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ON GENERALIZATIONS OF A PROBLEM OF DIOPHANTUS

YANN BUGEAUD AND KATALIN GYARMATI

Abstract. Let $k \geq 2$ be an integer and let $\mathcal{A}$ and $\mathcal{B}$ be two sets of integers. We give upper bounds for the number of perfect k-th powers of the form $ab+1$, with a in $\mathcal{A}$ and b in $\mathcal{B}$. We further investigate several related questions.

1. Introduction

The Greek mathematician Diophantus of Alexandria noted that the rational numbers 1/16, 33/16, 17/4, and 105/16 have the following property: the product of any two of them increased by 1 is a square of a rational number. Later, Fermat found that the set of four positive integers $\{1, 3, 8, 120\}$ shares the same property. A finite set of m positive integers $a_1 < \cdots < a_m$ such that $a_i a_j + 1$ is a perfect square whenever $1 \leq i < j \leq m$ is commonly called a Diophantine m-tuple. A famous conjecture asserts that there does not exist a Diophantine 5-tuple. This question has been nearly solved in a remarkable paper by Dujella [3], who proved that there does not exist a Diophantine 6-tuple and that the elements of any Diophantine 5-tuple are less than $10^{10^{26}}$. We direct the reader to [3] for further references.

This problem was extended to higher powers by Bugeaud and Dujella [2]. They proved that if $k \geq 3$ is a given integer and $\mathcal{A}$ is a set of positive integers such that $aa'+1$ is a perfect k-th power for all distinct a and a' in $\mathcal{A}$, then $\mathcal{A}$ has at most 7 elements. In the present paper, we investigate related questions and, among other results, we provide, for an arbitrary set $\mathcal{A}$ of positive integers, estimates for the number $n_{\mathcal{A}}$ of pairs (a, a') with a, a' in $\mathcal{A}$ such that $aa' + 1$ is a perfect k-th power. It is clear that, for all m, there exists a set $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ such that the $m-1$ integers $a_1a_2+1,\ a_2a_3+1, \ldots,\ a_{m-1}a_m+1$ are perfect k-th powers; for such sets $\mathcal{A}$, the number $n_{\mathcal{A}}$ is at least equal to the cardinality of $\mathcal{A}$ minus one. In the present paper, we combine results from

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[2] with graph theory (see Theorem 1) to give an upper estimate for $n_{\mathcal{A}}$ that is much sharper than the trivial bound (which is the square of the cardinality of $\mathcal{A}$).

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2. Results

Throughout this paper, the cardinality of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. Given an integer $k \geq 3$ and two finite sets $\mathcal{A}$ and $\mathcal{B}$, our first result provides us with an upper bound for the number of perfect k-th powers of the form $ab+1$, with a in $\mathcal{A}$ and b in $\mathcal{B}$.

Theorem 1. *Let $k \geq 3$ be an integer. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of positive integers with $|\mathcal{A}| \geq |\mathcal{B}|$ and set*

$$\mathcal{S} = \{(a,b) : \ a \in \mathcal{A}, b \in \mathcal{B}, \ ab+1 \quad \textit{is a } k\textit{-th power}\}.$$

We then have

$$|\mathcal{S}| \leq 2 \cdot 6^{1/3} \, |\mathcal{A}| \cdot |\mathcal{B}|^{2/3} + 4 \, |\mathcal{A}| \leq 7.64 \, |\mathcal{A}| \cdot |\mathcal{B}|^{2/3} \quad \textit{if } k = 3,$$
$$|\mathcal{S}| \leq 2\sqrt{3} \, |\mathcal{A}| \cdot |\mathcal{B}|^{1/2} + 2 \, |\mathcal{A}| \leq 5.47 \, |\mathcal{A}| \cdot |\mathcal{B}|^{1/2} \quad \textit{if } k \geq 4.$$

It follows from Theorem 1 that, if $\mathcal{A}$ and $\mathcal{B}$ have same cardinality (in particular, if $\mathcal{A} = \mathcal{B}$), then the number of pairs (a, b) with a in $\mathcal{A}$ and b in $\mathcal{B}$ such that $ab + 1$ is a k-th power for a fixed k is less than $8 \, |\mathcal{A}|^{5/3}$ if $k = 3$ and is less than $6 \, |\mathcal{A}|^{3/2}$ if $k \geq 4$. We further notice that there is no positive integer a such that $a^2 + 1$ is a perfect power, a result due to V. A. Lebesgue [9].

We were unable to treat the case $k = 2$ in Theorem 1. However, if the sets $\mathcal{A}$ and $\mathcal{B}$ are equal, it is possible to slightly improve the trivial estimate.

Theorem 2. *Let $\mathcal{A}$ be a set of positive integers with $|\mathcal{A}| \geq 6$. Then the set*

$$\{(a, a') : \ a, a' \in \mathcal{A}, \ a > a', \ aa' + 1 \quad \textit{is a square}\}$$

has at most $0.4 \, |\mathcal{A}|^2$ elements.

The results from [2] also enable us to improve upon Theorems 1 and 2 of Gyarmati, Sárközy and Stewart [6]. For any integer $k \geq 2$, set

$$V_k = \{x^\ell : x \in \mathbb{Z}^+ \text{ and } 2 \leq \ell \leq k\}.$$

THEOREM 3. *Let $k \geq 2$ be an integer. Let $\mathcal{A}$ be a set of positive integers with the property that $aa' + 1$ is in V_k whenever a and a' are distinct integers from $\mathcal{A}$. We then have*

$$|\mathcal{A}| < 85000\left(\frac{k}{\log k}\right)^2. \tag{1}$$

Theorem 3 considerably improves Theorem 2 of [6], where the authors obtained the upper bound

$$|\mathcal{A}| < 160\left(\frac{k}{\log k}\right)^2 \log\log\left(\max_{a\in\mathcal{A}} a\right), \tag{2}$$

instead of (1). We point out that the right-hand side of (2) depends on the maximum of the elements of $\mathcal{A}$, unlike the right-hand side of (1).

The next result follows from Theorem 3 by noticing that if x^k is a positive integer in $\{2, \ldots, N\}$, then k is at most equal to $(\log N)/(\log 2)$.

COROLLARY 1. *Let $\mathcal{A}$ be a set of positive integers at most equal to N. If $aa' + 1$ is a perfect power for all distinct integers a and a' in $\mathcal{A}$, then we have*

$$|\mathcal{A}| < 177000\left(\frac{\log N}{\log\log N}\right)^2. \tag{3}$$

Corollary 1 slightly refines Theorem 1 of [6], where the upper bound

$$|\mathcal{A}| < 340\,\frac{(\log N)^2}{\log\log N}$$

is proved, instead of (3).

In Theorem 3, we make the strong assumption that $aa' + 1$ is *always* a power. Our method also provides new results under the weaker assumption that $aa' + 1$ is a power for *many* pairs (a, a') in $\mathcal{A}^2$. For any integer $k \geq 3$, set

$$W_k = \{x^\ell : x \in \mathbb{Z}^+ \text{ and } 3 \leq \ell \leq k\},$$

and, if $k \geq 4$, define

$$X_k = \{x^\ell : x \in \mathbb{Z}^+ \text{ and } 4 \leq \ell \leq k\}.$$

THEOREM 4. *Let $k \geq 3$ be an integer. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of positive integers. If $ab + 1$ is in W_k for at least $15(\max\{|\mathcal{A}|, |\mathcal{B}|\})^{5/3}$ pairs (a, b) with a in $\mathcal{A}$ and b in $\mathcal{B}$, then*

$$\max\{|\mathcal{A}|, |\mathcal{B}|\} < \left(\frac{k}{\log k}\right)^6.$$

If $k \geq 4$ and if there exists $\alpha > 3/2$ such that $ab+1$ is in X_k for at least $(\max\{|\mathcal{A}|, |\mathcal{B}|\})^\alpha$ pairs (a,b) with a in $\mathcal{A}$ and b in $\mathcal{B}$, then

$$\max\{|\mathcal{A}|, |\mathcal{B}|\} < c(\alpha)\left(\frac{k}{\log k}\right)^{2/(2\alpha-3)},$$

for a suitable constant $c(\alpha)$, depending only on α.

Erdős [4] and Moser [12] posed the following additive analogue of the problem of Diophantus: Is it true that, for all m, there are integers $a_1 < a_2 < \cdots < a_m$ such that $a_i + a_j$ is a perfect square for all $i \neq j$? Rivat, Sárközy and Stewart [10] proved that, if $\mathcal{A}$ is contained in $\{1, 2, \ldots, N\}$ and $a + a'$ is a perfect square for all $a, a' \in \mathcal{A}$ with $a \neq a'$, then $|\mathcal{A}| \ll \log N$. We can also investigate what happens if the sums $a + a'$ are replaced by other polynomials in a and a', and perfect squares by higher powers (see, e.g., Gyarmati, Sárközy and Stewart [7]). First we study the case of $a - a'$. For a given integer $k \geq 3$ and an arbitrary set $\mathcal{A}$ of distinct positive integers, the set

$$\{(a, a') : \ a, a' \in \mathcal{A}, \ a > a', \ a - a' \ \text{ is a } k\text{-th power}\}$$

has at most $0.25\,|\mathcal{A}|^2$ elements, since the related graph (the graph whose vertices are the elements of $\mathcal{A}$ and two vertices are joined if, and only if, their difference is a k-th power) does not contain a triangle (apply Lemma 3 below). Indeed, otherwise we would have three elements a_1, a_2, a_3 in $\mathcal{A}$ such that $a_1 - a_2 = x^k$, $a_2 - a_3 = y^k$, $a_3 - a_1 = z^k$ for some integers x, y, z, and so $x^k + y^k + z^k = 0$. By Fermat's Last Theorem [13] this is not possible.

So far, we have studied problems for which shifted products $aa' + 1$ are perfect powers for many pairs (a, a') in $\mathcal{A}^2$. Theorem 5 below deals with the polynomial $a^2 + {a'}^2$.

Theorem 5. *There exists a positive integer N_0 with the following property: For any integer $N \geq N_0$ and any set $\mathcal{A}$ contained in $\{1, 2, \ldots, N\}$ such that $a^2 + {a'}^2$ is a perfect square for all $a, a' \in \mathcal{A}$, $a \neq a'$, we have $|\mathcal{A}| \leq 4(\log N)^{1/2}$.*

The remainder of the paper is organized as follows. Section 3 is devoted to auxiliary results taken from [2] and to classical results from graph theory. Proofs of Theorems 1–4 are given in Section 4, whereas Theorem 5 is established in Section 5.

3. Auxiliary results

We shall need the following lemmas, extracted from [2]. Their proofs rest heavily on Baker's theory of linear forms in logarithms.

LEMMA 1. *Assume that the integers* $0 < a < b < c < d_1 < \cdots < d_m$ *are such that* $ad_i + 1$, $bd_i + 1$ *and* $cd_i + 1$ *are perfect cubes for any* $1 \leq i \leq m$. *Then we have* $m \leq 6$.

Proof. This is [2, Theorem 3]. □

LEMMA 2. *Let* $k \geq 4$ *be an integer. Assume that the integers* $0 < a < b < c_1 < \cdots < c_m$ *are such that* $ac_i + 1$ *and* $bc_i + 1$ *are perfect* k*-th powers for any* $1 \leq i \leq m$. *Then there exists an effectively computable constant* $C_1(k)$ *depending only on* k, *such that* $m \leq C_1(k)$. *More precisely, we may take* $C_1(4) = 3$, $C_1(k) = 2$ *for* $5 \leq k \leq 176$, $C_1(k) = 1$ *for* $177 \leq k$.

Proof. This is [2, Theorems 1 and 2]. □

We further need two results from graph theory. Throughout this paper, for a graph G, we denote by $v(G)$ the number of its vertices and by $e(G)$ the number of its edges.

LEMMA 3. *Let* G *be a graph on* n *vertices having at least*

$$\frac{r-2}{2(r-1)}\, n^2$$

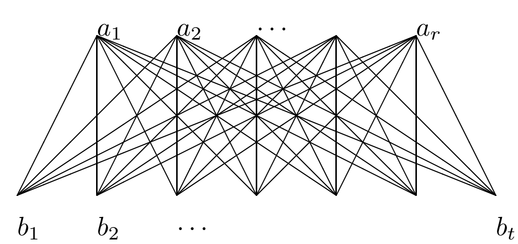
edges for some positive integer $r \geq 3$. *Then* G *contains a complete subgraph on* r *edges.*

Proof. This is a consequence of Turán's graph theorem (see, for example, [1, p. 294, Theorem 1.1]) combined with the upper bound

$$\sum_{0 \leq i < j < r-1} \left[\frac{n+i}{r-1}\right] \left[\frac{n+j}{r-1}\right] \leq \frac{r-2}{2(r-1)}\, n^2,$$

which follows from the method of Lagrange multipliers. □

LEMMA 4. *Assume that* $G(V_1, V_2)$ *is a bipartite graph with* $|V_1| = n \leq |V_2| = m$, *and the vertices are labelled by positive real numbers. Suppose that* $G(V_1, V_2)$ *does not contain a* $K_{r,t}$ *subgraph* G_0 *of the form*

$G_0 =$ 

with $a_i < b_j$ *for all* $1 \le i \le r$, $1 \le j \le t$ *(where the a's belong to* V_1 *and the b's belong to* V_2 *or vice versa). Then* G *has at most*

$$e(G) \le 2(t-1)^{1/r} m n^{1-1/r} + 2(r-1)m$$

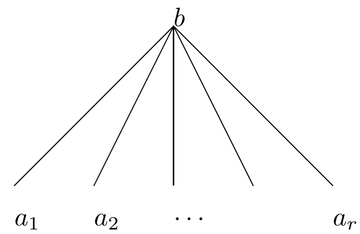
edges.

Proof. The proof is very similar to that of the Kővári–Sós–Turán theorem [8]. For any vertex x, set

$$d_x = \left|\{y \in v(G) :\ y < x,\ (x, y) \text{ is an edge in } G\}\right|,$$

$e_1 = \sum_{x \in V_1} d_x$ and $e_2 = \sum_{x \in V_2} d_x$. Then we have $e(G) = e_1 + e_2$. First we get an upper bound for e_1.

Denote by H the number of subgraphs G_1 of G of the form

$G_1 =$ 

with $b \in V_1$, $a_i \in V_2$ and $b > a_i$ for $1 \le i \le r$. Since the graph G does not contain G_0 we have

$$H \le (t-1)\binom{m}{r}, \tag{4}$$

by Dirichlet's *Schubfachprinzip*. We further have

$$H = \sum_{x \in V_1} \binom{d_x}{r}$$

and, by the Cauchy-Schwarz inequality, we get

$$H \ge n \binom{e_1/n}{r} \tag{5}$$

Combining (4) and (5) yields

$$e_1 \le (t-1)^{1/r} m n^{1-1/r} + (r-1)n,$$

and, similarly, exchanging the roles of V_1 and V_2 in the definition of G_1 ($b \in V_2$, $a_i \in V_1$ and $b > a_i$ for $1 \le i \le r$), we obtain

$$e_2 \le (t-1)^{1/r} n m^{1-1/r} + (r-1)m.$$

It then follows that

$$\begin{aligned} e(G) = e_1 + e_2 &\le 2\max\{(t-1)^{1/r} m n^{1-1/r},\ (t-1)^{1/r} n m^{1-1/r}\} + 2(r-1)m \\ &\le 2(t-1)^{1/r} m n^{1-1/r} + 2(r-1)m, \end{aligned}$$

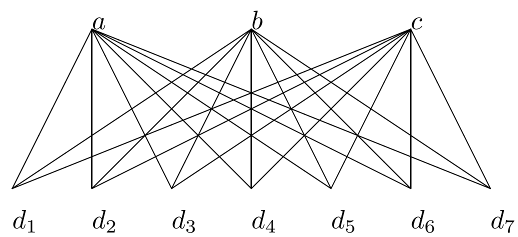
which completes the proof of the lemma. □

4. Proofs of Theorems 1–4

Proof of Theorem 1. Let $k \geq 2$ be an integer. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ denote the elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. We define a graph G on the $n+m$ vertices $v_1, \ldots, v_{n+m}$ in the following way. For any integers i and j with $1 \leq i \leq n$ and $1 \leq j \leq m$, an edge joins the vertices v_i and v_{n+j} if, and only if, $a_i b_j + 1$ is a perfect k-th power. No edge joins two vertices v_i and v_j if either $1 \leq i, j \leq n$ or $n+1 \leq i, j \leq n+m$.

For $k = 3$, Lemma 1 implies that G does not contain a subgraph G_0 defined by

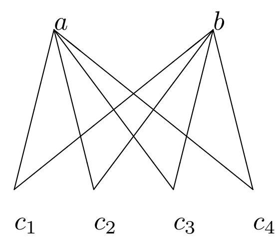
$G_0 =$



with $a < b < c < d_i$ for $1 \leq i \leq 7$.

When $k \geq 4$, Lemma 2 implies that the graph G does not contain a subgraph G_0 defined by

$G_0 =$



with $a < b < c_i$ for $1 \leq i \leq 4$.

These two remarks combined with Lemma 4 give Theorem 1. □

Proof of Theorem 2. Let $a_1, a_2, \ldots, a_n$ denote the elements of $\mathcal{A}$. We define a graph G on n vertices $v_1, \ldots, v_n$ as in the proof of Theorem 1. For any integers i and j with $1 \leq i < j \leq n$, an edge joins the vertices v_i and v_j if, and only if, $a_i a_j + 1$ is a square. By Dujella's result [3] recalled in the Introduction, the graph G does not contain K_6 as a subgraph. Lemma 3 then implies that G has at most $0.4n^2 = 0.4\,|\mathcal{A}|^2$ edges. This proves Theorem 2. □

Proof of Theorem 3. The proof of Theorem 3 is very similar to that of Theorem 2 from [6]. However, instead of introducing the sets $\mathcal{A}_m$ as in [6], we use Theorem 1 and we work directly with the complete graph G labelled by the elements of $\mathcal{A}$. We colour the edge joining the vertices a and a' by the smallest integer ℓ larger than one for which $aa'+1$ is a perfect ℓ-th power. Thus, each edge is coloured by a prime number. For $i = 2, 3, \dots, k$, let b_i denote the number of edges of G which are coloured with the integer i. Set $n = |\mathcal{A}|$ and assume that $n \geq 85000(k/\log k)^2$. By Theorem 2, we have $b_2 \leq 0.4n^2$. Thus $k \geq 3$ and

$$b_3 + \cdots + b_k \geq \frac{n(n-1)}{2} - \frac{2n^2}{5} = \frac{n^2}{10} - \frac{n}{2}.$$

Furthermore, we infer from Theorem 1 that $b_3 \leq 7.64n^{5/3}$. Consequently, we have $k \geq 5$. By Corollary 2 of Rosser and Schoenfeld [11], the number of prime numbers up to k is at most $(5k)/(4\log k)$. Thus, there exists a prime number p with $5 \leq p \leq k$ such that

$$b_p \geq \frac{4\log k}{5k}\left(\frac{n^2}{10} - \frac{n}{2} - 7.64n^{5/3}\right) \geq 5.5n^{3/2},$$

since $n > 85000(k/\log k)^2$. Let G_p be the subgraph of G whose vertices are those of G and whose edges are the edges of G coloured by the prime p. Theorem 1 implies that $b_p \leq 5.47n^{3/2}$, which is the desired contradiction. □

Proof of Theorem 4. Let $k \geq 3$ be an integer. Let $a_1, \dots, a_n$ and $b_1, \dots, b_m$ denote the elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. For simplicity, we assume that $m \geq n$. We define a graph G on the $n+m$ vertices $v_1, \dots, v_{n+m}$ in the following way. No edge joins two vertices v_i and v_j if either $1 \leq i, j \leq n$ or $n+1 \leq i, j \leq n+m$. For any integers i and j with $1 \leq i \leq n$ and $1 \leq j \leq m$, an edge joins the vertices v_i and v_{n+j} if, and only if, $a_i b_j + 1$ is a perfect cube or a higher power. We colour it with the smallest integer ℓ at least equal to 3 such that $ab+1$ is a perfect ℓ-th power. Observe that each edge is coloured by 4 or by an odd prime number. For any integer $i = 3, \dots, k$, denote by b_i the number of edges of G which are coloured by the integer i. Denoting by N the number of edges of G, we have

$$b_3 + \cdots + b_k = N.$$

By Theorem 1, we have $b_3 \leq 7.64\ m^{5/3}$. Since, by assumption, N is greater than $15\ m^{5/3}$, we get

$$b_4 + \cdots + b_k = N - b_3 \geq 7.36\ m^{5/3}.$$

Arguing now as in [6] and in the proof of Theorem 3, we infer that there exists an integer p with $4 \leq p \leq k$ such that

$$b_p \geq \left(\frac{4\log k}{5k}\right) 7.36\ m^{5/3} > 5.88\ m^{5/3}\frac{\log k}{k}.$$

By Theorem 1, we have $b_p \leq 5.47\ m^{3/2}$. Hence the desired result follows.

The proof of the second assertion of Theorem 4 follows along the same lines, but in this case we obtain

$$b_4 + \cdots + b_k = N \geq m^{\alpha}.$$

Thus, there exists an integer p with $4 \leq p \leq k$ such that

$$b_p \geq \frac{4 \log k}{5k} m^{\alpha}.$$

By Theorem 1 we have $b_p \leq 5.47 m^{3/2}$. Hence the desired result follows. □

5. Proof of Theorem 5

We begin by proving an auxiliary lemma.

LEMMA 5. *For any sufficiently large integer N and any set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ contained in $\{1, 2, \ldots, N\}$, there exists a prime p such that $p \equiv \pm 3 \pmod 8$, p divides at most $[n/3]$ numbers from the set $\mathcal{A}$, and p satisfies*

$$p \leq \frac{3}{\log 1.6} \log N.$$

Proof. We argue by contradiction. Suppose that all prime numbers $p \equiv \pm 3 \pmod 8$ with $p \leq \frac{3}{\log 1.6} \log N$ divide at least $[n/3]$ numbers from the set $\mathcal{A}$. Each of these primes satisfies

$$p^{[n/3]} \mid a_1 a_2 \ldots a_n.$$

Hence we get

$$\Big(\prod_{\substack{p \leq \frac{3}{\log 1.6} \log N \\ p \equiv \pm 3 \pmod 8}} p \Big)^{[n/3]} \mid a_1 a_2 \ldots a_n. \tag{6}$$

It follows from the prime number theorem for arithmetic progressions of small moduli that for all sufficiently large x we have

$$1.6^x < \prod_{p \leq x,\ p \equiv \pm 3 \pmod 8} p.$$

Thus, by (6), we get

$$N^n \leq \Big(1.6^{\frac{3}{\log 1.6} \log N}\Big)^{[n/3]} < \Big(\prod_{\substack{p \leq \frac{3}{\log 1.6} \log N \\ p \equiv \pm 3 \pmod 8}} p \Big)^{[n/3]} \leq a_1 a_2 \ldots a_n \leq N^n,$$

which is a contradiction. □

Let N and $\mathcal{A}$ be as in the statement of Lemma 5, and let p be a prime which satisfies the conclusion of that lemma. Assume that $a^2 + {a'}^2$ is a square for any a, a' in $\mathcal{A}$ with $a \neq a'$. Let us consider the numbers from the set $\mathcal{A}$ which are not divisible by p. These are $b_1, b_2, \ldots, b_t$, $t \geq \lceil 2n/3 \rceil$. If $b_i^2 \equiv b_j^2 \pmod p$ for $i \neq j$, then $b_i^2 + b_j^2 \equiv 2b_i^2$ is a quadratic residue modulo p. Therefore 2 is also a quadratic residue modulo p. But this contradicts the assumption $p \equiv \pm 3 \pmod 8$. Thus $b_1^2, b_2^2, \ldots, b_t^2$ are incongruent modulo p.

We further need the following lemma.

Lemma 6. *Let p be a prime number. Let $\mathcal{B}$ be a set of positive integers coprime with p and whose residues modulo p are all distinct. Assume that for all $b, b' \in \mathcal{B}$ with $b \neq b'$ the number $b + b'$ is a perfect square modulo p. Then we have $|\mathcal{B}| \leq p^{1/2} + 3$.*

Proof of Lemma 6. See [5]. □

We now have all the tools for the proof of Theorem 5. The sum of any two elements of the set $\{b_1^2, b_2^2, \ldots, b_t^2\}$ is a perfect square, so we get by Lemma 5 and Lemma 6 that

$$2n/3 \leq t \leq p^{1/2} + 3 \leq \left(\frac{3}{\log 1.6} \log N \right)^{1/2} + 3.$$

From this we obtain

$$|\mathcal{A}| = n \leq 4(\log N)^{1/2},$$

for N sufficiently large. This completes the proof of Theorem 5. □

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Yann Bugeaud, Université Louis Pasteur, U. F. R. de mathématiques, 7, rue René Descartes, 67084 Strasbourg, France

E-mail address: bugeaud@math.u-strasbg.fr

Katalin Gyarmati, University Eötvös Lorand, Algebra and Number Theory Department, Pázmány Péter sétány 1/c, H-1117 Budapest, Hungary

E-mail address: gykati@cs.elte.hu