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THE MCSHANE AND THE PETTIS INTEGRAL OF BANACH SPACE-VALUED FUNCTIONS DEFINED ON $\mathbb{R}^m$

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Abstract. In this paper, we define and study the McShane integral of functions mapping a compact interval I_0 in R^m into a Banach space X. We compare this integral with the Pettis integral and prove, in particular, that the two integrals are equivalent if X is reflexive and the unit ball of the dual X^* satisfies an additional condition (P). This gives additional information on an implicitly stated open problem of R.A. Gordon and on the work of D.H. Fremlin and J. Mendoza.

1. Introduction

The McShane integral of real-valued functions is a Riemann-type integral, which is equivalent to the Lebesgue integral. R.A. Gordon [5] generalized the definition of the McShane integral for real-valued functions to abstract functions from intervals in $\mathbb{R}$ to Banach spaces and proved that the McShane integral and Pettis integral are equivalent when f is strongly measurable or the Banach space X is separable and contains no copy of c_0. The relation between the Pettis integral and the McShane integral for arbitrary Banach spaces is unknown.

In this paper, we prove, among other results, the equivalence of Pettis and McShane integrability of functions mapping an m-dimensional compact interval I_0 into a reflexive Banach space with a certain additional condition on the closed unit ball of its dual, using the properties of the Pettis integral appearing in [1], [4], [10] and [14].

2. Definitions and basic properties

Throughout this paper X will denote a real Banach space with norm $\|\cdot\|$, and X^* denotes its dual.

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I_0 is a compact interval in R^m and Σ is the set of all μ-measurable subsets of I_0, μ stands for the Lebesgue measure.

$B(X^*) = \{x^* \in X^*;\ \|x^*\| \le 1\}$ is the unit ball in X^*.

We first extend the notion of partition of an interval. A *partial M-partition* D in I_0 is a finite collection of interval-point pairs (I, ξ) with non-overlapping intervals $I \subset I_0$, where $\xi \in I_0$ is the point associated with I. If we require, in addition, that $\xi \in I$ for this point, we get the concept of a *partial K-partition* D in I_0. We write $D = \{(I, \xi)\}$.

A partial M-partition $D = \{(I, \xi)\}$ in I_0 is an *M-partition* of I_0 if the union of all the intervals I equals I_0; a K-partition is defined similarly.

Let δ be a positive function defined on the interval I_0. A partial M-partition (K-partition) $D = \{(I, \xi)\}$ is said to be δ-fine if for each interval-point pair $(I, \xi) \in D$ we have $I \subset B(\xi, \delta(\xi))$, where $B(\xi, \delta(\xi)) = \{t \in R^m;\ \text{dist}(\xi, t) < \delta(\xi)\}$ and dist is the metric in R^m.

The m-dimensional volume of a given interval $I \subset I_0$ is denoted by $\mu(I)$.

Given an M-partition $D = \{(I, \xi)\}$, we write

$$f(D) = (D) \sum f(\xi)\mu(I)$$

for integral sums over D, whenever $f : I_0 \mapsto X$.

Definition 1. An X-valued function f is said to be *McShane integrable* on I_0 if there exists an element $S_f \in X$ such that for every $\varepsilon > 0$, there exists $\delta(t) > 0$, $t \in I_0$, such that for every δ-fine M-partition $D = \{(I, \xi)\}$ of I_0 we have

$$\left\| (D) \sum f(\xi)\mu(I) - S_f \right\| < \varepsilon.$$

We write $(M)\int_{I_0} f = S_f$, and call S_f the *McShane integral of f over I_0*.

f is said to be McShane integrable on a set $E \subset I_0$ if the function $f \cdot \chi_E$ is McShane integrable on I_0, where χ_E denotes the characteristic function of E. We write $(M)\int_E f = (M)\int_{I_0} f\chi_E = F(E)$ for the McShane integral of f on E, and denote the set of all McShane integrable functions $f : I_0 \mapsto X$ by $\mathcal{M}$.

Replacing the term "M-partition" by "K-partition" in this definition we obtain *Kurzweil-Henstock integrability* and the definition of the *Kurzweil-Henstock integral* $(K)\int_{I_0} f$. It is clear that if $f : I_0 \mapsto X$ is McShane integrable, then it is also Kurzweil-Henstock integrable.

The following theorems describe some of the basic properties of the McShane integral. The proofs of these results are virtually identical to the proofs for functions defined on one-dimensional intervals, and the reader is referred to R.A. Gordon [5][6] for the details.

Theorem 2. *A function $f : I_0 \to X$ is McShane integrable on I_0 if and only if for each $\varepsilon > 0$ there exists a positive function δ on I_0 such that*

$$\left\| (D_1) \sum f(\xi)\mu(I) - (D_2) \sum f(\eta)\mu(J) \right\| < \varepsilon$$

whenever $D_1 = \{(I, \xi)\}$ *and* $D_2 = \{(J, \eta)\}$ *are* δ*-fine* M*-partitions of* I_0.

THEOREM 3. *Let* f *and* g *be functions mapping* I_0 *into* X.

(a) *If* f *is McShane integrable on* I_0, *then* f *is McShane integrable on every subinterval of* $I \subset I_0$.

(b) *If* f *is McShane integrable on each of the intervals* I_1 *and* I_2, *where* I_1 *and* I_2 *are non-overlapping and* $I_1 \cup I_2 = I$ *is an interval, then* f *is McShane integrable on* I *and*

$$\int_I f = \int_{I_1} f + \int_{I_2} f.$$

(c) *If* f *and* g *are McShane integrable on* I_0 *and* α *and* β *are real numbers, then* $\alpha f + \beta g$ *is McShane integrable on* I_0 *and*

$$\int_{I_0} (\alpha f + \beta g) = \alpha \int_{I_0} f + \beta \int_{I_0} g.$$

THEOREM 4. *Let* $f : I_0 \to X$ *be McShane integrable on* I_0, *and let* $F(I) = \int_I f$ *for a subinterval* $I \subset I_0$; F *is an* X*-valued interval function called the primitive of* f *on* I_0. *Then for every* $\varepsilon > 0$ *there exists a positive function* δ *defined on* I_0 *such that for any* δ*-fine partial* M*-partition* $D = \{(I, \xi)\}$ *we have*

$$\left\| (D) \sum [f(\xi)\mu(I) - F(I)] \right\| < \varepsilon;$$

in particular, if $D' = \{(I', \xi')\}$ *is a* δ*-fine* M*-partition of* I_0, *then*

$$\left\| (D') \sum f(\xi')\mu(I') - F(I_0) \right\| \leq \varepsilon.$$

Theorem 4 is called the Saks-Henstock lemma for the McShane integral.

THEOREM 5. *Let* $f : I_0 \to X$ *be McShane integrable on* I_0.

(a) *If* $f = g$ *almost everywhere (with respect to the Lebesgue measure* μ *in* $\mathbb{R}^m$*) on* I_0, *then* g *is McShane integrable on* I_0 *and* $\int_{I_0} f = \int_{I_0} g$.

(b) *If* $E = \bigcup_{j=1}^{p} E_j$, *where* $\mu(E_j \cap E_i) = 0$ *for* $i \neq j$, $i, j \in \{1, \ldots, p\}$, *and* $\int_{E_j} f$, $j = 1, \ldots, p$, *exist, then* $\int_E f$ *exists and*

$$\int_E f = \sum_{j=1}^{p} \int_{E_j} f.$$

DEFINITION 6. A set $K \subset \mathcal{M}$ is called M*-equiintegrable* (*McShane-equiintegrable*) if for every $\varepsilon > 0$ there is a $\delta : I_0 \mapsto (0, +\infty)$ such that

$$\left\| (D) \sum f(\xi)\mu(I) - \int_{I_0} f \right\| < \varepsilon$$

for every δ-fine M-partition $D = \{(I, \xi)\}$ of I_0 and every $f \in K$.

Using the concept of M-equiintegrability we have the following convergence result for the McShane integral (see, e.g., [5]):

Theorem 7. *If the sequence of real functions* $f_n : I_0 \mapsto \mathbb{R}$, $n \in \mathbb{N}$, *is* M*-equiintegrable and*

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \textit{for} \quad t \in I_0,$$

then $f \in \mathcal{M}$ *and*

$$\lim_{n\to\infty} \int_{I_0} f_n = \int_{I_0} f.$$

This result can be proved similarly to the analogous theorem for the Kurzweil-Henstock integral (see, e.g., [5], [7]).

Definition 8. $f : I_0 \mapsto X$ is called *(strongly) measurable* if there is a sequence of simple functions (f_n) with $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in I_0$.

$f : I_0 \mapsto X$ is called *weakly measurable* if for each $x^* \in X^*$ the real function $x^*(f) : I_0 \mapsto \mathbb{R}$ is measurable.

Two functions $f, g : I_0 \mapsto X$ are called *weakly equivalent* on I_0 if for every $x^* \in X^*$ the relation

$$x^*(f(t)) = x^*(g(t))$$

holds for almost all $t \in I_0$.

Theorem 9. *If* $f : I_0 \to X$ *is McShane integrable on* I_0*, then:*

(a) *For each* x^* *in* X^*, $x^*(f)$ *is McShane integrable on* I_0 *and* $\int_{I_0} x^*(f) = x^*(\int_{I_0} f)$.
(b) $\{x^*(f);\ x^* \in B(X^*)\}$ *is* M*-equiintegrable on* I_0.
(c) f *is weakly measurable.*

Proof. Since $f : I_0 \to X$ is McShane integrable on I_0, for every $\varepsilon > 0$ there exists a positive function δ defined on I_0 such that for any δ-fine M-partition $D = \{(I, \xi)\}$ we have

$$\left\| (D) \sum f(\xi)\mu(I) - \int_{I_0} f \right\| < \varepsilon.$$

Hence for any $x^* \in X^*$ we have

$$\begin{aligned} &\left| (D) \sum x^*(f(\xi))\mu(I) - x^*\left(\int_{I_0} f \right) \right| \\ &\qquad \le \|x^*\| \ \left\| (D) \sum f(\xi)\mu(I) - \int_{I_0} f \right\| < \|x^*\|\varepsilon \end{aligned}$$

for any δ-fine M-partition $D = \{(I,\xi)\}$. Therefore (a) holds. If $x^* \in B(X^*)$, then the above inequality gives

$$\left|(D)\sum x^*(f(\xi))\mu(I) - x^*\left(\int_{I_0} f\right)\right| < \varepsilon$$

for every $x^* \in B(X^*)$, so the set $\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable on I_0 and f is weakly measurable because for every $x^* \in X^*$ the real function $x^*(f)$ is Lebesgue integrable (see, e.g., [9]). □

Corollary 10. *If $f : I_0 \mapsto X$ is McShane integrable on I_0 then:*

(a) *For every subinterval $I \subset I_0$ and for every $x^* \in X^*$ the function $x^*(f)$ is McShane (= Lebesgue) integrable on I and*

$$\int_I x^*(f) = x^*\left(\int_I f\right).$$

(b) *If $E = \bigcup_{j=1}^p I_j$, where I_j are non-overlapping subintervals of I_0, then f is McShane integrable on E with*

$$\int_E f = \sum_{j=1}^p \int_{I_j} f$$

and for every $x^* \in X^*$ we have

$$\int_E x^*(f) = \sum_{j=1}^p \int_{I_j} x^*(f) = x^*\left(\sum_{j=1}^p \int_{I_j} f\right) = x^*\left(\int_E f\right).$$

Proof. (a) follows easily from (a) in Theorem 3. For (b) we set $f = 0$ on the boundary of every I_j, $j = 1, \dots, p$, and take Theorem 5 into account. □

3. The Pettis integral and its relation to the McShane integral

We denote by $\mathcal{L}$ the set of Lebesgue integrable real functions on I_0 (with respect to the Lebesgue measure μ). It should be noted at this point that a real function f belongs to $\mathcal{L}$ if and only if it is McShane integrable, i.e., we have $\mathcal{L} = \mathcal{M}$ (see [7], [8], and [9]).

We use the notation $\mu(E)$ for the Lebesgue measure of a (Lebesgue) measurable set $E \subset I_0$.

In [1, Definition 10, p. 74] the following concept is introduced.

Definition 11. A set $K \subset \mathcal{L}$ is called *uniformly integrable* if

$$\lim_{\mu(E)\to 0} \int_E |f| = 0$$

uniformly for $f \in K$, where the sets $E \subset I_0$ are measurable sets.

For uniformly integrable sets we have the following well-known result (see, e.g., [11, p. 168]).

THEOREM 12 (Vitali theorem). *If on a measurable set $E \subset I_0$ a sequence $f_j \in \mathcal{L}$, $j \in \mathbb{N}$, is given such that f_j converges to f in measure, and if the sequence (f_j) is uniformly integrable, then $f \in \mathcal{L}$ and*

$$\lim_{j\to\infty} \int_E f_j = \int_E f.$$

We first give the basic definitions concerning the Pettis integral. We will take these from the book [1] of Diestel and Uhl or from the extensive survey paper [10] by K. Musiał.

DEFINITION 13. If $f : I_0 \mapsto X$ is weakly measurable such that $x^*(f) \in \mathcal{L}$ for all $x^* \in X^*$ and if for every measurable set $E \subset I_0$ there is an element $x_E \in X$ such that

$$x^*(x_E) = \int_E x^*(f),$$

then f is called *Pettis integrable* and the Pettis integral of f over E is the element $x_E \in X$. We write $x_E = (P)\int_E f$ and denote by $\mathcal{P}$ the set of all Pettis integrable functions.

Since (I_0, Σ, μ) is a finite perfect measure space we can use the result of Theorem 6 from [4] in the following form.

THEOREM 14. *The function $f : I_0 \mapsto X$ is Pettis integrable if and only if there is a sequence (f_n) of simple functions from I_0 into X with the following properties:*

(a) *The set $\{x^*(f_n);\ x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable.*
(b) *For each x^* in X^*, $\lim_{n\to\infty} x^*(f_n) = x^*(f)$ a.e. on I_0.*

The following result due to K. Musiał (see [10, Theorem 10.1]) gives important information on Pettis integrable functions.

Let X be an arbitrary normed space and let $f : I_0 \mapsto X$ be a Pettis integrable function. Then the following are equivalent:

(i) *$\{x^*(f); x^* \in B(X^*)\}$ is a separable subset of $\mathcal{L}$.*
(ii) *There exists a sequence (f_n) of simple functions from I_0 into X, such that for each $x^* \in X^*$ one of the following conditions is satisfied:*
    (a) *The sequence $\{x^*(f_n); n \in \mathbb{N}\}$ is uniformly integrable and converges to $x^*(f)$ almost everywhere.*
    (b) *The sequence $\{x^*(f_n); n \in \mathbb{N}\}$ is uniformly integrable and converges to $x^*(f)$ in measure.*
    (c) *$\{x^*(f_n); n \in \mathbb{N}\}$ is convergent to $x^*(f)$ in $\mathcal{L}$.*
    (d) *$\{x^*(f_n); n \in \mathbb{N}\}$ is convergent to $x^*(f)$ weakly in $\mathcal{L}$.*

(iii) $\nu_f(\Sigma) = \{(P)\int_E f \in X; E \in \Sigma\}$ *is a separable subset of* X.

In Musiał's paper [10] it is mentioned that the uniform integrability of the sets $\{x^*(f_n);\ n \in \mathbb{N}\}$ in the conditions (ii)(a) and (ii)(b) may be replaced by the uniform integrability of the set $\{x^*(f_n);\ x^* \in B(X^*), n \in \mathbb{N}\}$. This leads to the following formulation of the result of Musiał.

THEOREM 15. *Let X be an arbitrary normed space and let $f : I_0 \mapsto X$ be a Pettis integrable function. Then the following are equivalent:*

(i) $\{x^*(f); x^* \in B(X^*)\}$ *is a separable subset of* $\mathcal{L}$.

(ii) *There exists a sequence (f_n) of simple functions from I_0 into X, such that for each $x^* \in X^*$ one of the following conditions is satisfied:*

   (a) *The set $\{x^*(f_n);\ x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable and the sequence $x^*(f_n)$, $n \in \mathbb{N}$, converges to $x^*(f)$ almost everywhere.*

   (b) *The set $\{x^*(f_n);\ x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable and the sequence $x^*(f_n)$, $n \in \mathbb{N}$, converges to $x^*(f)$ in measure.*

   (c) $\{x^*(f_n); n \in \mathbb{N}\}$ *is convergent to* $x^*(f)$ *in* $\mathcal{L}$.

   (d) $\{x^*(f_n); n \in \mathbb{N}\}$ *is convergent to* $x^*(f)$ *weakly in* $\mathcal{L}$.

(iii) $\nu_f(\Sigma) = \{(P)\int_E f \in X; E \in \Sigma\}$ *is a separable subset of* X.

It is possible to combine Theorems 14 and 15 to obtain the following corollary.

COROLLARY 16. *For a function $f : I_0 \mapsto X$ the following are equivalent:*

(a) $f \in \mathcal{P}$.

(b) $\{x^*(f); x^* \in B(X^*)\}$ *is a separable subset of* $\mathcal{L}$.

(c) *There exists a sequence (f_n) of simple functions from I_0 into X such that for each $x^* \in X^*$ one of the following conditions is satisfied:*

   (i) *The set $\{x^*(f_n);\ x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable and the sequence $x^*(f_n)$, $n \in \mathbb{N}$, converges to $x^*(f)$ almost everywhere.*

   (ii) *The set $\{x^*(f_n);\ x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable and the sequence $x^*(f_n)$, $n \in \mathbb{N}$, converges to $x^*(f)$ in measure.*

   (iii) $\{x^*(f_n); n \in \mathbb{N}\}$ *is convergent to* $x^*(f)$ *in* $\mathcal{L}$.

   (iv) $\{x^*(f_n); n \in \mathbb{N}\}$ *is convergent to* $x^*(f)$ *weakly in* $\mathcal{L}$.

Proof. By Theorem 14 the Pettis integrability of f is equivalent to (ii)(a) of Theorem 15. Theorem 15 now gives the corollary. $\square$

D.H. Fremlin and J. Mendoza [3, Example 3C] give an example of a Pettis integrable function with values in $l^\infty(\mathbb{N})$ which is not McShane integrable.

We will consider the problem when a Pettis integrable function f with values in X is McShane integrable. By the above mentioned example of

Fremlin and Mendoza this is a problem related to the properties of the space X or some additional properties of the function f.

PROPOSITION 17. *If $f : I_0 \mapsto X$ is Pettis integrable then the set $\{x^*(f);\ x^* \in B(X^*)\}$ is uniformly integrable.*

Proof. By Theorem 14 there is a sequence of simple functions (f_n) such that for each $x^* \in X^*$ we have

$$\lim_{n\to\infty} x^*(f_n) = x^*(f) \text{ a.e. in } I_0,$$

and the set

$$\{x^*(f_n); x^* \in B(X^*), n \in \mathbb{N}\}$$

is uniformly integrable. Hence

$$\lim_{\mu(E)\to 0} \int_E |x^*(f_n)| = 0$$

uniformly for $x^* \in B(X^*)$, $n \in \mathbb{N}$, for measurable $E \subset I_0$. By the Vitali convergence theorem (Theorem 12) we have

$$\lim_{n\to\infty} \int_E |x^*(f_n)| = \int_E |x^*(f)|$$

for every measurable $E \subset I_0$. This yields

$$\lim_{\mu(E)\to 0} \int_E |x^*(f)| = \lim_{\mu(E)\to 0} \lim_{n\to\infty} \int_E |x^*(f_n)| = \lim_{n\to\infty} \lim_{\mu(E)\to 0} \int_E |x^*(f_n)| = 0$$

uniformly for $x^* \in B(X^*)$, and the statement is proved. □

LEMMA 18. *Assume that $f : I_0 \mapsto X$ is Pettis integrable. Then f is McShane integrable if and only if the set $\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable.*

Proof. By (b) in Theorem 9 the set $\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable provided $f \in \mathcal{M}$.

Assume that $\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable. Then, by definition, for every $\varepsilon > 0$ there exists $\delta(\xi) > 0$, $\xi \in I_0$, such that for every δ-fine M-partition $D = \{(I, \xi)\}$ of I_0 and $x^* \in B(X^*)$ we have

$$\left| (D) \sum x^*(f(\xi))\mu(I) - \int_{I_0} x^*(f) \right| < \varepsilon.$$

Since $f \in \mathcal{P}$, we have $\int_{I_0} x^*(f) = x^*((P)\int_{I_0} f)$, and

$$(D) \sum x^*(f(\xi))\mu(I) = x^* \Big((D) \sum f(\xi)\mu(I) \Big)$$

holds evidently. Hence for every δ-fine M-partition $D = \{(I,\xi)\}$ of I_0 and $x^* \in B(X^*)$ we have

$$\left| x^* \left((D) \sum f(\xi)\mu(I) - (P) \int_{I_0} f \right) \right| < \varepsilon,$$

and this yields immediately

$$\left\| (D) \sum f(\xi)\mu(I) - (P) \int_{I_0} f \right\| < \varepsilon$$

for every δ-fine M-partition $D = \{(I,\xi)\}$. Consequently we obtain that f is McShane integrable on I_0 (and $(M) \int_{I_0} f = (P) \int_{I_0} f$). $\square$

R.A. Gordon [5, Theorem 17] proved the following result:

THEOREM 19. *Let $f : I_0 \mapsto X$ be (strongly) measurable. If f is Pettis integrable on I_0 then f is McShane integrable on I_0.*

REMARK 20. In fact, Gordon proved the result given in Theorem 19 for the case when $I_0 = [a,b] \in \mathbb{R}$ is a one-dimensional interval. An inspection of the proof in [5] shows that the approach of Gordon can be adopted to the case of a compact interval I_0 in $\mathbb{R}^m$.

LEMMA 21. *Assume that $f : I_0 \mapsto X$ is Pettis integrable. If f is weakly equivalent to a measurable function $g : I_0 \mapsto X$ then for every sequence $x_m^* \in B(X^*)$, $m \in \mathbb{N}$, the set $\{x_m^*(f);\ m \in \mathbb{N}\}$ is M-equiintegrable.*

Proof. Since $g : I_0 \mapsto X$ is assumed to be measurable, by Theorem 19 the function g is McShane integrable, i.e., $g \in \mathcal{M}$.

By Theorem 9 the set $\{x^*(g);\ x^* \in B(X^*)\}$ is M-equiintegrable. Assuming that $(x_m^*) \in B(X^*)$, $m \in \mathbb{N}$, is an arbitrary sequence, the set $\{x_m^*(g);\ m \in \mathbb{N}\}$ is evidently also M-equiintegrable.

From the weak equivalence of f and g we obtain that for every $m \in \mathbb{N}$ there is a measurable $N_m \subset I_0$ with $\mu(N_m) = 0$ such that

$$x_m^*(f(t)) = x_m^*(g(t)) \text{ for } \ t \in I_0 \setminus N_m.$$

Let us put $N = \bigcup_{m=1}^{\infty} N_m$. Then $\mu(N) = 0$ and for every $m \in \mathbb{N}$ we have

$$x_m^*(f(t)) = x_m^*(g(t)) \text{ for } \ t \in I_0 \setminus N.$$

By Theorem 5 the function $g\chi_{I_0 \setminus N}$ is McShane integrable, and again by Theorem 9 we obtain that the set $\{x_m^*(g\chi_{I_0 \setminus N}); m \in \mathbb{N}\}$ is M-equiintegrable.

Let us set

$$f_1(t) = \begin{cases} f(t), & \text{for } t \in I_0 \setminus N, \\ 0, & \text{for } t \in N \end{cases}$$

and

$$f_2(t) = \begin{cases} 0, & \text{for } t \in I_0 \setminus N, \\ f(t), & \text{for } t \in N. \end{cases}$$

Then $f = f_1 + f_2$, $x_m^*(f_1(t)) = x_m^*(g(t)\chi_{I_0\setminus N}(t))$ for every $m \in \mathbb{N}$ and $t \in I_0$, and $f_2(t) = 0$ almost everywhere in I_0. This yields that $\{x_m^*(f_1); m \in \mathbb{N}\}$ is M-equiintegrable because of the M-equiintegrability of $\{x_m^*(g\chi_{I_0\setminus N}); m \in \mathbb{N}\}$. By Theorem 5 we have $f_2 \in \mathcal{M}$, and the set $\{x_m^*(f_2); m \in \mathbb{N}\}$ is also M-equiintegrable.

Assume that $\varepsilon > 0$ is given. By Definition 6 there is a function $\delta_1 > 0$ on I_0 such that

$$\left|(D)\sum x_m^*(f_1(t))\mu(I) - \int_{I_0} x_m^*(f_1)\right| < \varepsilon$$

for every δ_1-fine M-partition $D = \{(I,t)\}$ and every $m \in \mathbb{N}$, where

$$\int_{I_0} x_m^*(f_1) = \int_{I_0} x_m^*(f) = x_m^*\left((P)\int_{I_0} f\right).$$

Similarly there is a function $\delta_2 > 0$ on I_0 such that

$$\left|(D)\sum x_m^*(f_2(t))\mu(I)\right| < \varepsilon$$

for every δ_2-fine M-partition $D = \{(I,t)\}$ and every $m \in \mathbb{N}$, since $\int_{I_0} x_m^*(f_2) = 0$ for every $m \in \mathbb{N}$.

Taking now an arbitrary $\min(\delta_1, \delta_2)$-fine M-partition $D = \{(I,t)\}$, we get

$$\begin{aligned} &\left|(D)\sum x_m^*(f(t))\mu(I) - x_m^*\left(\int_{I_0} f\right)\right| \\ &\quad \le \left|(D)\sum x_m^*(f_1(t))\mu(I) - \int_{I_0} x_m^*(f_1)\right| + \left|(D)\sum x_m^*(f_2(t))\mu(I)\right| \\ &\quad < 2\varepsilon \end{aligned}$$

for every $m \in \mathbb{N}$. This proves the lemma. □

For the next lemma we need to assume that the ball $B(X^*)$ in X^* has the following property.

PROPERTY (P). There exists a sequence $\{x_m^* \in B(X^*);\ m \in \mathbb{N}\}$ such that for every $x^* \in B(X^*)$ there exists a subsequence $\{x_k^* \in B(X^*);\ k \in \mathbb{N}\}$ of $\{x_m^* \in B(X^*);\ m \in \mathbb{N}\}$ such that

$$x_k^*(x) \to x^*(x) \text{ for every } x \in X \text{ if } k \to \infty. \tag{1}$$

LEMMA 22. *Let $f : I_0 \mapsto X$ be Pettis integrable. If f is weakly equivalent to a measurable function $g : I_0 \mapsto X$ and X has the property (P) then the set $\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable.*

Proof. Assume that $x^* \in B(X^*)$ is given. Then by (1) we have

$$x_k^*(f(t)) \to x^*(f(t)) \text{ for every } t \in I_0 \text{ if } k \to \infty. \tag{2}$$

By Lemma 21 the set $\{x_m^*(f);\ m \in \mathbb{N}\}$ is M-equiintegrable and therefore also

$$\{x_k^*(f);\ k \in \mathbb{N}\} \text{ is } M\text{-equiintegrable.} \tag{3}$$

Applying the convergence theorem (Theorem 7), (2) and (3), we get

$$\lim_{k\to\infty} \int_{I_0} x_k^*(f) = \int_{I_0} x^*(f). \tag{4}$$

Assume now that $\varepsilon > 0$ is arbitrary. Then, by (2), for any $t \in I_0$ there is a $j_0 = j_0(\varepsilon, t) \in \mathbb{N}$ such that

$$k > j_0 \implies |x_k^*(f(t)) - x^*(f(t))| < \varepsilon. \tag{5}$$

Since the set $\{x_m^*(f);\ m \in \mathbb{N}\}$ is M-equiintegrable, by Definition 6 there is a positive function $\delta : I_0 \mapsto (0, \infty)$ such that

$$\left| \sum_{i=1}^{p} x_m^*(f(t_i))\mu(I_i) - \int_{I_0} x_m^*(f) \right| < \varepsilon \tag{6}$$

for every $m \in \mathbb{N}$, provided $D = \{(I_i, t_i);\ i = 1, \dots, p\}$ is a δ-fine M-partition of I_0.

Finally, (4) gives that there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have

$$\left| \int_{I_0} x_k^*(f) - \int_{I_0} x^*(f) \right| = \left| \int_{I_0} (x_k^*(f) - x^*(f)) \right| < \varepsilon. \tag{7}$$

Let $D = \{(I_i, t_i);\ i = 1, \dots, p\}$ be an arbitrary δ-fine M-partition of I_0 and let $k \in \mathbb{N}$ be such that $k > \max(k_0, j_0(\varepsilon, t_1), \dots, j_0(\varepsilon, t_p))$. Then using (5), (6) and (7) we obtain

$$\begin{aligned}
&\left| \sum_{i=1}^{p} x^*(f(t_i))\mu(I_i) - \int_{I_0} x^*(f) \right| \\
&\qquad \le \left| \sum_{i=1}^{p} [x^*(f(t_i)) - x_k^*(f(t_i))]\mu(I_i) \right| + \left| \sum_{i=1}^{p} x_k^*(f(t_i))\mu(I_i) - \int_{I_0} x_k^*(f) \right| \\
&\qquad\qquad + \left| \int_{I_0} x_k^*(f) - \int_{I_0} x^*(f) \right| \\
&\qquad < \sum_{i=1}^{p} |x^*(f(t_i)) - x_k^*(f(t_i))|\mu(I_i) + \varepsilon + \varepsilon \\
&\qquad < \varepsilon \sum_{i=1}^{p} \mu(I_i) + 2\varepsilon = \varepsilon(\mu(I_0) + 2).
\end{aligned}$$

Since $x^* \in B(X^*)$ and $\varepsilon > 0$ can be taken arbitrarily, we obtain the M-equiintegrability of $\{x^*(f);\ x^* \in B(X^*)\}$. $\square$

THEOREM 23. *If $f : I_0 \mapsto X$ is Pettis integrable on I_0, f is weakly equivalent to a measurable function $g : I_0 \mapsto X$, and X has property (P), then f is McShane integrable on I_0, i.e., $\mathcal{P} \subset \mathcal{M}$.*

Proof. By Lemma 22 the set $\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable. Lemma 18 implies that $f \in \mathcal{M}$ and the theorem is proved. $\square$

A weakly measurable function $f : I_0 \mapsto X$ is said to be *determined by a subspace* D of X if for each $x^* \in X^*$ which restricted to D equals zero (i.e., $x^*|_D = 0$) the function $x^*(f)$ equals zero almost everywhere on I_0.

G.F. Stefánsson [14] proved the following result.

PROPOSITION 24. *All weakly measurable functions determined by reflexive spaces are weakly equivalent to strongly measurable functions.*

Since every weakly measurable function $f : I_0 \mapsto X$ is determined by the space X itself, we conclude easily that the following holds.

PROPOSITION 25. *If the Banach space X is reflexive and $f : I_0 \mapsto X$ is weakly measurable then there exists a strongly measurable function $g : I_0 \mapsto X$ which is weakly equivalent to f.*

Using this result together with Theorem 23 we arrive at the following result.

THEOREM 26. *If the Banach space X is reflexive, X has the property (P) and $f : I_0 \mapsto X$ is Pettis integrable, then f is McShane integrable on I_0, i.e., $\mathcal{P} \subset \mathcal{M}$.*

REMARK 27. Stefánsson's Proposition 24 can be used to obtain the following stronger result.

If $f : I_0 \mapsto X$ is Pettis integrable on I_0, f is determined by a reflexive space and X has the property (P) then f is McShane integrable on I_0.

According to a remark in Stefánsson's paper [14, p. 412] an even more general statement holds.

If $f : I_0 \mapsto X$ is Pettis integrable on I_0, f is determined by a subspace of X which has the Radon-Nikodým Property, and X has the property (P), then f is McShane integrable on I_0.

4. McShane integrable functions are Pettis integrable

In this section we show that the converse inclusion $\mathcal{M} \subset \mathcal{P}$ also holds in the case of $I_0 \subset \mathbb{R}^m$. We first prove some some auxiliary results.

LEMMA 28. *If $f : I_0 \to X$ is McShane integrable on I_0, then for every $\varepsilon > 0$ there is an $\eta > 0$ such that for any finite collection $\{J_j : 1 \le j \le p\}$ of* *non-overlapping intervals in I_0 with $\sum_{j=1}^{p} \mu(J_j) < \eta$ we have*

$$\left\| \sum_{j=1}^{p} \int_{J_j} f \right\| < \varepsilon.$$

Proof. Let $\epsilon > 0$ be given. Since f is M-integrable on I_0, there exists a positive function δ on I_0 such that $\left\| (D) \sum f(\xi)\mu(I) - \int_{I_0} f \right\| < \epsilon$ whenever $D = \{(I, \xi)\}$ is an arbitrary δ-fine M-partition of I_0. Fix a δ-fine M-partition of I_0

$$D_0 = \{(I_i, t_i) : 1 \leq i \leq q\},$$

put $M = \max\{\|f(t_i)\| ; 1 \leq i \leq q\}$ and set $\eta = \varepsilon/(M + 1)$.

Suppose that $\{J_j : 1 \leq j \leq p\}$ is a finite collection of non-overlapping intervals in I_0 such that $\sum_{j=1}^{p} \mu(J_j) < \eta$. By subdividing these intervals if necessary, we may assume that for each j, $J_j \subseteq I_i$ for some i. For each i, $1 \leq i \leq q$, let $M_i = \{j; 1 \leq j \leq p \text{ with } J_j \subseteq I_i\}$ and let

$$D = \{(J_j, t_i) : j \in M_i, i = 1, \ldots, q\}.$$

Note that D is a δ-fine partial M-partition of I_0.

Using the Saks-Henstock Lemma (Theorem 4), we have

$$\begin{aligned} \left\| \sum_{j=1}^{p} \int_{J_j} f \right\| &\leq \left\| \sum_{j=1}^{p} \int_{J_j} f - f(t_i)\mu(J_j) \right\| + \sum_{j=1}^{p} \|f(t_i)\| \, \mu(J_j) \\ &\leq \epsilon + M \sum_{j=1}^{p} \mu(J_j) < \epsilon + M\eta < 2\epsilon. \end{aligned}$$

□

REMARK 29. The above lemma can be reformulated as follows.

If F is the primitive of a McShane integrable function $f : I_0 \mapsto X$, then for every $\varepsilon > 0$ there is an $\eta > 0$ such that for any finite collection $\{J_j : 1 \leq j \leq p\}$ of non-overlapping intervals in I_0 with $\sum_{j=1}^{p} \mu(J_j) < \eta$ we have $\left\| \sum_{j=1}^{p} F(J_j) \right\| < \varepsilon$.

LEMMA 30. *Assume that F is an X-valued interval function defined for intervals in I_0 such that for every $\varepsilon > 0$ there is an $\eta >$ such that for any finite collection $\{J_j : 1 \leq j \leq p\}$ of non-overlapping intervals in I_0 with $\sum_{j=1}^{p} \mu(J_j) < \eta$ we have $\left\| \sum_{j=1}^{p} F(J_j) \right\| < \varepsilon$. Then:*

(a) *For any sequence $\{I_i : i = 1, 2, \cdots\}$ of non-overlapping intervals $I_i \subset I_0$, $i \in \mathbb{N}$, with $\sum_{i=1}^{\infty} \mu(I_i) \leq \mu(I_0)$ the limit*

$$\lim_{n \to \infty} \sum_{i=1}^{n} F(I_i) = \sum_{i=1}^{\infty} F(I_i) \in X$$

exists.

(b) *If for the sequence* $\{I_i : i = 1, 2, \cdots\}$ *of non-overlapping intervals we have* $\sum_{i=1}^{\infty} \mu(I_i) < \eta$*, where* $\eta > 0$ *is the value of* η *corresponding to* $\varepsilon > 0$ *by the assumption, then* $\|\sum_{i=1}^{\infty} F(I_i)\| \leq \varepsilon$*.*

Proof. Suppose that $\{I_i : i = 1, 2, \cdots\}$ is a sequence of non-overlapping intervals with $\sum_{i=1}^{\infty} \mu(I_i) < \infty$. Assume that $\varepsilon > 0$ is given and that $\eta > 0$ is the value of η corresponding to ε by the assumption. Since $\sum_{i=1}^{\infty} \mu(I_i) < \infty$, there is an $N \in \mathbb{N}$ such that for $n > N$ we have $\sum_{i=n}^{\infty} \mu(I_i) < \eta$.

Assume that $n, m \in \mathbb{N}$, $N < n < m$. Then

$$\left\| \sum_{i=1}^{m} F(I_i) - \sum_{i=1}^{n} F(I_i) \right\| = \left\| \sum_{i=n+1}^{m} F(I_i) \right\| < \varepsilon,$$

because $\sum_{i=n+1}^{m} \mu(I_i) \leq \sum_{i=n}^{\infty} \mu(I_i) < \eta$. So $\sum_{i=1}^{n} F(I_i)$, $n \in \mathbb{N}$, is a Cauchy sequence in the Banach space X, and (a) is proved.

If $\sum_{i=1}^{\infty} \mu(I_i) < \eta$, then $\sum_{i=1}^{n} \mu(I_i) < \eta$ for every $n \in \mathbb{N}$ and therefore

$$\left\| \sum_{i=1}^{n} F(I_i) \right\| < \varepsilon \quad \text{for every} \quad n \in \mathbb{N}.$$

Since by (a) $\sum_{i=1}^{\infty} F(I_i) \in X$ exists, we obtain $\| \sum_{i=1}^{\infty} F(I_i) \| \leq \varepsilon$ and (b) is proved. □

LEMMA 31. *If* f *is McShane integrable on* I_0*, then for every open set* $G \subset I_0$ *there is an element* $x_G \in X$ *such that*

$$\int_G x^*(f) = x^*(x_G)$$

for every $x^* \in X^*$*.*

Proof. From Theorem 9 it follows that f is weakly measurable and for every $x^* \in X^*$ the real function $x^*(f)$ is McShane and therefore also Lebesgue integrable.

Given a λ, $0 < \lambda < 1$, an interval I in R^m is called λ-regular if

$$r(I) = \frac{\mu(I)}{[d(I)]^m} > \lambda,$$

where $d(I) = \sup\{|x - y|;\ x, y \in I\}$, $|x - y| = \max\{|x_1 - y_1|, \ldots, |x_m - y_m|\}$, and $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$. ($r(I)$ is the regularity of the interval I.)

Suppose that G is an open subset of I_0. For $t \in G$ let $\delta(t) > 0$ be such that $B(t, \delta(t)) \subset G$ and assume that $0 < \delta_n(t) < \delta(t)$ for $n \in \mathbb{N}$, where $\delta_n(t) > \delta_{n+1}(t)$, $\delta_n(t) \to 0$ for $n \to \infty$.

Let $0 < \lambda < 1$ be fixed. For $n \in \mathbb{N}$ define

$$\Phi_n = \{I \subset I_0, I \text{ is an interval};\ t \in I \subset B(t, \delta_n(t)), r(I) > \lambda, t \in G\}.$$

Then $\Phi = \bigcup_{n=1}^{\infty} \Phi_n$ is a Vitali cover of G, and if $I \in \Phi$ then $I \subset G$.

By the Vitali covering theorem (see, e.g., [12, Proposition 9.2.4]) there is a sequence E_n, $n \in \mathbb{N}$, where E_n is a finite union of non-overlapping intervals belonging to Φ, such that

$$\mu(G \setminus E_n) < \frac{1}{n},$$

i.e., $\mu(G \setminus E_n) \to 0$ for $n \to \infty$ and $E_n \subset G$ for any $n \in \mathbb{N}$.

Set $E_0 = \bigcup_{n=1}^{\infty} E_n$. Since $G \setminus E_0 \subset G \setminus E_n$ for every $n \in \mathbb{N}$, we have $\mu(G \setminus E_0) \le \mu(G \setminus E_n) < 1/n$ for every $n \in \mathbb{N}$ and consequently $\mu(G \setminus E_0) = 0$. This yields $\mu(E_0) = \mu(G)$.

Let us set $F_n = \bigcup_{i=1}^{n} E_i$. Then clearly $F_n \nearrow E_0$ for $n \to \infty$, and for every $n \in \mathbb{N}$ the set F_n can be expressed as a finite union of non-overlapping intervals in $\mathbb{R}^m$.

Set $F_0 = \emptyset$ and define $K_n = F_n \setminus F_{n-1}^o$, where F_{n-1}^o is the interior of the set F_{n-1}. We have $E_0 = \bigcup_{n=1}^{\infty} K_n$, $K_n^o \cap K_l^o = \emptyset$ for $n \neq l$, and again K_n can be expressed as a finite union of non-overlapping intervals in $\mathbb{R}^m$, i.e.,

$$K_n = \bigcup_{i=1}^{p_n} I_i^n,$$

while $\{I_i^n;\ i = 1, \dots, p_n, n \in \mathbb{N}\}$ forms an at most countable system of non-overlapping intervals contained in E_0.

Since $\bigcup_{n=1}^{p} K_n \subset E_0$, $p \in \mathbb{N}$, we have

$$\sum_{n=1}^{p} \mu(K_n) = \mu\left(\bigcup_{n=1}^{p} K_n\right) \le \mu(E_0) = \mu(G) \le \mu(I_0) < \infty.$$

This gives

$$\sum_{n=1}^{\infty} \mu(K_n) = \sum_{n=1}^{\infty} \mu\left(\sum_{i=1}^{p_n} I_i^n\right) = \sum_{n=1}^{\infty} \sum_{i=1}^{p_n} \mu(I_i^n) < \infty,$$

and by Lemmas 28 and 30 we obtain the existence of the limit

$$\lim_{m\to\infty} \sum_{n=1}^{m} \sum_{i=1}^{p_n} F(I_i^n) = \lim_{m\to\infty} \sum_{n=1}^{m} F(K_n) = x_G \in X,$$

where F is the McShane primitive of f.

Given $x^* \in X^*$, the real function $x^*(f)$ is McShane integrable on I_0, and therefore it is also Lebesgue integrable on I_0. Hence the Lebesgue integral $\int_G x^*(f)$ exists and

$$\int_G x^*(f) = \int_{E_0} x^*(f),$$

because $\mu(G \setminus E_0) = 0$ and $E_0 \subset G$. Further we have

$$\begin{aligned}\int_{E_0} x^*(f) &= \int_{\cup_{n=1}^{\infty} K_n} x^*(f) = \int_{\cup_{n=1}^{\infty} \bigcup_{i=1}^{p_n} I_i^n} x^*(f) \\ &= \lim_{m\to\infty} \int_{\bigcup_{n=1}^{m} \bigcup_{i=1}^{p_n} I_i^n} x^*(f) = \lim_{m\to\infty} x^*\left(\int_{\bigcup_{n=1}^{m} \bigcup_{i=1}^{p_n} I_i^n} f\right) \\ &= \lim_{m\to\infty} x^*\left(\int_{\bigcup_{n=1}^{m} K_n} f\right) = \lim_{m\to\infty} x^*\left(\sum_{n=1}^{m} F(K_n)\right) = x^*(x_G),\end{aligned}$$

and $\int_G x^*(f) = x^*(x_G)$ for every $x^* \in X^*$. The proof is complete. $\square$

LEMMA 32. *If f is McShane integrable on I_0, then for every closed set $H \subset I_0$ there is an element $x_H \in X$ such that*

$$\int_H x^*(f) = x^*(x_H)$$

for every $x^* \in X^*$.

Proof. If $H \subset I_0$ is closed then $I_0 \setminus H$ is open and for every $x^* \in X^*$ we have (cf. Theorem 9 and Lemma 31)

$$\begin{aligned}x^*((M)\int_{I_0} f) = \int_{I_0} x^*(f) &= \int_H x^*(f) + \int_{I_0 \setminus H} x^*(f) \\ &= \int_H x^*(f) + x^*(x_{I_0\setminus H}),\end{aligned}$$

where for the open set $I_0 \setminus H$ the element $x_{I_0\setminus H} \in X$ is given by Lemma 31. Hence

$$\int_H x^*(f) = x^*((M)\int_{I_0} f - x_{I_0\setminus H}),$$

and we can take $x_H = (M)\int_{I_0} f - x_{I_0\setminus H} \in X$. $\square$

LEMMA 33. *If f is McShane integrable on I_0, $G \subset I_0$ is open, then for every $\varepsilon > 0$ there is an $\eta > 0$ such that if $\mu(G) < \eta$, then $\|x_G\| < \varepsilon$, where $x_G \in X$ is such that $\int_G x^*(f) = x^*(x_G)$ for every $x^* \in X^*$.*

Proof. As in the proof of Lemma 31 we see that there exists a sequence of sets $K_n \subset G$, $n \in \mathbb{N}$, which are finite unions of non-overlapping intervals and satisfy $K_n^o \cap K_l^o = \emptyset$ for $n \neq l$, such that for every $x^* \in X^*$ we have

$$\int_G x^*(f) = \lim_{m\to\infty} x^*\left(\sum_{n=1}^{m} F(K_n)\right) = x^*(x_G).$$

By Lemma 28 and Lemma 30(b), for every $\varepsilon > 0$ there is an $\eta > 0$ such that if $\sum_{n=1}^{\infty} \mu(K_n) < \eta$ then $\| \sum_{n=1}^{\infty} F(K_n) \| = \| x_G \| < \varepsilon$. Hence the lemma is proved. $\square$

THEOREM 34. *If f is McShane integrable on I_0, then f is Pettis integrable, i.e., $\mathcal{M} \subset \mathcal{P}$.*

Proof. By Definition 13 and Theorem 9 it only remains to prove is that for every measurable subset E of I_0 there is an element $x_E \in X$ which satisfies $x^*(x_E) = \int_E x^*(f)$ for every $x^* \in X^*$.

Suppose that E is a measurable subset of I_0. Then there exists a sequence of open sets $G_n \subset I_0$, $n \in \mathbb{N}$, such that

$$E \subset \cdots \subset G_{n+1} \subset G_n \subset \cdots \quad \text{and} \quad \mu(G_n \setminus E) < \frac{1}{2n}$$

and a sequence of closed sets $H_n \subset I_0$, $n \in \mathbb{N}$, such that

$$\cdots \subset H_n \subset H_{n+1} \subset \cdots \subset E \quad \text{and} \quad \mu(E \setminus H_n) < \frac{1}{2n}.$$

Then $G_n \setminus H_n$, $n \in \mathbb{N}$, are open sets and

$$\mu(G_n \setminus H_n) = \mu(G_n \setminus E \cup E \setminus H_n) = \mu(G_n \setminus E) + \mu(E \setminus H_n) < \frac{1}{n}.$$

Since $G_n \subset I_0$, $n \in \mathbb{N}$, are open sets, by Lemma 31 there exist x_{G_n}, $n \in \mathbb{N}$, such that

$$\int_{G_n} x^*(f) = x^*(x_{G_n}) \quad \text{for every} \quad x^* \in X^*.$$

Let $\varepsilon > 0$ be given and let $\eta > 0$ be the value corresponding to $\varepsilon/2$ by Lemma 33. Then there exists $N \in \mathbb{N}$, such that $1/n < \eta$ if $n \geq N$ and therefore $\mu(G_n \setminus H_n) < \eta$ for $n \geq N$.

Assume that $m_1, m_2 > N$. Then

$$\begin{aligned}
\left|x^*(x_{G_{m_1}} - x_{G_{m_2}})\right| &= \left|\int_{G_{m_1}} x^*(f) - \int_{G_{m_2}} x^*(f)\right| \\
&= \left|\int_{G_{m_1}} x^*(f) - \int_{H_N} x^*(f) - \int_{G_{m_2}} x^*(f) + \int_{H_N} x^*(f)\right| \\
&= \left|\int_{G_{m_1}\setminus H_N} x^*(f) - \int_{G_{m_2}\setminus H_N} x^*(f)\right| \\
&\leq \left|\int_{G_{m_1}\setminus H_N} x^*(f)\right| + \left|\int_{G_{m_2}\setminus H_N} x^*(f)\right| \\
&\leq \|x^*\| \left(\|x_{G_{m_1}\setminus H_N}\| + \left\|x_{G_{m_2}\setminus H_N}\right\|\right) \leq \|x^*\|\varepsilon
\end{aligned}$$

because $G_{m_1} \setminus H_N \subset G_N \setminus H_N$ and $\mu(G_{m_1} \setminus H_N) \leq \mu(G_N \setminus H_N) < \eta$, and similarly $\mu(G_{m_2} \setminus H_N) < \eta$.

Hence for every $x^* \in B(X^*)$ we have $|x^*(x_{G_{m_1}} - x_{G_{m_2}})| < \varepsilon$ provided $m_1, m_2 > N$ and therefore $\|x_{G_{m_1}} - x_{G_{m_2}}\| < \varepsilon$ for $m_1, m_2 > N$. The sequence $x_{G_n} \in X$, $n \in \mathbb{N}$, is therefore Cauchy, and consequently the limit $\lim_{m\to\infty} x_{G_m} = x_E \in X$ exists.

Moreover, we have $E \subset \bigcap_{m=1}^{\infty} G_m$ and $\bigcap_{m=1}^{\infty} G_m \setminus E \subset G_n \setminus E$ for every $n \in \mathbb{N}$ and therefore $\mu(\bigcap_{m=1}^{\infty} G_m \setminus E) \le \mu(G_n \setminus E) < 1/2n$, $n \in \mathbb{N}$, and $\mu(\bigcap_{m=1}^{\infty} G_m \setminus E) = 0$. Hence

$$
\begin{aligned}
x^*(x_E) &= \lim_{m\to\infty} x^*(x_{G_m}) = \lim_{m\to\infty} \int_{G_m} x^*(f) \\
&= \int_{\bigcap_{m=1}^{\infty} G_m} x^*(f) = \int_E x^*(f)
\end{aligned}
$$

for all $x^* \in X^*$.

This holds for any measurable $E \subset I_0$. Therefore, by definition, f is Pettis integrable, $(P)\int_E f = x_E$, and the theorem is proved. $\square$

5. Remarks on previous results

Let us note that Theorem 19 of R.A. Gordon yields the following result (see [5]).

Theorem 35. *If the Banach space X is separable and $f : I_0 \mapsto X$, where $I_0 \subset \mathbb{R}$, is Pettis integrable, then f is McShane integrable on I_0, i.e., $\mathcal{P} \subset \mathcal{M}$.*

Let us mention that if X is a separable Banach space, the weak measurability of a function $f : I_0 \mapsto X$ is equivalent to its (strong) measurability and X has the property (P). For this fact see the Pettis theorem and its proof in [15, V.5]. Hence, if the conditions of Gordon's Theorem 35 are fulfilled, then also the conditions of our Theorem 23 hold, and Theorem 23 implies Theorem 35.

On the other hand, the property (P) does not imply the separability of the space X. From this point of view our Theorem 23 is slightly more general than Gordon's Theorem 35.

D.H. Fremlin and J. Mendoza [3, Theorem 2C] proved our Theorem 34 for the case of one-dimensional intervals I_0.

From Theorems 26, 34 and 35 we obtain the following result:

Theorem 36. *Let the Banach space X be reflexive with the property (P) or separable. Then $f : [a, b] \mapsto X$ is McShane integrable on $[a, b]$ if and only if f is Pettis integrable, i.e., $\mathcal{M} = \mathcal{P}$.*

D.H. Fremlin [2, Theorem 8] proved the following result for one-dimensional intervals (i.e., for the case $m = 1$):

Theorem 37. *Let X be a Banach space. Then $f : [a, b] \mapsto X$ is McShane integrable on $[a, b]$ if and only if it is Pettis integrable and Henstock-Kurzweil integrable.*

By Theorems 26, 35 and 37 this yields that if the Banach space X is reflexive with the property (P) or separable, then every Pettis integrable function $f : [a, b] \mapsto X$ is automatically Henstock-Kurzweil integrable.

Let us denote by $\mathcal{B}$ the set of all Bochner integrable functions $f : I_0 \mapsto X$ (see, e.g., [1], [10]). In our paper [13] we gave some characterizations of $\mathcal{B}$ and we proved the following result.

Theorem 38. *The inclusion $\mathcal{B} \subset \mathcal{M}$ holds in general, and we have $\mathcal{B} = \mathcal{M}$ if and only if the Banach space X is finite dimensional.*

Since finite dimensional Banach spaces are separable, we can combine this with Theorems 26 and 35 to obtain the following result.

Theorem 39. *The inclusion $\mathcal{B} \subset \mathcal{P}$ holds in general, and we have $\mathcal{B} = \mathcal{P}$ if and only if the Banach space X is finite dimensional.*

Added in proof. Luisa Di Piazza and David Preiss pointed out that the assumption of reflexivity of the Banach space X together with the property (P) imply the separability of X. Therefore the result of Theorem 26 is a consequence of Gordon's result presented in Theorem 35. The authors express their thanks to L. Di Piazza and D. Preiss for this essential observation as well as for their progress in characterizing Banach spaces for which the Pettis and McShane integrals coincide. As we know, their sophisticated and deep results in this direction will be published soon.

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