CONVOLUTIONS OF EQUICONTRACTIVE SELF-SIMILAR MEASURES ON THE LINE

DE-JUN FENG, NHU T. NGUYEN, AND TONGHUI WANG

ABSTRACT. Let μ be a self-similar measure on $\mathbb{R}$ generated by an equicontractive iterated function system. We prove that the Hausdorff dimension of μ^{*n} tends to 1 as n tends to infinity, where μ^{*n} denotes the n-fold convolution of μ. Similar results hold for the L^q dimension and the entropy dimension of μ^{*n}.

1. Introduction

Let $\mu_1, \ldots, \mu_n$ ($n \geq 2$) be a family of Borel probability measures on $\mathbb{R}$. Recall that the *convolution* $\mu_1 * \ldots * \mu_n$ of $\mu_1, \ldots, \mu_n$ is defined by

$$\mu_1 * \ldots * \mu_n(E) = \int_{\mathbb{R}^n} \chi_E(x_1 + \ldots + x_n) d\mu_1(x_1) \ldots d\mu_n(x_n)$$

for any Borel set $E \subset \mathbb{R}$, where χ_E denotes the characteristic function of E. In particular if $\mu_1 = \cdots = \mu_n = \mu$, then

$$\mu^{*n} := \underbrace{\mu * \cdots * \mu}_{n}$$

is called the *n-fold convolution* of μ.

It is well known that if μ is absolutely continuous with a density function f, then μ^{*n} is absolutely continuous with density f^{*n} for each $n \geq 2$, where f^{*n} denotes the n-fold convolution of f. However, if μ is a singular measure, μ^{*n} may be still singular for all n. In this case it is interesting to describe the asymptotic behavior of the "degree of singularity" of μ^{*n} as n tends to infinity. There are some widely used indices for describing the degree of singularity of measures, such as the Hausdorff dimension, the L^q dimension and the entropy dimension.

Recall that for a Borel probability measure η on $\mathbb{R}$, the *upper Hausdorff dimension* and the *lower Hausdorff dimension* of η are defined, respectively,

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by
$$\overline{\dim}_H \eta = \inf\{\dim_H E : \ E \text{ is a Borel set with } \eta(E) = 1\}$$
and
$$\underline{\dim}_H \eta = \inf\{\dim_H E : \ E \text{ is a Borel set with } \eta(E) > 0\},$$
where $\dim_H E$ denotes the Hausdorff dimension of E. (See [1], [2], [8] for the definition and properties of the Hausdorff dimension.) For $q > 1$, the *upper L^q-dimension* of η is defined by
$$\overline{\dim}_q \eta = \limsup_{r \to 0} \frac{\log \int \eta([x - r, x + r])^q dx}{(q - 1) \log r} - \frac{1}{q - 1}.$$
The *lower L^q-dimension* $\underline{\dim}_q \eta$ can be defined similarly by taking the lower limit. The *upper entropy dimension* of η is defined by
$$\overline{\dim}_e \eta = \limsup_{n \to \infty} \frac{H_n(\eta)}{\log 2^n},$$
where
$$H_n(\eta) = - \sum_{k=-\infty}^{\infty} \eta\left([2^{-n}k, 2^{-n}(k+1))\right) \log \eta\left([2^{-n}k, 2^{-n}(k+1))\right).$$
The *lower entropy dimension* $\underline{\dim}_e \eta$ is defined similarly by taking the lower limit.

As we will show, the sequences $\underline{\dim}_H \mu^{*n}$, $\overline{\dim}_H \mu^{*n}$, $\underline{\dim}_q \mu^{*n}$, $\overline{\dim}_q \mu^{*n}$, $\underline{\dim}_e \mu^{*n}$ and $\overline{\dim}_e \mu^{*n}$ are increasing in n and bounded from above by 1 (see Corollary 2.4). However, it is a rather subtle question to determine the limits of these sequences in general. In this paper, we provide precise values for the above limits for the class of equicontractive self-similar measures on $\mathbb{R}$. Suppose
$$\phi_i(x) = \rho x + d_i \qquad (i = 1, \ldots, m)$$
is a family of equicontractive similitudes on $\mathbb{R}$ with $0 < \rho < 1$, $m \geq 2$, and $d_1 < d_2 < \cdots < d_m$. Usually, $\{\phi_i\}_{i=1}^m$ is called an *equicontractive iterated function system*. For a given probability weight $\{p_i\}_{i=1}^m$ (i.e., $p_i > 0$ and $\sum_i p_i = 1$), it was proved by Hutchinson [5] that there is a unique Borel probability measure ν on $\mathbb{R}$ such that

$$(1.1) \qquad \nu = \sum_{i=1}^m p_i \nu \circ \phi_i^{-1}.$$

The measure ν is called an *equicontractive self-similar measure*.

We can formulate our result as follows:

THEOREM 1.1. *Let ν be an equicontractive self-similar measure on $\mathbb{R}$. Then*

$$(1.2) \quad \lim_{n \to \infty} \underline{\dim}_H \nu^{*n} = \lim_{n \to \infty} \overline{\dim}_H \nu^{*n} = \lim_{n \to \infty} \underline{\dim}_e \nu^{*n} = \lim_{n \to \infty} \overline{\dim}_e \nu^{*n} = 1$$

and

$$\text{(1.3)} \qquad \lim_{n \to \infty} \underline{\dim}_q \nu^{*n} = \lim_{n \to \infty} \overline{\dim}_q \nu^{*n} = 1 \qquad (1 < q \leq 2).$$

We remark that under the condition of Theorem 1.1, ν^{*n} is an equicontractive self-similar measure for each $n \geq 1$ (cf. [3, Proposition 3.1]). It follows from a result of Peres and Solomyak ([9, Theorem 1.1]) that

$$\underline{\dim}_e \nu^{*n} = \overline{\dim}_e \nu^{*n}, \qquad \underline{\dim}_q \nu^{*n} = \overline{\dim}_q \nu^{*n} \quad (q > 1).$$

Also, it is known that $\underline{\dim}_H \nu^{*n} = \overline{\dim}_H \nu^{*n}$ (see, e.g., [4, p. 200]).

Lindenstrauss, Meiri and Peres [7] have considered the measure-theoretic entropy of convolutions of ergodic measures on the circle $\mathbb{R}/\mathbb{Z}$. Let $\{\mu_i\}$ be a sequence of invariant and ergodic measures on $\mathbb{R}/\mathbb{Z}$ with respect to the transformation $\sigma_p : x \mapsto px(\bmod 1)$, where p is an integer greater than 1. Then Lindenstrauss, Meiri and Peres proved that the measure-theoretic entropy $h(\mu_1 * \cdots * \mu_n, \sigma_p)$ tends to $\log p$ as n tends to infinity, under a sharp condition

$$\sum_{i=1}^{\infty} \frac{h_i}{|\log h_i|} = \infty,$$

where $h_i = h(\mu_i, \sigma_p)/\log p$. We remark that one can use the above deep result to deduce (1.2) if ν is a self-similar measure for the special iterated function system

$$\phi_i(x) = \frac{1}{p}(x + i - 1), \qquad i = 1, \cdots, p.$$

We organize the paper as follows. In Section 2 we establish a sufficient condition for a probability measure on $\mathbb{R}$ to satisfy the conclusion of Theorem 1.1. In Section 3 this condition will be shown to hold for equicontractive self-similar measures, completing the proof of Theorem 1.1. Our proof is based on some classical properties of Fourier transforms of Borel probability measures as well as some basic properties of energy functions. We also use some properties of Fourier transforms of self-similar measures developed by Strichartz [10], [11], [12], and Lau and Wang [6].

2. Probability measures satisfying (1.2) and (1.3)

For a Borel probability measure η, the Fourier transformation $\widehat{\eta}$ is a complex-valued function on $\mathbb{R}$ defined by

$$\widehat{\eta}(t) = \int e^{-itx} d\eta(x).$$

For any integer $n > 0$ let

$$\text{(2.1)} \qquad \alpha_n = \alpha_n(\eta) = \limsup_{T \to \infty} \frac{\log \int_{|t|<T} |\widehat{\eta}(t)|^n dt}{\log T}.$$

In this section we establish the following fact, which is the first step in our proof of Theorem 1.1.

PROPOSITION 2.1.   *Suppose that η is a Borel probability measure on $\mathbb{R}$ with compact support. If $\lim_{n\to\infty} \alpha_n = 0$, then η satisfies (1.2) and (1.3), where ν is replaced by η.*

Although the condition in the above proposition appears to be rather technical and hard to check, we can verify it for the class of equicontractive self-similar measures. This will prove Theorem 1.1.

We prove several lemmas before giving a proof of Proposition 2.1.

LEMMA 2.2.   *Let η_1 and η_2 be Borel probability measures on $\mathbb{R}$. Then:*
  (i) $\underline{\dim}_H \eta_1 * \eta_2 \geq \underline{\dim}_H \eta_1$, $\overline{\dim}_H \eta_1 * \eta_2 \geq \overline{\dim}_H \eta_1$.
  (ii) *For any $q > 1$, $\overline{\dim}_q \eta_1 * \eta_2 \geq \overline{\dim}_q \eta_1$ and $\underline{\dim}_q \eta_1 * \eta_2 \geq \underline{\dim}_q \eta_1$.*
  (iii) *If furthermore η_1 and η_2 are compactly supported, then $\overline{\dim}_e \eta_1 * \eta_2 \leq 1$, and $\overline{\dim}_e \eta_1 * \eta_2 \geq \overline{\dim}_e \eta_1$, $\underline{\dim}_e \eta_1 * \eta_2 \geq \underline{\dim}_e \eta_1$.*

Proof. Suppose $\eta_1 * \eta_2(E) > 0$ for some Borel set $E \subset \mathbb{R}$. Then

$$\int \eta_1(E - x) d\eta_2(x) = \eta_1 * \eta_2(E) > 0,$$

which implies that $\eta_1(E - x) > 0$ for a set of x with positive η_2 measure. Thus there is at least one point $x_0 \in \mathbb{R}$ such that $\eta_1(E - x_0) > 0$. Hence $\dim_H E = \dim_H(E - x_0) \geq \underline{\dim}_H \eta_1$, from which we obtain $\underline{\dim}_H \eta_1 * \eta_2 \geq \underline{\dim}_H \eta_1$.

Now suppose $\eta_1 * \eta_2(F) = 1$ for some Borel set $F \subset \mathbb{R}$. Then

$$\int \eta_1(F - y) d\eta_2(y) = \eta_1 * \eta_2(F) = 1,$$

which implies that $\eta_1(F - y) = 1$ for η_2 almost all $y \in \mathbb{R}$. Thus there is at least one point $y_0 \in \mathbb{R}$ such that $\eta_1(F - y_0) = 1$. Hence $\dim_H F = \dim_H(F - y_0) \geq \overline{\dim}_H \eta_1$, from which we obtain $\overline{\dim}_H \eta_1 * \eta_2 \geq \overline{\dim}_H \eta_1$.

To see (ii), we note that by the Hölder inequality we have

$$\int \eta_1 * \eta_2([x - r, x + r])^q dx = \int \left(\int \eta_1([x - y - r, x - y + r]) d\eta_2(y) \right)^q dx$$

$$\leq \int \int \eta_1([x - y - r, x - y + r])^q d\eta_2(y) dx$$

$$= \int \int \eta_1([x - y - r, x - y + r])^q dx d\eta_2(y)$$

$$= \int \eta_1([x - r, x + r])^q dx.$$

This implies (ii).

To prove (iii), define $f(x) = -x \log x$ for $x \in \mathbb{R}^+$. It is easy to see that

$$(2.2) \qquad f(x+y) \leq f(x) + f(y) \leq 2f\left(\frac{x+y}{2}\right) = f(x+y) + (x+y) \log 2$$

for all $x, y \in \mathbb{R}^+$. Since η_1 is compactly supported,

$$\sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k + z, 2^{-n}(k+1) + z])\right) < \infty \quad \text{for any } n \in \mathbb{N}, \; z \in \mathbb{R}.$$

Now fix n and z. Denote by z_0 the unique real number satisfying $0 \leq z_0 < 2^{-n}$ and $2^n(z_0 - z) \in \mathbb{Z}$. Using (2.2), we have

$$\sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k + z, 2^{-n}(k+1) + z])\right)$$

$$= \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k + z_0, 2^{-n}(k+1) + z_0])\right)$$

$$\geq \sum_{k=-\infty}^{\infty} \Big[f\left(\eta_1([2^{-n}k + z_0, 2^{-n}(k+1)])\right)$$

$$+ f\left(\eta_1([2^{-n}(k+1), 2^{-n}(k+1) + z_0])\right)$$

$$- \eta_1([2^{-n}k + z_0, 2^{-n}(k+1) + z_0]) \log 2 \Big]$$

$$= \sum_{k=-\infty}^{\infty} \Big[f\left(\eta_1([2^{-n}k + z_0, 2^{-n}(k+1)])\right)$$

$$+ f\left(\eta_1([2^{-n}k, 2^{-n}k + z_0])\right) \Big] - \log 2$$

$$\geq \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k, 2^{-n}(k+1)])\right) - \log 2$$

$$= H_n(\eta_1) - \log 2.$$

A similar argument yields

$$H_n(\eta_1) \geq \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k + z, 2^{-n}(k+1) + z])\right) - \log 2.$$

Therefore we have

$$(2.3) \qquad \left| H_n(\eta_1) - \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k + z, 2^{-n}(k+1) + z])\right) \right| \leq \log 2.$$

Similarly, using (2.2) again, we can deduce that

$$H_n(\eta_1) \leq H_{n+1}(\eta_1) \leq H_n(\eta_1) + \log 2.$$

By the above inequality and the definition of entropy dimension, we have $\overline{\dim}_e \eta_1 \le 1$. Note that $\eta_1 * \eta_2$ is also compactly supported, and therefore

$$\overline{\dim}_e \eta_1 * \eta_2 \le 1.$$

By the convexity of f, we have

$$
\begin{aligned}
H_n(\eta_1 * \eta_2) &= \sum_{k=-\infty}^{\infty} f\left(\eta_1 * \eta_2([2^{-n}k, 2^{-n}(k+1)))\right) \\
&= \sum_{k=-\infty}^{\infty} f\left(\int \eta_1([2^{-n}k - z, 2^{-n}(k+1) - z))d\eta_2(z)\right) \\
&\ge \sum_{k=-\infty}^{\infty} \int f\left(\eta_1([2^{-n}k - z, 2^{-n}(k+1) - z))\right) d\eta_2(z) \\
&= \int \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k - z, 2^{-n}(k+1) - z))\right) d\eta_2(z) \\
&\ge \int \left(H_n(\eta_1) - \log 2\right) d\eta_2(z) = H_n(\eta_1) - \log 2,
\end{aligned}
$$

from which the last two inequalities in (iii) follow.   □

In the following lemma we cite some known facts about the relationship between various dimensions of a measure.

LEMMA 2.3.   *Suppose η is a Borel probability measure on $\mathbb{R}$ with compact support. Then:*
  (i) $\underline{\dim}_q \eta \le \underline{\dim}_H \eta \le \underline{\dim}_e \eta \le \overline{\dim}_e \eta \le 1$ *for any $q > 1$.*
  (ii) $\overline{\dim}_q \eta \le 1$ *for any $q > 1$. Furthermore $\underline{\dim}_q \eta$ and $\overline{\dim}_q \eta$ are monotone decreasing in $q > 1$.*

We remark that part (i) of the above lemma was proved by Fan, Lau and Rao [4, Theorem 1.4], while part (ii) was proved by Strichartz [12, Theorem 2.8 and Lemma 2.9].

As a corollary of Lemma 2.2 and Lemma 2.3 we have:

COROLLARY 2.4.   *Suppose η is a Borel probability measure on $\mathbb{R}$ with compact support. Then the sequences $\overline{\dim}_H \eta^{*n}$, $\underline{\dim}_H \eta^{*n}$, $\overline{\dim}_q \eta^{*n}$, $\underline{\dim}_q \eta^{*n}$, $\overline{\dim}_e \eta^{*n}$ and $\underline{\dim}_e \eta^{*n}$ are increasing in n. Each of these sequences is bounded from above by 1.*

The following lemma is used to prove Proposition 2.1.

LEMMA 2.5.   *For a Borel probability measure η on $\mathbb{R}$ with compact support, we have*

$$\underline{\dim}_H \eta \ge 1 - \alpha, \text{ and } \underline{\dim}_2 \eta = 1 - \alpha,$$

where

$$\alpha = \alpha_2 = \limsup_{T \to \infty} \frac{\log \int_{|t| < T} |\widehat{\eta}(t)|^2 dt}{\log T}.$$

Although the assertion $\underline{\dim}_H \eta \geq 1 - \alpha$ can be derived from the assertion $\underline{\dim}_2 \eta = 1 - \alpha$ using Lemma 2.3 (i), for the sake of self-containedness we will give a direct proof of both assertions.

We divide the proof into three parts, Claims 2.6, 2.7, and 2.8 below. In the proof of Claims 2.7 and 2.8 we adopt some ideas due to Lau and Wang [6].

CLAIM 2.6. $\underline{\dim}_H \eta \geq 1 - \alpha$.

Proof. Recall that for $t \geq 0$ the t-energy $I_t(\eta)$ of η is defined by

$$I_t(\eta) = \iint |x - y|^{-t} d\eta(x) d\eta(y).$$

It is well known (cf. [8, Theorem 8.7]) that if E is a Borel set with $\eta(E) > 0$, then $I_s(\eta) = \infty$ for any $s > \dim_H E$. This implies that

$$\underline{\dim}_H \eta \geq \sup\{s \geq 0 : I_s(\eta) < \infty\}.$$

Recall (cf. [8, Lemma 12.12]) that for each $0 < t < 1$, there is a positive constant $c(t)$ (independent of η) such that

$$I_t(\eta) = c(t) \int |x|^{t-1} |\widehat{\eta}(x)|^2 dx.$$

Therefore

$$\underline{\dim}_H \eta \geq \sup \left\{ s \in (0, 1) : \int |x|^{s-1} |\widehat{\eta}(x)|^2 dx < \infty \right\}.$$

Consequently, to prove $\underline{\dim}_H \eta \geq 1 - \alpha$ it suffices to establish the following inequality:

(2.4) $$\int |x|^{\beta - 1} |\widehat{\eta}(x)|^2 dx < \infty \text{ for any } \beta \in (0, 1 - \alpha).$$

To see (2.4), take $\epsilon > 0$ so that $\beta < 1 - \alpha - 2\epsilon$. By the definition of α, there exists an integer $N > 0$ such that

$$\int_{|x| < T} |\widehat{\eta}|^2 dx \leq T^{\alpha + \epsilon} \text{ for any } T > N.$$

It follows that

$$\int_{|x|\geq N} |x|^{\beta-1}|\widehat{\eta}(x)|^2 dx \leq \sum_{i=1}^{\infty}\int_{N+i-1\leq|x|\leq N+i}|x|^{-\alpha-2\epsilon}|\widehat{\eta}(x)|^2 dx$$

$$\leq \sum_{i=1}^{\infty}(N+i-1)^{-\alpha-2\epsilon}\int_{N+i-1\leq|x|\leq N+i}|\widehat{\eta}(x)|^2 dx$$

$$\leq \sum_{i=1}^{\infty}(N+i-1)^{-\alpha-2\epsilon}(N+i)^{\alpha+\epsilon}<\infty.$$

Since $\beta > 0$, we have

$$\int_{|x|<N} |x|^{\beta-1}|\widehat{\eta}|^2 dx \leq \int_{|x|<N}|x|^{\beta-1}dx < \infty.$$

The above two inequalities prove (2.4). $\square$

CLAIM 2.7. $\underline{\dim}_2\eta \geq 1-\alpha$.

Proof. Let

$$V_\gamma(r;\eta) = \frac{1}{r^{1+\gamma}}\int \eta([x-r,x+r])^2 dx \qquad \text{for any } \gamma, r \geq 0.$$

The claim is a simple consequence of the following fact, proved by Lau and Wang (see the proof of Proposition 3.2 in [6]):

$$(2.5) \qquad\qquad V_\gamma(r;\eta) \leq C(\gamma)I_\gamma(\eta) \qquad \text{for every } r > 0,$$

where $C(\gamma)$ is a positive constant depending on γ only.

For the reader's convenience, we include a brief proof of (2.5):

$$V_\gamma(r;\eta) = \frac{1}{r^{1+\gamma}}\int \eta([x-r,x+r])^2 dx$$

$$= \frac{1}{r^{1+\gamma}}\int\int\int \chi_{[x-r,x+r]}(y)\chi_{[x-r,x+r]}(z)d\eta(y)d\eta(z)dx$$

$$= \frac{1}{r^{1+\gamma}}\int\int \mathcal{L}^1([y-r,y+r]\cap[z-r,z+r])d\eta(y)d\eta(z)$$

$$\leq \frac{1}{r^{1+\gamma}}\int\int_{|y-z|\leq 2r}2rd\eta(y)d\eta(z)$$

$$\leq 2^{1+\gamma}\int\int\frac{1}{|y-z|^\gamma}d\eta(y)d\eta(z) = 2^{1+\gamma}I_\gamma(\eta),$$

which proves (2.5).

Now take $\beta < 1-\alpha$. Since $I_\beta(\eta) < \infty$, $V_\beta(r;\eta)$ has a uniform upper bound, and by the definition of $\underline{\dim}_2\eta$ we have $\underline{\dim}_2\eta \geq \beta$. Since $\beta < 1-\alpha$ is arbitrary, $\underline{\dim}_2\eta \geq 1-\alpha$. $\square$

CLAIM 2.8. $\underline{\dim}_2\eta \leq 1-\alpha$.

Proof. First we prove

$$(2.6) \qquad \int \eta([x-r,x+r])^2 dx = \frac{2}{\pi} \int |\widehat{\eta}(t)|^2 \frac{\sin^2(rt)}{t^2} dt \quad \text{for every } r > 0.$$

To see (2.6), fix $r > 0$ and define $f(x) = \eta([x-r, x+r])$. Then $f(x)$ is a Borel measurable function with compact support. By the Fubini Theorem,

$$\widehat{f}(t) = \int e^{-itx} f(x) dx = \int e^{-itx} \int_{|x-y| \leq r} d\eta(y) dx$$

$$= \int \int_{|x-y| \leq r} e^{-itx} dx d\eta(y)$$

$$= \int \frac{2e^{-ity} \sin(tr)}{t} d\eta(y) = \frac{2\sin(tr)}{t} \widehat{\eta}(t).$$

Therefore (2.6) follows from the following equality, known as the Plancherel formula (cf. [8]):

$$\int |\widehat{f}(t)|^2 dx = 2\pi \int |f(x)|^2 dx.$$

Now since $\sin^2(tr) \geq \frac{4}{\pi^2}(tr)^2$ for $|tr| \leq 1$, by (2.6) we have

$$\frac{8\pi^3}{r^2} \int \eta([x-r, x+r])^2 dx \geq \int_{|t| \leq 1/r} |\widehat{\eta}(t)|^2 dt.$$

Therefore, by the definition of $\underline{\dim}_2 \eta$, we have $\underline{\dim}_2 \eta \leq 1 - \alpha$. $\qquad\square$

Proof of Proposition 2.1. Since $|\widehat{\eta^{*n}}(x)| = |\widehat{\eta}(x)|^n$, by Lemma 2.5 we have

$$\underline{\dim}_H \eta^{*n} \geq 1 - \alpha_{2n} \quad \text{and} \quad \underline{\dim}_2 \eta^{*n} = 1 - \alpha_{2n}.$$

Since $\lim_{n \to \infty} \alpha_n = 0$,

$$\lim_{n \to \infty} \underline{\dim}_H \eta^{*n} = 1 \text{ and } \underline{\dim}_2 \eta^{*n} = 1.$$

Combining this with Lemma 2.3 yields the desired result. $\qquad\square$

3. Proof of Theorem 1.1

Let ν be an equicontractive self-similar measure defined as in (1.1), and let $\alpha_n = \alpha_n(\nu)$ be defined as in (2.1). By Proposition 2.1, it suffices to prove $\lim_{n \to \infty} \alpha_n = 0$.

It is well known that the Fourier transform of ν is given by

$$\widehat{\nu}(x) = \prod_{n=0}^{\infty} P(\rho^n x),$$

where ρ is the common contractive ratio of ϕ_i and $P(x) = \sum_{j=1}^m p_j e^{-id_j x}$ (see [11, p. 342]). Note that $d_j \neq d_k$ for $j \neq k$ and

$$|P(x)|^2 = \sum_{j=1}^m p_j^2 + \sum_{1 \leq k < j \leq m} 2p_k p_j \cos((d_j - d_k)x)$$

$$= 1 - \sum_{1 \leq k < j \leq m} 2p_k p_j \Big(1 - \cos((d_j - d_k)x)\Big).$$

We define $\Phi(x) = 1 - 2p_1 p_2 \Big(1 - \cos(2\pi x)\Big)$. Then Φ is a periodic function with period 1. By the above equality,

$$|P(x)|^2 \leq \Phi\left(\frac{d_2 - d_1}{2\pi} x\right).$$

Hence

(3.1) $$|\widehat{\nu}(x)|^2 \leq \prod_{n=0}^\infty \Phi\left(\frac{d_2 - d_1}{2\pi} \rho^n x\right).$$

For a given positive integer ℓ and $0 < \delta < 1$, let $r = r(\ell, \delta)$ be a positive integer such that

(3.2) $\qquad \Phi_r(x) < \delta \qquad$ for any $x \in \left[k + \frac{1}{3}\rho^\ell,\ k + 1 - \frac{1}{3}\rho^\ell\right]$ and $k \in \mathbb{Z}$,

where $\Phi_r(x) := (\Phi(x))^r$. Let $q(\ell)$ be the smallest integer $s \geq \rho^{-\ell}$, and write $\Lambda = \{0, 1, \dots, q(\ell) - 1\}$. For $j \in \Lambda$, define

$$I_j := \left[\frac{1 - \rho^\ell}{q(\ell) - 1} j,\quad \frac{1 - \rho^\ell}{q(\ell) - 1} j + \rho^\ell\right].$$

It is clear that $\bigcup_{j \in \Lambda} I_j = [0, 1]$, and for any $k \in \mathbb{Z}$, $y \in \mathbb{R}$ we have

$$\#\left\{j \in \Lambda:\ \left[k - \frac{1}{3}\rho^\ell, k + \frac{1}{3}\rho^\ell\right] \bigcap (I_j + y) \neq \emptyset\right\} \leq 2,$$

where $\#A$ denotes the cardinality of A. This combined with (3.2) yields

(3.3) $$\#\left\{j \in \Lambda:\ \max_{x \in I_j + y} \Phi_r(x) \geq \delta\right\} \leq 2,$$

for any $y \in \mathbb{R}$.

Now define a family of maps $\{\psi_j\}_{j \in \Lambda}$ on $\mathbb{R}$ by

$$\psi_j(x) = \rho^\ell x + \frac{1 - \rho^\ell}{q(\ell) - 1} j, \qquad j \in \Lambda.$$

Then $\psi_j([0,1]) = I_j$ and $[0,1] = \bigcup_{j \in \Lambda} \psi_j([0,1])$. Iterating the last equality n times we get

$$[0,1] = \bigcup_{j_1, \dots, j_n \in \Lambda} \psi_{j_1} \circ \cdots \circ \psi_{j_n}([0,1]).$$

For simplicity we write $I_{j_1\ldots j_n} = \psi_{j_1} \circ \cdots \circ \psi_{j_n}([0,1])$. By (3.3), for any $k \in \mathbb{N}$ and $j_1, \ldots, j_k \in \Lambda$ we have

$$(3.4) \qquad \# \left\{ j_{k+1} \in \Lambda : \max_{x \in I_{j_1 \cdots j_k j_{k+1}}} \Phi_r(\rho^{-k\ell} x) \geq \delta \right\} \leq 2.$$

By (3.1), we have for any integer $n \in \mathbb{N}$,

$$\int_0^{\frac{2\pi}{d_2-d_1} \rho^{-n\ell}} |\widehat{\nu}(x)|^{2r} dx \leq \int_0^{\frac{2\pi}{d_2-d_1} \rho^{-n\ell}} \prod_{j=0}^{\infty} \Phi_r\left(\frac{(d_2-d_1)\rho^j x}{2\pi} \right) dx$$

$$= \frac{2\pi}{d_2-d_1} \int_0^{\rho^{-n\ell}} \prod_{j=0}^{\infty} \Phi_r(\rho^j x) dx$$

$$\leq \frac{2\pi}{d_2-d_1} \int_0^{\rho^{-n\ell}} \prod_{j=1}^{n} \Phi_r(\rho^{j\ell} x) dx$$

$$= \frac{2\pi}{d_2-d_1} \rho^{-n\ell} \int_0^1 \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x) dx$$

$$\leq \frac{2\pi}{d_2-d_1} \rho^{-n\ell} \sum_{j_1,\ldots,j_n \in \Lambda} \int_{I_{j_1\ldots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x) dx$$

$$\leq \frac{2\pi}{d_2-d_1} \sum_{j_1,\ldots,j_n \in \Lambda} \max_{x \in I_{j_1\ldots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x).$$

Note that for any fixed indices $j_1, \ldots, j_{n-1}$ we have

$$\max_{x \in I_{j_1\ldots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x) \leq \max_{x \in I_{j_1\ldots j_{n-1}}} \prod_{j=0}^{n-2} \Phi_r(\rho^{-j\ell} x) \max_{y \in I_{j_1\ldots j_n}} \Phi_r(\rho^{-(n-1)\ell} y).$$

Hence by (3.4),

$$\sum_{j_n \in \Lambda} \max_{x \in I_{j_1\ldots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x) \leq \max_{x \in I_{j_1\ldots j_{n-1}}} \prod_{j=0}^{n-2} \Phi_r(\rho^{-j\ell} x)(2 + \delta q(\ell)).$$

Thus by induction

$$\sum_{j_1,\ldots,j_n \in \Lambda} \max_{x \in I_{j_1\ldots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x) \leq (2 + \delta q(\ell))^n.$$

Therefore

$$\int_0^{\frac{2\pi}{d_2-d_1} \rho^{-n\ell}} |\widehat{\nu}(x)|^{2r} dx \leq \frac{2\pi}{d_2-d_1} (2 + \delta q(\ell))^n.$$

Similarly

$$\int_{-\frac{2\pi}{d_2-d_1}\rho^{-n\ell}}^{0} |\widehat{\nu}(x)|^{2r}dx \leq \frac{2\pi}{d_2-d_1}\left(2+\delta q(\ell)\right)^n.$$

Thus

$$\int_{|x|<\frac{2\pi}{d_2-d_1}\rho^{-n\ell}} |\widehat{\nu}(x)|^{2r}dx \leq \frac{4\pi}{d_2-d_1}\left(2+\delta q(\ell)\right)^n,$$

which implies (see (2.1))

$$\alpha_{2r} = \limsup_{T\to\infty} \frac{\log \int_{|x|<T}|\widehat{\nu}(x)|^{2r}dx}{\log T} \leq \frac{\log(2+\delta q(\ell))}{\log \rho^{-\ell}}.$$

Now letting first $\delta \to 0$ and then $\ell \to \infty$, we finally obtain $\lim_{r\to\infty}\alpha_{2r}=0$ and so $\lim_{r\to\infty}\alpha_r=0$. Therefore by Proposition 2.1 we get the desired results.

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De-Jun Feng, Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China
  E-mail address: dfeng@math.tsinghua.edu.cn
  Current address: School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA
  E-mail address: djfeng@math.gatech.edu

Nhu T. Nguyen, Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA
  E-mail address: nnguyen@nmsu.edu

Tonghui Wang, Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA
  E-mail address: twang@nmsu.edu