order of integration we get

$$S_M = \int_{-\infty}^\infty e^{i\eta y} \int_0^\infty e^{-\epsilon\tau} e^{-\frac{\eta^2}{2\tau}} \sqrt{\tau} d\tau e^{\eta(x+r)} \frac{1}{e^{2\eta}+e^{-2\eta}}\, d\eta \tag{4.20}$$

$$= \frac{c}{\epsilon\sqrt{\epsilon}} \int_{-\infty}^\infty e^{i\eta y} \int_0^\infty e^{-\tau} e^{-\frac{\eta^2\epsilon}{2\tau}} \sqrt{\tau}\, d\tau e^{\eta(x+r)} \frac{1}{e^{2\eta}+e^{-2\eta}}\, d\eta, \tag{4.21}$$

where the second line is obtained by putting $\tau' = \epsilon\tau$. Then we use

$$\int_0^\infty e^{-\tau} e^{-\frac{\eta^2}{4\tau}} \sqrt{\tau}\, d\tau = \frac{\pi}{2}(|\eta|+1)e^{-|\eta|},$$

which is proved by an argument similar to one in [16], and

$$\int_{-\infty}^\infty \frac{e^{i\eta x}}{(1+x^2)^2}\, dx = \frac{\pi}{2}(|\eta|+1)e^{-|\eta|},$$

which is proved by doing a contour integral. We get to

$$S_M = \frac{c}{\epsilon^{3/2}} \int_{-\infty}^\infty e^{i\eta y}\left(\sqrt{2\epsilon}|\eta|+1\right) e^{-\sqrt{2\epsilon}|\eta|} e^{n(x+r)} \frac{1}{e^{2\eta}+e^{-2\eta}}\, d\eta$$
$$= \frac{c}{\epsilon^{3/2}}\Bigg[\int_{-\infty}^0 e^{i\eta y}\left(-\sqrt{2\epsilon}\,\eta+1\right) e^{\eta\left(\sqrt{2\epsilon}+(x+r+2)\right)} \frac{1}{e^{4\eta}+1} d\eta$$
$$+ \int_0^\infty e^{i\eta y}\left(\sqrt{2\epsilon}\,\eta+1\right) e^{-\eta\left(\sqrt{2\epsilon}-(x+r-2)\right)} \frac{1}{1+e^{-4\eta}} d\eta\Bigg].$$

Then we write

$$\frac{1}{e^{-4|\eta|}+1} = 1 - \frac{e^{-4|\eta|}}{1+e^{-4|\eta|}}.$$

The term corresponding to the term 1 will be called S_{M_1}, the remaining term E_1.

For S_{M_1}: By a change of variables and the Identity Theorem, we get

$$\begin{aligned}
&S_{M_1}\\
&= \frac{c}{\epsilon^{3/2}}\Bigg[\sqrt{2\epsilon}\frac{1}{(iy+\sqrt{2\epsilon}+(x+r-2)+4)^2} + \frac{1}{(iy+\sqrt{2\epsilon}+(x+r-2)+4)}\\
&\qquad +\sqrt{2\epsilon}\frac{1}{(-iy+\sqrt{2\epsilon}-(x+r-2))^2} + \frac{1}{(-iy+\sqrt{2\epsilon}-(x+r-2))^2}\Bigg]\\
&= c\frac{\left[(4(2\epsilon))^{3/2}+16(2\epsilon)+16(\sqrt{2\epsilon})+4(x-r)^2+16D+4y^2-i(8y(x+r))\right]}{\epsilon^{3/2}\left[(x-r)^2+y^2+4D-i(2y(x+r))\right]^2},
\end{aligned}$$

where

$$D = \sqrt{2\epsilon}-(x+r-2) \approx \left\{\begin{array}{ll} \frac{(x-r)^2}{\sqrt{\epsilon}} & \text{if } x+r-2\geq 0\\ \sqrt{\epsilon} & \text{if } x+r-2\leq 0\end{array}\right\} \lesssim \sqrt{\epsilon}.$$

Then we have

$$|S_{M_1}| \approx \frac{1}{\epsilon^{3/2}}\frac{\left[\epsilon^{3/2}+\epsilon^{1/2}+y^2+|y(x+r)|\right]}{\left[(x-r)^2+y^2+D+|y(x+r)|\right]^2}.$$

Here we used the fact that $D^2, (x-r)^2 \leq c\epsilon$. We postpone the discussion of the error E_1 for now.

Next allow $|t|\geq 0$. Now in place of ϵ, we have $\epsilon+it$. To get from (4.20) to (4.21), we will now need a change of contour. We then proceed along the same lines as above, with an additional use of the Identity theorem. We arrive at

$$S_M = \frac{c}{(\epsilon+it)^{3/2}}\int_{-\infty}^{\infty} e^{i\eta y}\frac{(\sqrt{2(\epsilon+it)}|\eta|+1)e^{-\sqrt{2(\epsilon+it)}|\eta|}}{e^{2\eta}+e^{-2\eta}}e^{\eta(x+r)}\,d\eta.$$

As above, we write

$$\frac{1}{e^{2\eta}+e^{-2\eta}} = e^{-2|\eta|}\frac{1}{e^{-4|\eta|}+1} = e^{-2|\eta|}\left(1-\frac{e^{-4|\eta|}}{e^{-4|\eta|}+1}\right).$$

Taking care with the complex $\epsilon+it$, we arrive at

$$|S_{M_1}| \approx \frac{\left[\left(\mathrm{Re}((\epsilon+it)^{3/2})+\mathrm{Re}((\epsilon+it)^{1/2})+D+D_{\mathrm{e}}+y^2\right)^2+(\mathrm{Im}_a)^2\right]^{1/2}}{(\epsilon+|t|)^{3/2}\left[\left((x-r)^2+y^2+D+D_{\mathrm{e}}\right)^2+(\mathrm{Im}_b)^2\right]},$$

as claimed.

The error E_1. Using

$$\left| e^{-\sqrt{2(\epsilon+it)}|\eta|} e^{\eta(x+r)} e^{-2|\eta|} \right| \leq 1,$$

we can easily show that

$$|E_1| \leq \frac{c}{\left|(\epsilon+it)^{3/2}\right|} \left|\sqrt{2(\epsilon+it)}+1\right|. \tag{4.22}$$

If $|\epsilon+it|>2$, this is bounded by a constant. If $|\epsilon+it|\leq 2$, integration by parts shows that in addition to (4.22), we have

$$|E_1| \leq \frac{c}{\left|(\epsilon+it)^{3/2}\right| y^2}.$$

This finishes the proof of the lemma. □

4.4. Proof of Lemma 4.2.

4.4.1. *t dominant.* We assume $x, r \geq 1/2$, so $\epsilon = (x-1)^2+(r-1)^2$. In this subsection we assume $|t| \geq c|x+r-2|^2$. Then we show that $|S| \leq c_1 \frac{1}{t^2} + |E|$. In fact, we show even more:

$$|S| \leq c_1 \frac{1}{((x-r)^2+|t|)^2} + |E|. \tag{4.23}$$

We will be able to get our result for any positive c, though the constant c_1 above depends on c. Since $|x+r-2|^2 \leq \epsilon$, this result is true if $t \geq c\epsilon$.

Proof. We again use Lemma 2.1. We also use the easily shown fact that

$$\left| \int_0^{\frac{1}{(x-r)^2+|t|}} e^{-\epsilon\tau} \int_{-\infty}^{\infty} e^{\eta(x+r)} (Mc_0+E_0) d\tau \right| \leq c\left(\frac{1}{((x-r)^2+|t|)^2} + 1 \right).$$

So we need only look at

$$\int_{\frac{1}{(x-r)^2+|t|}}^{\infty} e^{-\tau(\epsilon+it)} \int_{-\infty}^{\infty} \frac{e^{\frac{-\eta^2}{2\tau}} e^{\eta(x+r-2)} e^{i\eta y}}{e^{-4\eta}+1}\, d\eta \sqrt{\tau}\, d\tau$$

$$= \lim_{\delta\to 0} \int_{\frac{1}{(x-r)^2+|t|}}^{\infty} e^{-\tau\left(\frac{(x-r)^2}{2}+it\right)} \int_{-\infty}^{\infty} e^{-\frac{(\eta-\tau(x+r-2))^2}{2\tau}} \frac{e^{i\eta y}}{e^{-4\eta}+1}\, d\eta \sqrt{\tau}\chi(\delta\tau)\, d\tau,$$

where $\chi(\tau)$ is a Schwartz cutoff function which is 1 for $\tau \leq 1$ and 0 for $\tau \geq 2$. Now we will integrate by parts:

$$\int_{\frac{1}{(x-r)^2+|t|}}^{\infty} \underbrace{e^{-\tau\left(\frac{(x-r)^2}{2}+it\right)}}_{dv} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{(\eta-\tau(x+r-2))^2}{2\tau}} \frac{e^{i\eta y}}{e^{-4\eta}+1}\, d\eta \sqrt{\tau}\chi(\delta\tau)}_{u}\, d\tau.$$

One can show that

$$|u| \le c\tau\chi(\delta\tau),$$
$$\left|\frac{d^p u}{d\tau^p}\right| \le c\frac{1}{\tau^{p-1}}(\sqrt{\tau}|x+r-2|+1)^p\chi(\delta\tau).$$

Using this and the assumption that $|t| \ge c|x+r-2|^2$, we integrate by parts five times to obtain the result. □

4.4.2. *y dominant.* Here we assume $|y| \ge \sqrt{|t|}, \sqrt{\epsilon}$, and as above, $\epsilon \le c|t|$ for positive c. We then prove $|S| \le c_1 \frac{1}{|t|^{3/2}|y|}$. In fact, we prove more:

$$|S_M| \le c\frac{1}{((x-r)^2+|t|)^{3/2}|y|}.$$

Proof. We again use Lemma 2.1, so we can start with

$$S_M = \int_0^\infty e^{-\tau\left(\frac{(x-r)^2}{2}+it\right)} \int_{-\infty}^\infty e^{-\frac{(\eta-\tau(x+r-2))^2}{2\tau}} \frac{e^{i\eta y}}{e^{-4\eta}+1}\, d\eta\sqrt{\tau}\, d\tau. \tag{4.24}$$

We could use the above section, but we can get a better result by doing the following if $|y| \ge \sqrt{|t|}$. This explanation will be similar to the one in Section 4.4.1. The only difference is that we will bring down a power of $|y|$ first, and then do integration by parts in the τ-integral. We look at the inner integral in (4.24). We will integrate by parts:

$$\int_{-\infty}^\infty \underbrace{e^{i\eta y}}_{dv} \underbrace{\frac{e^{-\frac{1}{2\tau}(\eta-\tau(x+r-2))^2}}{1+e^{-4\eta}}}_{u}\, d\eta.$$

The boundary terms are 0, so we get for χ a cutoff as above that S_M is a constant times the limit as δ goes to 0 of

$$\int_0^\infty e^{-\tau\left(\frac{(x-r)^2}{2}+it\right)}$$
$$\times \int_{-\infty}^\infty \frac{e^{i\eta y}}{y} e^{-\frac{(\eta-\tau(x+r-2))^2}{2\tau}} \left[\frac{\frac{2(\eta-\tau(x+r-2))}{2\tau}}{1+e^{-4\eta}} - \frac{4e^{-4\eta}}{(1+e^{-4\eta})^2}\right] d\eta\sqrt{\tau}\chi(\delta\tau)d\tau.$$

A quick calculation shows

$$\left|\int_0^{\frac{1}{(x-r)^2+|t|}} (\cdot)d\tau\right| \le c\int_0^{\frac{1}{(x-r)^2+|t|}} \frac{1}{|y|}\sqrt{\tau}d\tau = \frac{c}{|y|((x-r)^2+|t|)^{3/2}},$$

and so we need only do integration by parts for

$$\int_{\frac{1}{(x-r)^2+|t|}}^{\infty} \underbrace{e^{-\tau\left(\frac{(x-r)^2}{2}+it\right)}}_{dv}$$
$$\times \underbrace{\int_{-\infty}^{\infty} \frac{e^{i\eta y}}{y} e^{-\frac{(\eta-\tau(x+r-2))^2}{2\tau}} \left[\frac{\frac{2(\eta-\tau(x+r-2))}{2\tau}}{1+e^{-4\eta}} - \frac{4e^{-4\eta}}{(1+e^{-4\eta})^2} \right] d\eta \sqrt{\tau}\chi(\delta\tau)}_{u} \, d\tau.$$

In a manner similar to the above, one can show that

$$|u| \leq c\sqrt{\tau}\frac{1}{|y|}\chi(\delta\tau),$$
$$\left|\frac{d^p u}{d\tau^p}\right| \leq c\frac{1}{|y|\tau^p}\sqrt{\tau}(\sqrt{\tau}|x+r-2|+1)^p\chi(\delta\tau).$$

We then use this as we integrate by parts four times, and the result follows. □

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