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PARAMETER-DEPENDENT OPERATORS AND RESOLVENT EXPANSIONS ON CONIC MANIFOLDS

PAUL LOYA

ABSTRACT. The goal of this paper is to present, from the b-calculus perspective, a program for analyzing generalized resolvent families and heat kernels of pseudodifferential operators on conic manifolds. To carry this out, a new class of parameter-dependent cone pseudodifferential operators is developed and studied.

1. Introduction

This paper further develops the work initiated in [12], henceforth called 'Part I'. The purpose of Part I was to give a comprehensive treatment, from the b-calculus viewpoint, of a space of pseudodifferential operators depending on a parameter $\lambda \in \mathbb{C}$ that captures operators of the form $B(A-\lambda)^{-N}$ for $N \in \mathbb{N}$, where A and B are cone (or Fuchs Type) differential operators on a compact manifold with boundary. This space describes in a precise way the structure of the Schwartz kernels of such operators, especially their uniform asymptotics as λ tends to infinity in the spectral parameter domain. This precise structure allows the coefficients occurring in the trace asymptotics of $B(A-\lambda)^{-N}$ as $|\lambda| \to \infty$ to be identified in terms of residue traces and zeta-functions introduced in [18], [24], [10], and can be used to express the complex powers BA^z as an entire family of cone pseudodifferential operators and to describe the meromorphic properties of its Schwartz kernel both on and off the diagonal [14], [13]. In certain applications however, most notably to the non-commutative residue, one requires B to be a cone pseudodifferential operator. In this paper we extend the machinery of Part I so as to allow such pseudodifferential factors. This involves the introduction of a new parameter-dependent space of pseudodifferential operators, whose development and properties are the main focus of this work.

We first describe our class of operators. Let X be a compact manifold with boundary and let x be a boundary defining function. A cone differential

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operator of order m is an operator of the form $A = x^{-m}P$, where P is an m-th order totally characteristic, or b-differential operator. We denote this space of cone operators by $x^{-m}\mathrm{Diff}_b^m(X)$. Thus, A is a usual differential operator on the interior of X such that in any collar $X \cong [0,\varepsilon)_x \times \partial X$ we have

$$A = x^{-m} \sum_{k=0}^{m} P_{m-k}(x)(xD_x)^k,$$

where $P_{m-k}(x)$ is a family of differential operators of order $m-k$ on ∂X depending smoothly in x. The weight factor x^{-m} could be replaced with x^{-b} for any real number b.

In order to develop an elliptic theory of cone operators, appropriate spaces of pseudodifferential operators have been developed by various authors; to name a few: Melrose [17], Plamenevskij [20], Rempel and Schulze [22], and Schulze [27]. The operators we choose to work with are the b-pseudodifferential operators of Melrose [17]. Thus, our class of cone pseudodifferential operators of order $b, q \in \mathbb{R}$ is the space $x^{-b}\Psi_b^q(X)$, where $\Psi_b^q(X)$ is the space of b-pseudodifferential operators of order q.[1]

We now discuss the role of parameter-dependency. Gil [5] defines a natural ellipticity condition which ensures that for an operator $A \in x^{-m}\mathrm{Diff}_b^m(X)$, where m is positive, the resolvent $(A-\lambda)^{-1}$ exists as an operator between appropriate weighted Sobolev spaces for all λ sufficiently large in some sector $\Lambda \subset \mathbb{C}$; cf. Section 3.1. We remark that this ellipticity condition encompasses a wide class of cone operators that includes elliptic self-adjoint ones. Fix such an operator A, and let $B \in x^{-b}\Psi_b^q(X)$ with $b, q \in \mathbb{R}$. Then the generalized resolvent $B(A-\lambda)^{-1}$ can be described very precisely as a cone pseudodifferential operator depending smoothly on the parameter λ, as long as λ is within any *bounded*, but arbitrary, subset of Λ [17, Ch. 6]. However, for applications to the noncommutative residue, spectral asymptotics, spectral geometry, index theory, etc., it is not enough to understand the generalized resolvent for finite spectral parameter; the asymptotics as $|\lambda| \to \infty$ in Λ are also needed. This is exactly the reason why it is necessary to incorporate λ as a symbolic variable at the start within the very definition of cone pseudodifferential operators. We mention that the development of parameter-dependent operators has a long history originating from Seeley's paper [28] and continued in a variety of areas of geometric analysis; cf. Grubb [7], Grubb and Seeley [8], Rempel and Schulze [21], and Schrohe [23].

There are two main approaches to incorporating the spectral parameter into the definition of cone operators. The first systematic method was initiated by Gil [5], which relies on techniques from the 'edge theory' of Schulze. The second method was developed in Part I of this paper [12] using the geometric 'b-calculus' approach of Melrose. As explained in [9], the methods of Melrose

[1] The subscript b is for boundary and is not related to the number b in the factor x^{-b}.

and Schulze are essentially equivalent, although their presentations are quite different. Gil expresses his parameter-dependent operators as two parts, an interior part and a boundary part. The interior part is a usual parameter-dependent operator in the sense of Seeley [28] (see also Agmon and Kannai [1] and Shubin [29]). The boundary part is an operator-valued symbol within the edge symbolic calculus introduced by Schulze [26], [27]. The geometric approach of Melrose handles the singularities and parameter-dependency on a global scale by tailoring these features into the geometry of the Schwartz kernels of the operators. These two complementary viewpoints each have their advantages. For instance, the methods of Schulze can be extended quite directly to manifolds with higher singularities [25]. The methods of Melrose are capable to extract with great precision the geometric structure of the Schwartz kernels. We mention that the resolvents of self-adjoint second order cone differential operators can be analyzed without the formal development of such calculi; cf. Brüning and Seeley [2], Cheeger [3], [4], Lesch [10], and Mooers [19]. We especially note that the methods of Mooers are the closest to those of this paper: she also relies on the geometric techniques of Melrose [17, Ch. 7] to determine the structure of the heat kernel of the cone Laplacian.

The calculi of Gil and Part I are able to capture $B(A-\lambda)^{-1}$ only when B is differential. In this paper we extend the results of Part I to allow B to be pseudodifferential. The added ingredient (see Section 3.2) is a new parameter-dependent space of operators which is a 'conic' version of the one introduced in [11]. Using the precise description of the Schwartz kernel of $B(A-\lambda)^{-1}$ within the new parameter-dependent space, an application of Melrose's pushforward theorem [16] yields the following trace expansion.

Trace Expansion. *Let $B \in x^{-b}\Psi_b^q(X)$, $b, q \in \mathbb{R}$ with $b < m$, and $A \in x^{-m}\mathrm{Diff}_b^m(X)$ be as above. Then for N sufficiently large, $B(A-\lambda)^{-N}$ is trace class and*

$$(1.1)\quad \operatorname{Tr} B(A-\lambda)^{-N} \sim_{|\lambda|\to\infty} \sum_{k=0}^{\infty} \Big\{ a_k + b_k \log\lambda + c_k (\log\lambda)^2 \Big\} \lambda^{(q+n-k)/m-N} + \sum_{k=0}^{\infty} \Big\{ d_k + e_k \log\lambda \Big\} \lambda^{(b-k)/m-N} + \sum_{k=0}^{\infty} f_k \lambda^{-k-N}.$$

Moreover, $b_k = 0$ unless $k \in (\mathbb{N}_0 + q + n - b) \cup (m\mathbb{N}_0 + q + n)$; $c_k = 0$ unless $k \in m\mathbb{N}_0 \cap (\mathbb{N}_0 - b) + q + n$; and $e_k = 0$ unless $k \in m\mathbb{N}_0 + b$.

The $\log^2$ terms in the expansion (1.1) are in general nonzero; this follows from a joint report with Juan Gil [6] on the noncommutative residue of cone operators. On the other hand, as shown in Part I, when B is differential there are no $\log^2$ terms.

In Section 2, we review some fundamental aspects of analysis on manifolds with corners, including 'blow-ups' and b-pseudodifferential operators. In Section 3.1, we review the parameter-dependent spaces introduced in Part I, and in Section 3.2, we introduce our new parameter-dependent space. The main result of this paper is Theorem 3.7, which expresses $B(A-\lambda)^{-1}$, where B is pseudodifferential, as an element of our new space. In order to prove this, we need to analyze the composition of B with $(A-\lambda)^{-1}$. We give a direct 'geometric' proof of this composition result in Section 4. In Section 5, we prove the trace expansion (1.1). As mentioned already, once the structure of $B(A-\lambda)^{-1}$ is identified within the new parameter-dependent space, the proof of (1.1) is basically just an application of Melrose's pushforward theorem. We review this pushforward theorem in the Appendix.

Finally, I thank the referee for helpful comments in improving this paper.

2. b-pseudodifferential operators

In this section, we review various ideas useful for analysis on manifolds with corners. References include [17], [15] or even Part I [12].

2.1. Manifolds with corners and asymptotic expansions. An n-dimensional manifold with corners X is a topological space with C^∞ structure given by local charts of the form $[0,1)^k \times (-1,1)^{n-k}$, where k can run between 0 and n depending on where the chart is located in the manifold. We assume that each of the boundary hypersurfaces has a boundary defining function. Thus, given a hypersurface H, there is a nonnegative function $\varrho \in C^\infty(X)$ that vanishes only on H where it has a nonzero differential.

We now review asymptotic expansions. Let $\mathcal{U} = [0,1)^k_x \times (-1,1)^{n-k}_y$. Then the space $S^a(\mathcal{U})$, where $a \in \mathbb{R}$, consists of those functions u on $\mathcal{U}$ of the form

$$u(x,y) = x_1^a \cdots x_k^a \, v(x,y),$$

where for each α and β, $(x\partial_x)^\alpha \partial_y^\beta v(x,y)$ is a bounded function. Let $\mathbb{N}_0 = \{0,1,\ldots\}$ and $\mathbb{N} = \{1,2,\ldots\}$. An index set is a subset $E \subset \mathbb{C} \times \mathbb{N}_0$ such that $(z,k) \in E \Rightarrow (z,\ell) \in E$ for all $0 \le \ell \le k$ and such that given any $M \in \mathbb{R}$, $\{(z,k) \in E \,:\, \operatorname{Re} z \le M\}$ is finite. If in addition, $(z,k) \in E \Rightarrow (z+\ell,k) \in E$ for all $\ell \in \mathbb{N}_0$, then E is a C^∞ index set. For simplicity, we use the word 'index set' to mean 'C^∞ index set' unless stated otherwise. Given an index set E, a function u on $\mathcal{U}$ has an asymptotic expansion at $x_1 = 0$ with index set E if it has the following expansion property: For each $N \in \mathbb{N}$ we can write

$$u(x,y) = \sum_{(z,k)\in E,\, \operatorname{Re} z \le N} x_1^z \, (\log x_1)^k \, u_{(z,k)}(x',y) \;+\; x_1^N \, u_N(x,y), \tag{2.1}$$

where for some $a \in \mathbb{R}$ independent of N, $u_N(x,y) \in S^a(\mathcal{U})$ and $u_{(z,k)}(x',y) \in S^a(\mathcal{U}')$, where $x = (x_1, x')$ and $\mathcal{U}' = [0,1)^{k-1}_{x'} \times (-1,1)^{n-k}_y$. Note that if $E = \varnothing$, then the expansion property (2.1) holds for each N if and only if

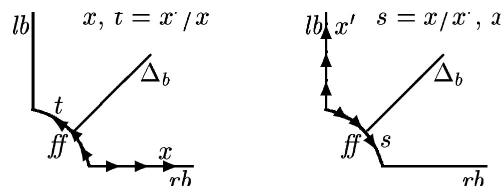

FIGURE 1. Each of these coordinates together with coordinates on Y^2 define projective coordinates on X_b^2 near *ff*.

u vanishes to infinite order at $x_1 = 0$. An asymptotic expansion at any $x_i = 0$ is defined similarly. On any manifold with corners X one can define asymptotic expansions at a hypersurface H with index set E by reference to local coordinates. Thus, a function u on X has an asymptotic expansion at H with index set E if u is smooth on the interior of X and if for any patch $\mathcal{U} = [0,1)_{x_1} \times \mathcal{U}'$ on X, where $\mathcal{U}' = [0,1)_{x'}^{k-1} \times (-1,1)_y^{n-k}$ and $H \cap \mathcal{U} = \{x_1 = 0\}$, the function $\varphi(x,y)\, u(x,y)$ for any $\varphi \in C_c^\infty(\mathcal{U})$ has an asymptotic expansion at $x_1 = 0$ with index set E in the sense described above. In [11] we show that this expansion property is defined independent of the coordinates chosen.

2.2. ***b*-pseudodifferential operators.** Let X be an n-dimensional compact manifold with connected boundary $Y = \partial X$. Fix a boundary defining function x that gives a decomposition $X \cong [0,1)_x \times Y$.

Recall that the manifold X_b^2 is the manifold X^2 blown-up at Y^2. All this means is that coordinates on X_b^2 include both polar coordinate charts around Y^2 and the usual charts on $X^2 \setminus Y^2$. For example, denoting the coordinates on the right factor of X^2 by primes, we have $X^2 \cong [0,1)_x \times [0,1)_{x'} \times Y^2$ near Y^2. Then taking polar coordinates in the $[0,1)_x \times [0,1)_{x'}$ factor along with coordinates on Y^2 give coordinates on X_b^2 near the blown-up face. Figure 1 shows two illustrations of X_b^2 as it is usually drawn. Convenient coordinates to work with are projective coordinates, which are also shown in Figure 1. In X_b^2, the left boundary *lb* is the face coming from $x = 0$, the right boundary *rb* is the face coming from $x' = 0$, the front face *ff* is the face created in the blow-up, and Δ_b is the b-diagonal coming from the diagonal in X^2.

A *b-measure* is a density of the form $x^{-1}\times$ a smooth positive density on X. We henceforth fix a b-measure ν on X and we let ν' denote the lift of ν to X^2 under the right projection $X^2 \ni (p,q) \mapsto q \in X$. Given $m \in \mathbb{R}$, the space of *b-pseudodifferential operators*, $\Psi_b^m(X)$ consists of operators A on $C^\infty(X)$ that have a Schwartz kernel K_A satisfying the following two conditions:

(1) Given $\varphi \in C_c^\infty(X_b^2 \setminus \Delta_b)$, the kernel φK_A is of the form $k\,\nu'$, where k is a smooth function on X_b^2 vanishing to infinite order at *lb* and *rb*.

(2) Given a coordinate patch of X_b^2 overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}^n_w$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$ and given $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, we have

$$\varphi K_A = \int e^{iw\cdot\xi}\, a(y,\xi)\, đ\xi \cdot \nu', \quad đ\xi = \frac{1}{(2\pi)^n} d\xi,$$

where $a(y,\xi)$ is a classical symbol of order m.

One can check that this space of operators is defined independent of the choice of b-measure ν. The b-Sobolev space of order $m \in \mathbb{R}$, $H_b^m(X)$, is the space of distributions u such that for all $A \in \Psi_b^m(X)$, $Au \in L_b^2(X)$, the space of functions that are square integrable with respect to ν.

3. Parameter-dependent operators

3.1. Review of Part I. In this section, we review the parameter-dependent operators introduced in [12]. Taking notation from [7], we define $\dot{\leq}$ to read "less than or equal to a constant times". A *sector* is a closed angle of $\mathbb{C}$ with vertex at the origin.

We begin by defining a class of parameter-dependent symbols due to Seeley. Given $m \in \mathbb{R}$ and $d \in \mathbb{N}$, we denote by $S_\Lambda^{m,d}(\mathbb{R}^n)$ the space of functions $a \in C^\infty(\Lambda \times \mathbb{R}^n)$ satisfying the following estimates: for each α, β,

$$\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda,\xi) \;\dot{\leq}\; (1 + |\lambda|^{1/d} + |\xi|)^{m-d|\alpha|-|\beta|}. \tag{3.1}$$

The corresponding classical subspace is defined as follows: Given $m \in \mathbb{R}$ and $d \in \mathbb{N}$, the space $S_{\Lambda,c\ell}^{m,d}(\mathbb{R}^n)$ consists of those $a(\lambda,\xi) \in S_\Lambda^{m,d}(\mathbb{R}^n)$ such that

$$a(\lambda,\xi) \sim \sum_{j=0}^{\infty} \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi), \tag{3.2}$$

where $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) = 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) = 1$ outside a neighborhood of 0, where $a_{m-j}(\lambda,\xi)$ is a smooth function of $(\lambda,\xi) \in \Lambda \times \mathbb{R}^n \setminus \{(0,0)\}$ such that $a_{m-j}(\delta^d\lambda, \delta\xi) = \delta^{m-j} a_{m-j}(\lambda,\xi)$ for all $\delta > 0$, and finally, where the asymptotic sum (3.2) means that for each $N \in \mathbb{N}$,

$$a(\lambda,\xi) - \sum_{j=0}^{N-1} \chi(\lambda,\xi)\, a_{m-j}(\lambda,\xi) \in S_\Lambda^{m-N,d}(\mathbb{R}^n).$$

Example 3.1. These symbol spaces are designed to capture the local symbols of $(A-\lambda)^{-1}$, where $A \in x^{-m}\mathrm{Diff}_b^m(X)$. Let $a(\xi)$ be an elliptic homogeneous polynomial on $\mathbb{R}^n$ of order m that never takes values in a sector Λ for $\xi \neq 0$. Given $N \in \mathbb{N}_0$, one can check that $\chi(\lambda,\xi)\,(a(\xi)-\lambda)^{-N} \in S_{\Lambda,c\ell}^{-Nm,m}(\mathbb{R}^n)$.

Before defining our operator spaces, we fix some terminology. Let $\overline{\Lambda}$ denote the stereographic compactification of Λ. We denote by $\partial_\infty\overline{\Lambda} \subset \overline{\Lambda}$ the boundary in the limit as $|\lambda| \to \infty$ in Λ. For a function u on $\Lambda \times M$, where M is a compact manifold with corners, we say that u vanishes to infinite order as $|\lambda| \to \infty$ if considered as a function on the compact manifold $\overline{\Lambda} \times M$, u has an expansion at the hypersurface $\partial_\infty \overline{\Lambda} \times M$ with index set $\varnothing$. We say that $u(\lambda, q)$ can be expanded in q at a hypersurface H of M with index set E if considered as a function on $\overline{\Lambda} \times M$, u has an expansion at $\overline{\Lambda} \times H$ with index set E.

We are now ready to define our basic parameter-dependent operators. The definition is similar to that of b-pseudodifferential operators given in Section 2.2. Let ϱ be a boundary defining function for *ff* of X_b^2 (e.g., $\varrho = x + x'$ is such a function). We henceforth fix one such function. Recall that ν' denotes the fixed b-measure ν lifted to X^2 under the right projection $X^2 \ni (p, q) \mapsto q \in X$.

Given $m \in \mathbb{R}$ and $d \in \mathbb{N}$, $\Psi_{c,\Lambda}^{m,d}(X)$ consists of parameter-dependent operators $A(\lambda)$ defined for $\lambda \in \Lambda$ that have a Schwartz kernel $K_{A(\lambda)}$ satisfying the following two conditions:

1. Given $\varphi \in C_c^\infty(X_b^2 \setminus \Delta_b)$, the kernel $\varphi K_{A(\lambda)}$ is of the form $k(\varrho^d \lambda, q)\, \nu'$, where $k(\lambda, q)$ is a smooth function of $(\lambda, q) \in \Lambda \times X_b^2$ vanishing to infinite order as $|\lambda| \to \infty$ and in q at the sets lb and rb.
2. Given a coordinate patch of X_b^2 overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}_w^n$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$ and given $\varphi \in C_c^\infty(\mathcal{U} \times \mathbb{R}^n)$, we have
$$\varphi K_{A(\lambda)} = \int e^{iw\cdot\xi}\, a(\varrho^d \lambda, y, \xi)\, đ\xi \cdot \nu',$$
where $y \mapsto a(\lambda, y, \xi)$ is smooth with values in $S_{\Lambda,c\ell}^{m,d}(\mathbb{R}^n)$.

One can check that the space $\Psi_{c,\Lambda}^{m,d}(X)$ is defined independent of the choice of boundary defining function ϱ. The function $k(\lambda, q)$ in (1) is assumed to be smooth up to the sides of the sector Λ, that is, when considered on $\overline{\Lambda} \times X_b^2$, k has an expansion at the sides of $\overline{\Lambda}$ in $\overline{\Lambda} \times X_b^2$ with index set $\mathbb{N}_0$. To avoid repetition, the kernels $k(\lambda, q)$ mentioned in all definitions henceforth are assumed to be smooth up to the sides of the sector Λ.

In view of Example 3.1, one may think that $(A - \lambda)^{-N} \in x^{Nm} \Psi_{c,\Lambda}^{-Nm,m}(X)$, where $A \in x^{-m}\mathrm{Diff}_b^m(X)$. However, because of the boundary spectrum of A, this is only true up to certain operators of order $-\infty$ defined as follows.

To define our second space of parameter-dependent operators, let $\mathcal{E} = (E_{lb}, E_{rb}, E_{ff}, E)$ be a set of four index sets. We define $\Psi_{c,\Lambda}^{-\infty,d,\mathcal{E}}(X)$ as the class of parameter-dependent operators $A(\lambda)$ defined for $\lambda \in \Lambda$ that have a Schwartz kernel $K_{A(\lambda)}$ satisfying the following two conditions:

1. The kernel $K_{A(\lambda)}$ is of the form $k(\lambda, q)\, \nu'$, where $k(\lambda, q)$ is a smooth function of $(\lambda, q) \in \Lambda \times \mathrm{int}(X_b^2)$ such that if q is restricted to a compact subset C of X_b^2 disjoint to *ff*, then $k(\lambda, q)$ vanishes to infinite order as $|\lambda| \to \infty$ and can be expanded in q at the sets $C \cap lb$ and $C \cap rb$ with index sets E_{lb} and E_{rb}, respectively. Moreover, if λ is restricted

to a compact subset of Λ, then $k(\lambda, q)$ can be expanded in q at the sets *lb*, *rb*, and *ff* with index sets E_{lb}, E_{rb}, and E_{ff}, respectively.

(2) Given a collar $[0,\varepsilon)_\varrho \times \mathit{ff}_y$ of *ff* in X^2_b, $k(\lambda, q)$ can be written in the form $k(r, \theta, v, y)$, where $r = |\lambda|^{-1/d}$, $v = \varrho|\lambda|^{1/d}$, and $\theta = \lambda/|\lambda|$. Moreover, k is smooth in θ; and k has expansions at $r = 0$ with index set E, at $v = 0$ with index set E_{ff}, at $y \in lb$ with index set E_{lb}, and at $y \in rb$ with index set E_{rb}. Finally, k vanishes to infinite order as $v \to \infty$; that is, in terms of the variable $w = 1/v$, k has an expansion at $w = 0$ with index set $\varnothing$.

Our final space of operators are parameter-dependent integral operators. Denote by $lb = Y \times X$ and $rb = X \times Y$, the left and right boundaries of X^2. Let $\mathcal{E} = (E_{lb}, E_{rb})$ be a family of index sets on X^2. We denote by $\Psi^{-\infty,\mathcal{E}}_\Lambda(X)$, the space of parameter-dependent operators $A(\lambda)$ whose Schwartz kernels are of the form $K_A = k(\lambda, q)\,\nu'$, where $k(\lambda, q)$ is a smooth function of $(\lambda, q) \in \Lambda \times \mathrm{int}(X^2)$ that vanishes to infinite order as $|\lambda| \to \infty$ and can be expanded in q at the sets *lb* and *rb* with index sets E_{lb} and E_{rb}, respectively.

We are now ready to describe the main result of Part I concerning the resolvent of cone differential operators. Let $A = x^{-m}P \in x^{-m}\mathrm{Diff}^m_b(X)$, where we henceforth assume that $m > 0$. If P is an elliptic b-differential operator, then $\mathrm{spec}_c(A)$ is the set of points where the conormal symbol (or normal operator) of P fails to be invertible [17, Ch. 5.2]. In the decomposition $X \cong [0,1)_x \times Y$ near Y, writing $A = x^{-m}\sum_{k=0}^m P_{m-k}(x)(xD_x)^k$, where $P_{m-k}(x)$ are differential operators on Y depending smoothly on x, we define

$$I(A) = \rho^{-m}\sum_{k=0}^m P_{m-k}(0)(\rho D_\rho)^k.$$

This operator is called the *indicial operator* of A and it models the infinitesimal behavior of A at Y. Then $I(A)$ is an operator on $Y^\wedge = [0,\infty)_\rho \times Y$. The natural space of functions on which it acts is defined as follows. Let $\chi \in C^\infty_c([0,\infty))$ with $\chi(\rho) = 1$ near $\rho = 0$. For $\ell \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$, the space $H^{\ell,\alpha}_c(Y^\wedge)$ consists of distributions u on $Y^\wedge$ such that $\chi u \in \rho^\alpha H^\ell_b(Y^\wedge)$ and such that given any coordinate patch $\mathcal{U}$ on Y diffeomorphic to an open subset of $\mathbb{S}^{n-1}$ and function $\varphi \in C^\infty_c(\mathcal{U})$, we have $(1-\chi)\varphi\, u \in H^\ell(\mathbb{R}^n)$, where $(0,\infty) \times \mathbb{S}^{n-1}$ is identified with $\mathbb{R}^n \setminus \{0\}$ via polar coordinates.

We say that A is *fully elliptic* with respect to $\alpha \in \mathbb{R}$ on a sector Λ if the following three conditions hold:

(1) ${}^b\sigma_m(P)(\xi) - \lambda$ is invertible for all $\xi \neq 0$ and $\lambda \in \Lambda$. Here, ${}^b\sigma_m(P)(\xi)$ is the totally characteristic principal symbol of P (see [17, Ch. 4]);
(2) $\alpha \notin -\mathrm{Im}\,\mathrm{spec}_c(A)$;
(3) $I(A) - \lambda : H^{m,\alpha}_c(Y^\wedge) \longrightarrow H^{0,\alpha-m}_c(Y^\wedge)$ is invertible for all $\lambda \in \Lambda$ sufficiently large.

Theorem 3.2. ([12, Th. 6.1]) *Let $A \in x^{-m}\mathrm{Diff}_b^m(X)$ be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ. Then for all $\lambda \in \Lambda$ sufficiently large,*

$$A - \lambda : x^{\alpha} H_b^s(X) \longrightarrow x^{\alpha-m} H_b^{s-m}(X)$$

is invertible and for any $N \in \mathbb{N}$, we have

$$\begin{aligned}(A-\lambda)^{-N} &\in x^{Nm}\Psi_{c,\Lambda}^{-Nm,m}(X) \\ &+ x^{Nm}\Psi_{c,\Lambda}^{-\infty,m,\mathcal{E}_N(\alpha)}(X) \ + \ x^{Nm}\Psi_{\Lambda}^{-\infty,\mathcal{F}_N(\alpha)}(X),\end{aligned}$$

where $\mathcal{E}_N(\alpha) = (E_{N,lb}(\alpha), E_{N,rb}(\alpha), E_{N,\mathit{ff}}(\alpha), \mathbb{N}_0)$ is an index family satisfying

$$E_{N,lb}(\alpha) > \alpha - Nm, \quad E_{N,rb}(\alpha) > -(\alpha - m), \quad E_{N,\mathit{ff}}(\alpha) \geq m - Nm, \tag{3.3}$$

and where $\mathcal{F}_N(\alpha) = (E_{N,lb}(\alpha), E_{N,rb}(\alpha))$.

Remark 3.3. It is possible to give (complicated) explicit expressions for the index sets of $\mathcal{E}_N(\alpha)$ in terms of $\mathrm{spec}_c(A)$; see [12, Sec. 3.2] for the details.

3.2. A new parameter-dependent operator space. We now analyze the composition $B(A-\lambda)^{-1}$, where $B \in x^{-b}\Psi_b^q(X)$, $b, q \in \mathbb{R}$. It turns out that $B(A-\lambda)^{-1}$ is an operator in a new parameter-dependent space. We begin by reviewing a class of parameter-dependent symbols introduced in [11]. A similar class of symbols was defined by Grubb and Seeley [8].

Given $m, p, d \in \mathbb{R}$ with $p/d \in \mathbb{Z}$ and $d > 0$, the class $S_{\Lambda,r}^{m,p,d}(\mathbb{R}^n)$ consists of functions $a(\lambda,\xi)$ smooth on $\Lambda \times \mathbb{R}^n$ such that if we define $\widetilde{a}(z,\xi) = z^{p/d}a(1/z,\xi)$, then $\widetilde{a}(z,\xi)$ is smooth at $z = 0$, and the following estimates are satisfied: given any α and β,

$$\begin{aligned}&\partial_\lambda^\alpha \partial_\xi^\beta a(\lambda,\xi) \ \dot{\leq}\ (1 + |\lambda|^{1/d} + |\xi|)^{p-d|\alpha|}(1+|\xi|)^{m-p-|\beta|}, \\ &\partial_z^\alpha \partial_\xi^\beta \widetilde{a}(z,\xi) \ \dot{\leq}\ (1 + |z||\xi|^d)^{p/d-|\alpha|}(1+|\xi|)^{d|\alpha|+m-p-|\beta|}, \quad |z| \leq 1.\end{aligned} \tag{3.4}$$

The subscript 'r' in $S_{\Lambda,r}^{m,p,d}(\mathbb{R}^n)$ stands for 'resolvent'.

The classical subspace is defined as follows: The space $S_{\Lambda,rc\ell}^{m,p,d}(\mathbb{R}^n)$ consists of those $a(\lambda,\xi) \in S_{\Lambda,r}^{m,p,d}(\mathbb{R}^n)$ such that

$$a(\lambda,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\, a_{m-j}(\lambda,\xi), \tag{3.5}$$

where $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is such that $\chi(\xi) = 0$ near $\xi = 0$ and $\chi(\xi) = 1$ outside a neighborhood of 0, and where $a_{m-j}(\lambda,\xi)$ is smooth for $(\lambda,\xi) \in \Lambda \times (\mathbb{R}^n \setminus \{0\})$ and anisotropic homogeneous, i.e., $a_{m-j}(\delta^d\lambda, \delta\xi) = \delta^{m-j}a_{m-j}(\lambda,\xi)$ for all $\delta > 0$, and is such that if we define $\widetilde{a}_{m-j}(z,\xi) = z^{p/d}a_{m-j}(1/z,\xi)$, then $\widetilde{a}_{m-j}(z,\xi)$ is smooth at $z = 0$. The asymptotic sum (3.5) means that for each

$N \in \mathbb{N}$,

$$a(\lambda,\xi) - \sum_{j=0}^{N-1} \chi(\xi)\, a_{m-j}(\lambda,\xi) \in S^{m-N,p,d}_{\Lambda,r}(\mathbb{R}^n). \tag{3.6}$$

EXAMPLE 3.4. These spaces are designed to capture the local symbols of $B(A-\lambda)^{-1}$. Let $a(\xi)$ be an elliptic homogeneous polynomial on $\mathbb{R}^n$ of order $m \in \mathbb{N}$ that never takes values in a sector Λ for $\xi \neq 0$, and let $b(\xi)$ be a homogeneous function of degree $q \in \mathbb{R}$. Given $N \in \mathbb{N}_0$, set

$$a_b(\lambda,\xi) = b(\xi)(a(\xi)-\lambda)^{-N}.$$

Then one can check that $\chi(\xi)\, a_b(\lambda,\xi) \in S^{q-Nm,-Nm,m}_{\Lambda,rc\ell}(\mathbb{R}^n)$. Here, the cut-off $\chi(\xi)$ is needed because $b(\xi)$ is in general not smooth at $\xi = 0$.

We now define our new parameter-dependent space of operators. Recall that ϱ denotes a boundary defining function for *ff* and ν' denotes the fixed b-measure ν lifted to X^2 under the right projection $X^2 \ni (p,q) \mapsto q \in X$.

DEFINITION 3.5. Given $m,p,d \in \mathbb{R}$ with $p/d \in \mathbb{Z}$ and $d > 0$, $\Psi^{m,p,d}_{c,\Lambda}(X)$ consists of parameter-dependent operators $A(\lambda)$ that have a Schwartz kernel $K_{A(\lambda)}$ satisfying the following two conditions:

(1) Given $\varphi \in C^\infty_c(X^2_b \setminus \Delta_b)$, the kernel $\varphi K_{A(\lambda)}$ is of the form $k(\varrho^d\lambda, q)\,\nu'$, where $k(\lambda,q)$ is a smooth function of $(\lambda,q) \in \Lambda \times X^2_b$ that vanishes to infinite order in q at the sets *lb* and *rb* and is such that if we define $\widetilde{k}(z,q) = z^{p/d} k(1/z,q)$, then $\widetilde{k}(z,q)$ is smooth at $z = 0$.

(2) Given a coordinate patch of X^2_b overlapping Δ_b of the form $\mathcal{U}_y \times \mathbb{R}^n_w$ such that $\Delta_b \cong \mathcal{U} \times \{0\}$ and given $\varphi \in C^\infty_c(\mathcal{U} \times \mathbb{R}^n)$, we have

$$\varphi K_{A(\lambda)} = \int e^{iw\cdot\xi}\, a(\varrho^d\lambda, y, \xi)\, đ\xi \cdot \nu',$$

where $y \mapsto a(\lambda,y,\xi)$ is smooth with values in $S^{m,p,d}_{\Lambda,rc\ell}(\mathbb{R}^n)$.

One can check that this space of operators is defined independent of the choice of boundary defining function ϱ.

THEOREM 3.6. *Let $A \in x^{-m}\mathrm{Diff}^m_b(X)$ be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ. Then for all $\lambda \in \Lambda$ sufficiently large,*

$$A - \lambda : x^\alpha H^s_b(X) \longrightarrow x^{\alpha-m} H^{s-m}_b(X)$$

is invertible, and moreover, for any $N \in \mathbb{N}$,

$$(A-\lambda)^{-N} \in x^{Nm}\Psi^{-Nm,-Nm,m}_{c,\Lambda}(X) + x^{Nm}\Psi^{-\infty,m,\mathcal{E}_N(\alpha)}_{c,\Lambda}(X) \;+\; x^{Nm}\Psi^{-\infty,\mathcal{F}_N(\alpha)}_{\Lambda}(X),$$

where $\mathcal{E}_N(\alpha)$ and $\mathcal{F}_N(\alpha)$ are the same index families in Theorem 3.2.

Proof. To see why this theorem holds, let $\chi(\lambda,\xi) \in C^\infty(\Lambda \times \mathbb{R}^n)$ with $\chi(\lambda,\xi) = 0$ near $(\lambda,\xi) = 0$ and $\chi(\lambda,\xi) = 1$ outside a neighborhood of 0, and let $a(\xi)$ be an elliptic homogeneous polynomial on $\mathbb{R}^n$ of order m that never takes values in Λ for $\xi \neq 0$. Then according to Example 3.1, $\chi(\lambda,\xi)\,(a(\xi) - \lambda)^{-N} \in S^{-Nm,m}_{\Lambda,c\ell}(\mathbb{R}^n)$. On the other hand, by Example 3.4, it follows that $\chi(\lambda,\xi)\,(a(\xi) - \lambda)^{-N} \in S^{-Nm,-Nm,m}_{\Lambda,rc\ell}(\mathbb{R}^n)$ also.

Now according to [12, Th. 6.1] (see Theorem 3.2) we have

$$\begin{aligned}(A-\lambda)^{-N} \in\ & x^{Nm}\Psi^{-Nm,m}_{c,\Lambda}(X)\\ &+ x^{Nm}\Psi^{-\infty,m,\mathcal{E}_N(\alpha)}_{c,\Lambda}(X) \ + \ x^{Nm}\Psi^{-\infty,\mathcal{F}_N(\alpha)}_{\Lambda}(X),\end{aligned}$$

where the first term on the right is constructed in Lemma 6.4 of loc. cit. by an explicit local symbolic construction. However, in view of the last example considered in the previous paragraph, reviewing the explicit construction it is evident that the first term lies in $x^{Nm}\Psi^{-Nm,-Nm,m}_{c,\Lambda}(X)$ also. For further details we refer the reader to [12, Lem. 6.4]. ☐

The main result of this paper is now just a consequence of Theorem 3.6 and the composition propositions 4.1 and 4.2 of the next section:

THEOREM 3.7. *Let $A \in x^{-m}\mathrm{Diff}^m_b(X)$ be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ. Then given any $B \in x^{-b}\Psi^q_b(X)$, $b, q \in \mathbb{R}$, for $\lambda \in \Lambda$ sufficiently large, we have for any $N \in \mathbb{N}$,*

$$\begin{aligned}B(A-\lambda)^{-N} \in\ & x^{Nm-b}\Psi^{q-Nm,-Nm,m}_{c,\Lambda}(X)\\ &+ x^{Nm-b}\Psi^{-\infty,m,\mathcal{E}_N(\alpha)}_{c,\Lambda}(X) \ + \ x^{Nm-b}\Psi^{-\infty,\mathcal{F}_N(\alpha)}_{\Lambda}(X),\end{aligned}$$

where $\mathcal{E}_N(\alpha)$ and $\mathcal{F}_N(\alpha)$ are the same index families in Theorem 3.2.

4. Composition properties

We now analyze the composition of b-pseudodifferential operators with our parameter-dependent spaces. One can show that the space $\Psi^{*,d}_{c,\Lambda}(X)$ is not closed under such compositions and this is exactly the reason for the need of a new parameter-dependent space. (However, one can check that $\Psi^{*,d}_{c,\Lambda}(X)$ is closed under compositions with b-differential operators; cf. [12].) In the following propositions, $\alpha, \alpha', m, m', p, d \in \mathbb{R}$ with $p/d \in \mathbb{Z}$ and $d > 0$.

PROPOSITION 4.1.
We have

$$x^\alpha\Psi^m_b(X) \circ x^{\alpha'}\Psi^{m',p,d}_{c,\Lambda}(X) \subset x^{\alpha+\alpha'}\Psi^{m+m',p,d}_{c,\Lambda}(X),$$

and

$$x^{\alpha'}\Psi^{m',p,d}_{c,\Lambda}(X) \circ x^\alpha\Psi^m_b(X) \subset x^{\alpha+\alpha'}\Psi^{m+m',p,d}_{c,\Lambda}(X).$$

The next proposition shows that the spaces $\Psi^{-\infty,d,\mathcal{E}}_{c,\Lambda}(X)$ and $\Psi^{-\infty,\mathcal{E}}_{\Lambda}(X)$ are closed under compositions with b-pseudodifferential operators.

Proposition 4.2.

(A) *For $d \in \mathbb{N}$ we have*

$$x^{\alpha}\Psi^m_b(X) \circ x^{\alpha'}\Psi^{-\infty,d,\mathcal{E}}_{c,\Lambda}(X) \subset x^{\alpha+\alpha'}\Psi^{-\infty,d,\mathcal{E}}_{c,\Lambda}(X),$$

and

$$x^{\alpha'}\Psi^{-\infty,d,\mathcal{E}}_{c,\Lambda}(X) \circ x^{\alpha}\Psi^m_b(X) \subset x^{\alpha'}\Psi^{-\infty,d,\mathcal{F}}_{c,\Lambda}(X),$$

where $\mathcal{F} = (E_{lb}, E_{rb} + \alpha, E_{ff} + \alpha, E + \alpha)$.

(B) *We have*

$$x^{\alpha}\Psi^m_b(X) \circ x^{\alpha'}\Psi^{-\infty,\mathcal{E}}_{\Lambda}(X) \subset x^{\alpha+\alpha'}\Psi^{-\infty,E_{lb},E_{rb}}_{\Lambda}(X),$$

$$x^{\alpha'}\Psi^{-\infty,\mathcal{E}}_{\Lambda}(X) \circ x^{\alpha}\Psi^m_b(X) \subset x^{\alpha'}\Psi^{-\infty,E_{lb},E_{rb}+\alpha}_{\Lambda}(X).$$

We prove Proposition 4.1 and Part (A) of Proposition 4.2 in Section 4.2. In order to prove them, we first need to review how b-pseudodifferential operators are composed. The proofs in Section 4.2 are quite technical and may be omitted at a first reading.

Proof of (B) in Proposition 4.2. As $\Psi^m_b(X)x^{\alpha'} = x^{\alpha'}\Psi^m_b(X)$, to prove the first statement in (B) we may assume that $\alpha = \alpha' = 0$. Let $A \in \Psi^m_b(X)$ and let $B \in \Psi^{-\infty,\mathcal{E}}_{\Lambda}(X)$. Then $K_B = B(\lambda, p, q)\nu(q)$, where $B(\lambda, p, q)$ is a function on $\Lambda \times X \times X$ that has expansions at $p \in Y$ and $q \in Y$ with index sets E_{lb} and E_{rb}, respectively, and vanishes to infinite order as $|\lambda| \to \infty$. Since the kernel of AB is just A acting on the variable p of $B(\lambda, p, q)\nu(q)$, the mapping properties of b-pseudodifferential operators (see [17]) imply that K_{AB} has the same asymptotic properties as K_B. By considering transposes, one can show that the second statement in (B) follows from the first. □

4.1. How b-pseudodifferential operators are composed. Let A and B be operators on $C^\infty(X)$ with Schwartz kernels K_A and K_B, respectively, that are smooth on X^2 and vanish to infinite order at ∂X^2. Then we know that AB is also a smoothing operator, and

$$K_{AB}(u, w) = \int_{v \in X} K_A(u, v)K_B(v, w). \tag{4.1}$$

We can write this purely in terms of pullbacks and pushforwards of distributions as follows. Let $\pi_F, \pi_S, \pi_C : X^3 \to X^2$ be the maps

$$\pi_F(u, v, w) = (u, v), \quad \pi_S(u, v, w) = (v, w), \quad \pi_C(u, v, w) = (u, w).$$

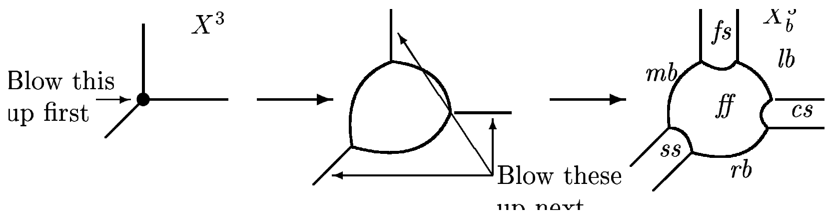

FIGURE 2. How X_b^3 is defined.

(F, S, and C stand for 'first', 'second', and 'composite'.) Writing $K_A = k_A\nu'$ and $K_B = k_B\nu'$, where k_A and k_B are smooth functions on X^2 vanishing to infinite order at ∂X^2, we have

$$(\pi_C^*\nu\, \pi_F^* K_A\, \pi_S^* K_B)(u,v,w) = k_A(u,v)\, k_B(v,w)\, \nu(u)\nu(v)\nu(w),$$

where on the left-hand side, ν represents the fixed b-measure on X lifted to X^2 under the left projection, that is, $\nu(u,w) = \nu(u)$ for all $(u,w) \in X^2$. In particular, $\pi_C^*\nu\, \pi_F^* K_A\, \pi_S^* K_B$ is a density on X^3 and so its pushforward to X^2 under π_C is well-defined. By (4.1) and the definition of $(\pi_C)_*$ it follows that

$$\nu K_{AB} = (\pi_C)_*(\pi_C^*\nu\, \pi_F^* K_A\, \pi_S^* K_B). \tag{4.2}$$

This equality shows that we can determine the Schwartz kernel of AB by analyzing pullbacks, products, and pushforwards of their Schwartz kernels. If $A, B \in \Psi_b^*(X)$, then we would like to use (4.2) to show that $AB \in \Psi_b^*(X)$. However, in this case K_A and K_B are distributions that naturally live on X_b^2 rather than on X^2. Thus, we need to rewrite (4.2) so that it is valid for kernels on X_b^2. To do so, we introduce the blown-up manifold X_b^3.

The manifold X_b^3 is defined by blowing-up (that is, introducing polar coordinates around) the manifold Y^3 in X^3 and then blowing-up (that is, introducing polar coordinates around) the submanifolds coming from the codimension two corners of X^3. The manifold X_b^3 along with its various faces are shown in Figure 2. The abbreviations *lb*, *rb*, *mb*, *fs*, *ss*, *cs*, and *ff* for the hypersurfaces shown in the picture of X_b^3 are for left boundary, right boundary, middle boundary, first side, second side, composite side, and front face, respectively (not that these names are important).

Let $\pi_{F,b}, \pi_{S,b}, \pi_{C,b} : X_b^3 \to X_b^2$ be the maps $\pi_F, \pi_S, \pi_C : X^3 \to X^2$ expressed in the polar coordinates of X_b^3 and X_b^2. Then we can express the composition (4.2) in terms of these new maps:

$$\nu K_{AB} = (\pi_{C,b})_*(\pi_{C,b}^*\nu\, \pi_{F,b}^* K_A\, \pi_{S,b}^* K_B). \tag{4.3}$$

Written in this way, the function νK_{AB} on the left-hand side is understood to be the kernel νK_{AB} lifted to X_b^2 (that is, written in terms of the polar coordinates given on the space X_b^2), and on the right-hand side ν, K_A, and

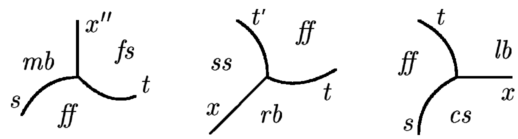

FIGURE 3. Some projective coordinates on X_b^3.

K_B are understood to be lifted to X_b^2. The formula (4.3) is the key to proving composition properties of b-pseudodifferential operators.

4.2. Composition of parameter-dependent operators. The proofs of Propositions 4.1 and 4.2 are based on the composition formula (4.3). We use local coordinates to analyze the pullbacks and pushforwards appearing in that formula. For simplicity, we assume that $X = [0,1)_x$ and that $\Lambda = [0,\infty)$. The argument for any compact manifold with boundary and parameter domain is not essentially different; the main difference is the annoying appearance of other variables (e.g., the variables on the boundary Y and the variables θ and y that appear in (2) in the definition of $\Psi_{c,\Lambda}^{-\infty,d,\mathcal{E}}(X)$) that make the proof complicated notationally. Composition proofs with all the variables appearing can be found in [12, Lem. 4.3] and [12, Th. 4.4].

Proof of (A) in Proposition 4.2. As $\Psi_b^m(X)x^{\alpha'} = x^{\alpha'}\Psi_b^m(X)$, to prove the first statement in (A) we may assume that $\alpha = \alpha' = 0$. Moreover, considering transposes one can show that the second statement in (A) follows from the first. Let $A \in \Psi_b^m(X)$ and $B \in \Psi_{c,\Lambda}^{-\infty,d,\mathcal{E}}(X)$ be supported near $x = 0$. We show that $AB \in \Psi_{c,\Lambda}^{-\infty,d,\mathcal{E}}(X)$. To prove this, we use the composition formula (4.3) above.

We use projective coordinates on X_b^2 as shown in Figure 1:

$$(s,x'),\ s = x/x' \quad \text{coordinates near } lb \text{ of } X_b^2; \tag{4.4}$$

$$(x,t),\ t = x'/x \quad \text{coordinates near } rb \text{ of } X_b^2; \tag{4.5}$$

and we assume that $\nu = |dx/x|$. Recall that we are assuming $X = [0,1)$. The coordinates (4.4) and (4.5) will be used throughout the following arguments. Let x, x', x'' be the left, middle, and right coordinates of X^3.

Step 1: Assume that $\pi_{C,b}^*\nu\,\pi_{F,b}^*K_A\,\pi_{S,b}^*K_B$ is supported near the intersection of mb, ff, and fs of X_b^3. In this region of X_b^3 we use the coordinates (s,t,x''), where $s = x/x''$ and $t = x'/x$, seen in the left-hand picture in Figure 3. In these coordinates, $\pi_{C,b}$ and $\pi_{S,b}$ map near lb in X_b^2 and in terms of the coordinates (4.4) on X_b^2 near lb, are given by

$$\pi_{S,b}(s,t,x'') = (st,x''); \quad \pi_{C,b}(s,t,x'') = (s,x''). \tag{4.6}$$

Also note that $\pi_{F,b}$ maps near rb in X_b^2 and in the coordinates (4.5),

$$\pi_{F,b}(s,t,x'') = (sx'',t). \tag{4.7}$$

Near rb in X_b^2, we can write $K_A = A(x,t)|dx'/x'|$, where $A(x,t)$ is smooth in x and t and vanishes to infinite order at $t=0$. Near lb in X_b^2, we can write $K_B = B(r,s,v')|dx'/x'|$, where $r=\lambda^{-1/d}$ and $v'=x'/r$, and where $B(r,s,v')$ has expansions at $r=0$, $s=0$, $v'=0$, and $v'=\infty$, with index sets E, E_{lb}, E_{ff}, and $\varnothing$, respectively. Using the formulas for $\pi_{S,b}$, $\pi_{C,b}$, and $\pi_{F,b}$ in (4.6) and (4.7), it follows that

$$\pi_{C,b}^*\nu\,\pi_{F,b}^*K_A\,\pi_{S,b}^*K_B = A(sx'',t)B(r,st,x''/r)\Big|\frac{ds\,dt\,dx''}{stx''}\Big|.$$

As $\pi_{C,b}(s,t,x'') = (s,x'')$ is a fibration, we have

$$\begin{aligned}\nu K_{AB} = (\pi_{C,b})_*(\pi_{C,b}^*\nu\,\pi_{F,b}^*K_A\,\pi_{S,b}^*K_B) &= \int A(sx',t)B(r,st,x'/r)\frac{dt}{t}\cdot\Big|\frac{ds\,dx'}{sx'}\Big| \\ &= C(r,s,v')\Big|\frac{ds\,dx'}{sx'}\Big|,\end{aligned}$$

where $C(r,s,v') = \int A(srv',t)B(r,st,v')dt/t$ with $v'=x'/r$. From this formula for $C(r,s,v')$ it is straightforward to check that the asymptotic properties of $A(x,t)$ and $B(r,s,v')$ imply that $C(r,s,v')$ has expansions at $r=0$, $s=0$, $v'=0$, and $v'=\infty$, with index sets E, E_{lb}, E_{ff}, and $\varnothing$, respectively.

Step 2: Assume that $\pi_{C,b}^*\nu\,\pi_{F,b}^*K_A\,\pi_{S,b}^*K_B$ is supported near the intersection of *ss*, *rb*, and *ff* of X_b^3. Here we use the coordinates (x,t,t'), where $t=x'/x$ and $t'=x''/x'$, seen in the middle picture in Figure 3. In these coordinates, $\pi_{F,b}$, $\pi_{S,b}$, and $\pi_{C,b}$ all map near rb in X_b^2. Moreover, in the coordinates (4.5) on X_b^2, we have

$$\pi_{F,b}(x,t,t') = (x,t); \quad \pi_{S,b}(x,t,t') = (xt,t'); \quad \pi_{C,b}(x,t,t') = (x,tt'). \tag{4.8}$$

Near rb in X_b^2, we can write $K_A = A(x,t)|dx'/x'|$, where $A(x,t)$ is smooth in x and t and vanishes to infinite order at $t=0$. Near rb in X_b^2, we can write $K_B = B(r,v,t)|dx'/x'|$, where $v=x/r$, and where $B(r,v,t)$ has expansions at $r=0$, $v=0$, $v=\infty$, and $t=0$, with index sets E, E_{ff}, $\varnothing$, and E_{rb}, respectively. Using the formulas in (4.8), we have

$$\pi_{C,b}^*\nu\,\pi_{F,b}^*K_A\,\pi_{S,b}^*K_B = A(x,t)B(r,xt/r,t')\Big|\frac{dx\,dt\,dt'}{xtt'}\Big|.$$

Hence, as $\pi_{C,b}(x,t,t') = (x,tt')$, by (A.5) of the Appendix,

$$\begin{aligned}\nu K_{AB} = (\pi_{C,b})_*(\pi_{C,b}^*\nu\,\pi_{F,b}^*A\,\pi_{S,b}^*B) &= \int A(x,t')B(r,xt'/r,t/t')\frac{dt'}{t'}\cdot\Big|\frac{dx\,dt}{xt}\Big| \\ &= C(r,v,t)\Big|\frac{dx\,dt}{xt}\Big|,\end{aligned}$$

where $C(r,v,t) = \int c(r,v,t/t',t')dt'/t'$ with $c(r,v,t,t') = A(rv,t')B(r,vt',t)$. Now the asymptotic properties of $A(x,t)$ and $B(r,v,t)$ imply that $c(r,v,t,t')$ has expansions at $r=0$, $v=0$, $v=\infty$, $t=0$, and $t'=0$ with index sets E, E_{ff}, $\varnothing$, E_{rb}, and $\varnothing$, respectively. It follows that $C(r,v,t)$ has expansions at $r=0$, $v=0$, and $v=\infty$, with index sets E, E_{ff}, and $\varnothing$, respectively; and by Proposition A.3, $C(r,v,t)$ has an expansion at $t=0$ with index set E_{rb}.

Step 3: Assume that $\pi^*_{C,b}\nu\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B$ is supported near the intersection of *ff*, *cs*, and *lb* of X^3_b. We now use the coordinates (s,x',t), where $s = x/x''$ and $t = x''/x'$, seen in the right-hand picture in Figure 3. In these coordinates, $\pi_{C,b}$ and $\pi_{F,b}$ map near *lb* in X^2_b, and in terms of the coordinates (4.4) on X^2_b, we have

$$\pi_{F,b}(s,x',t) = (st,x'); \quad \pi_{C,b}(s,x',t) = (s,x't). \tag{4.9}$$

Also note that $\pi_{S,b}$ maps near *rb* in X^2_b and in the coordinates (4.5),

$$\pi_{S,b}(s,x',t) = (x',t). \tag{4.10}$$

Near *lb* in X^2_b, we can write $K_A = A(s,x')|dx'/x'|$, where $A(s,x')$ is smooth in s and x' and vanishes to infinite order at $s=0$. If $v = x/r$, then near *rb* in X^2_b, $K_B = B(r,v,t)|dx'/x'|$, where $B(r,v,t)$ has expansions at $r=0$, $v=0$, $v=\infty$, and $t=0$, with index sets E, E_{ff}, $\varnothing$, and E_{rb}, respectively. Using (4.9) and (4.10), it follows that

$$\pi^*_{C,b}\nu\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B = A(st,x')B(r,x'/r,t)\Big|\frac{dsdtdx'}{stx'}\Big|.$$

Hence, as $\pi_{C,b}(s,x',t) = (s,x't)$, by (A.5) of the Appendix,

$$\begin{aligned}\nu K_{AB} = (\pi_{C,b})_*(\pi^*_{C,b}\nu\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B) &= \int A(st,\frac{x'}{t})B(r,\frac{x'}{rt},t)\frac{dt}{t}\cdot\Big|\frac{dsdx'}{sx'}\Big| \\ &= C(r,s,v')\Big|\frac{dsdx'}{sx'}\Big|,\end{aligned}$$

where $C(r,s,v') = \int c(r,s,v'/t,t)dt/t$ with $c(r,s,v',t) = A(st,rv')B(r,v',t)$. Now the asymptotic properties of $A(s,x')$ and $B(r,v,t)$ imply that $c(r,s,v',t)$ has expansions at $r=0$, $s=0$, $v'=0$, $v'=\infty$, and $t=0$, with index sets E, $\varnothing$, E_{ff}, $\varnothing$, and $\varnothing$, respectively. It follows that $C(r,s,v')$ has expansions at $r=0$, $v'=\infty$, and $s=0$, with index sets E, $\varnothing$, and $\varnothing$, respectively; and by Proposition A.3, $C(r,s,v')$ has an expansion at $v'=0$ with index set E_{ff}.

Step 4: We have thus far shown that near three of the six codimension three corners of X^3_b, $\pi^*_{C,b}\nu\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B$ pushes forward under $\pi_{C,b}$ to be in $\nu\cdot\Psi^{-\infty,d,\mathcal{E}}_{c,\Lambda}(X)$. Similar arguments show that near the other three codimension three corners of X^3_b, $\pi^*_{C,b}\nu\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B$ pushes forward under $\pi_{C,b}$ to be in $\nu\cdot\Psi^{-\infty,d,\mathcal{E}}_{c,\Lambda}(X)$. Hence, we are left to prove that if $\pi^*_{C,b}\nu\,\pi^*_{F,b}K_A\,\pi^*_{S,b}K_B$ is supported in a neighborhood of $\pi^{-1}_{F,b}(\Delta_b)$ as shown in Figure 4, then it pushes

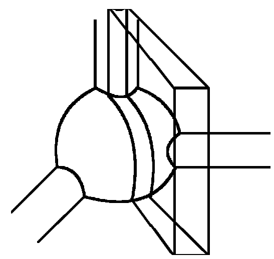

FIGURE 4. A neighborhood of $\pi_{F,b}^{-1}(\Delta_b)$, where X_b^3 is diffeomorphic to $X_b^2 \times \mathbb{R}_u$.

forward under $\pi_{C,b}$ to be in $\nu \cdot \Psi_{c,\Lambda}^{-\infty,d,\mathcal{E}}(X)$. To see this, we choose such a neighborhood so that (see Figure 4),

$$X_b^3 \cong X_b^2 \times \mathbb{R}_u, \quad \pi_{F,b}^{-1}(\Delta_b) \cong X_b^2 \times \{0\}_u, \tag{4.11}$$

where $\pi_{C,b}(p,u) = p$ and $\pi_{S,b}(p,u) = p$ for all $(p,u) \in X_b^2 \times \mathbb{R}_u$. Note that on the decomposition (4.11) (again see Figure 4), $\pi_{F,b}^* \varrho = \varrho \varrho_{lb}$, where ϱ_{lb} is a boundary defining function for *lb* of X_b^2. Let $X_b^2 \cong [0,1)_\varrho \times [-1,1]_y$, where $lb = \{y = -1\}$ and $rb = \{y = 1\}$. Then by the definitions of $\Psi_b^m(X)$ and $\Psi_{c,\Lambda}^{-\infty,d,\mathcal{E}}(X)$, on the decomposition (4.11) we can write

$$\pi_{C,b}^* \nu \, \pi_{F,b}^* K_A \, \pi_{S,b}^* K_B = \int e^{iu\cdot\xi} a(\varrho\varrho_{lb},\xi)\, đ\xi \cdot B(r,\varrho/r,y)|du|\nu\,\nu', \tag{4.12}$$

where $a(\varrho,\xi)$ is a symbol of order m, and $B(r,w,y)$ with $w = \varrho/r$ has expansions at $r = 0$, $w = 0$, $w = \infty$, $y = -1$, and $y = 1$, with index sets E, E_{ff}, $\varnothing$, E_{lb}, and E_{rb}, respectively. Thus, as $\pi_{C,b}(p,u) = p$, we have

$$\begin{aligned}\nu K_{AB} = (\pi_{C,b})_*(\pi_{C,b}^* \nu \, \pi_{F,b}^* K_A \, \pi_{S,b}^* K_B) &= a(\varrho\varrho_{lb},0) B(r,\varrho/r,y)\nu\,\nu' \\ &= C(r,w,y)\nu\,\nu',\end{aligned}$$

where $C(r,w,y) = a(\varrho\varrho_{lb},0)B(r,w,y)$ (with $\varrho\varrho_{lb}$ written in terms of r, w, and y). Since $a(\varrho,0)$ is smooth at $\varrho = 0$, the asymptotics of $B(r,w,y)$ imply that $C(r,w,y)$ has expansions at $r = 0$, $w = 0$, $w = \infty$, $y = -1$, and $y = 1$ with index sets E, E_{ff}, $\varnothing$, E_{lb}, and E_{rb}, respectively. The proof of (A) in Proposition 4.2 is now complete. □

Proof of Proposition 4.1. As in the proof of Proposition 4.2, it suffices to prove only the first statement in Proposition 4.1, and we may assume that $\alpha = \alpha' = 0$. Thus, let $A \in \Psi_b^m(X)$ and let $B \in \Psi_{c,\Lambda}^{m',p,d}(X)$.

Step 1: We can write $A = A_1 + A_2$, where K_{A_1} is supported near Δ_b and away from *lb* and *rb*, and where $A_2 \in \Psi_b^{-\infty}(X)$. Similarly, we can write $B = B_1 + B_2$, where K_{B_1} is supported near Δ_b and away from *lb* and *rb*, and where $B_2 \in \Psi^{-\infty,p,d}_{c,\Lambda}(X)$. Thus, $AB = A_1B_1 + A_1B_2 + A_2B_1 + A_2B_2$. Arguments very similar to those used in the proof of Part (A) in Proposition 4.2 can be used to show that $A_1B_2, A_2B_2 \in \Psi^{-\infty,p,d}_{c,\Lambda}(X)$. Thus, it remains to show that $A_1B_1 \in \Psi^{m+m',p,d}_{c,\Lambda}(X)$ and $A_2B_1 \in \Psi^{-\infty,p,d}_{c,\Lambda}(X)$.

Step 2: Consider first A_2B_1. By taking transposes, it suffices to prove that $B_1A_2 \in \Psi^{-\infty,p,d}_{c,\Lambda}(X)$. Since K_{B_1} is supported near Δ_b, we can use the decomposition (4.11). Thus, using the same notation and arguments found in the derivation of (4.12), we have

$$\pi^*_{C,b}\nu\, \pi^*_{F,b}K_{B_1}\, \pi^*_{S,b}K_{A_2} = \int e^{iu\cdot\xi} b(\varrho^d \varrho^d_{lb}\lambda, \varrho\varrho_{lb}, \xi)\, đ\xi \cdot A(\varrho, y)|du|\nu\, \nu',$$

where $b(\lambda, \varrho, \xi)$ is a classical parameter-dependent polyhomogeneous symbol of order m', p, d, and where $A(\varrho, y)$ is smooth and vanishes to infinite order at $y = -1$ and $y = 1$. Thus, as $\pi_{C,b}(p, u) = p$, we have

$$\begin{aligned}\nu K_{B_1A_2} = (\pi_{C,b})_*(\pi^*_{C,b}\nu\, \pi^*_{F,b}K_{B_1}\, \pi^*_{S,b}K_{A_2}) &= b(\varrho^d \varrho^d_{lb}\lambda, \varrho\varrho_{lb}, 0)A(\varrho, y)\nu\, \nu' \\ &= C(\varrho^d\lambda, \varrho, \varrho_{lb}, y)\nu\, \nu',\end{aligned}$$

where $C(\lambda, \varrho, \varrho_{lb}, y) = b(\varrho^d_{lb}\lambda, \varrho\varrho_{lb}, 0)A(\varrho, y)$. Since $b(\lambda, \varrho, 0)$ is such that if we define $\widetilde{b}(z, \varrho, \xi) = z^{p/d}b(1/z, \varrho, \xi)$, then $\widetilde{b}(z, \varrho, \xi)$ is smooth at $z = 0$ and since $A(\varrho, y)$ vanishes to infinite order at *lb* and *rb*, it follows that $C(\lambda, \varrho, \varrho_{lb}, y)$ vanishes to infinite order at *lb* and *rb* and is such that if we define $\widetilde{C}(z, \varrho, \varrho_{lb}, y) = z^{p/d}C(1/z, \varrho, \varrho_{lb}, y)$, then $\widetilde{C}(z, \varrho, \varrho_{lb}, y)$ is smooth at $z = 0$. Thus by definition, $B_1A_2 \in \Psi^{-\infty,p,d}_{c,\Lambda}(X)$.

Step 3: It remains to prove that $A_1B_1 \in \Psi^{m+m',p,d}_{c,\Lambda}(X)$. Since (x, t), where $t = x'/x$, are coordinates on X^2_b away from *lb*, it follows that (x, w), where $w = \log(x'/x)$ are coordinates on X^2_b away from *lb* and *rb*, such that $\Delta_b = \{w = 0\}$. Thus, we can write

$$K_{A_1} = \varphi(w) \int e^{iw\cdot\xi}\, a(x, \xi)\, đ\xi \cdot \nu',\ \ K_{B_1} = \int e^{iw\cdot\xi}\, b(x^d\lambda, x, \xi)\, đ\xi \cdot \nu',$$

where $\varphi(w) \in C^\infty_c(\mathbb{R})$ with $\varphi(w) = 1$ near $w = 0$, $a(x, \xi)$ is a classical symbol of order m, and $b(\lambda, x, \xi)$ is a classical parameter-dependent symbol of order m', p, d. A computation shows that the composition A_1B_1 has a kernel given explicitly by

$$K_{A_1B_1} = \int e^{iw\cdot\xi} c(x^d\lambda, x, \xi)\, đ\xi \cdot \nu',$$

where $c(\lambda, x, \xi) = \int a(x, \xi - \eta)\widetilde{b}(\lambda, x, \eta, \xi)\, đ\eta$ with

$$\widetilde{b}(\lambda, x, \eta, \xi) = \int e^{-iw\cdot\eta}\varphi(w)\, b(e^{wd}\lambda, xe^w, \xi)\, dw.$$

A straightforward computation (e.g., taking the Taylor series of $a(x,\xi-\eta)$ at $\eta=0$) shows that $x \mapsto c(\lambda,x,\xi)$ is smooth with values in $S^{m+m',p,d}_{\Lambda,rc\ell}(\mathbb{R})$. $\square$

5. Application: resolvent expansion

We now prove the trace expansion (1.1). Let $A \in x^{-m}\mathrm{Diff}^m_b(X)$ be fully elliptic with respect to $\alpha \in \mathbb{R}$ on a sector Λ and let $B \in x^{-b}\Psi^q_b(X)$, where $b, q \in \mathbb{R}$ with $b < m$. Then for $N \in \mathbb{N}$ such that $q - Nm < -n$, by Theorem 3.7 and the definitions of the various spaces of parameter-dependent operators, it follows that for $\lambda \in \Lambda$ sufficiently large, $B(A-\lambda)^{-N}$ exists and is trace class on $x^{\alpha-m}L^2_b(X)$. Moreover,

$$\operatorname{Tr} B(A-\lambda)^{-N} = \operatorname{Tr} F(\lambda) + \operatorname{Tr} G(\lambda) + \operatorname{Tr} H(\lambda),$$

where in the notation of Theorem 3.7, $F \in x^{Nm-b}\Psi^{q-Nm,-Nm,m}_{c,\Lambda}(X)$, $G \in x^{Nm-b}\Psi^{-\infty,m,\mathcal{E}_N(\alpha)}_{c,\Lambda}(X)$, and $H \in x^{Nm-b}\Psi^{-\infty,\mathcal{F}_N(\alpha)}_{\Lambda}(X)$. To investigate the traces $\operatorname{Tr} F(\lambda)$, $\operatorname{Tr} G(\lambda)$, and $\operatorname{Tr} H(\lambda)$, we shall need the following proposition, whose proof can be found in [11, Prop. 5.49].

Proposition 5.1. *Let $\varphi \in C^\infty(\Lambda)$ be holomorphic. Suppose that for some index set E, as $|\lambda| \to \infty$ in Λ, $\varphi(\lambda) \sim \sum_{(z,k)\in E} |\lambda|^{-z}(\log|\lambda|)^k \varphi_{(z,k)}(\theta)$, where $\varphi_{(z,k)}(\theta)$ is smooth in $\theta = \lambda/|\lambda|$. Then in fact, for some $\psi_{(z,k)} \in \mathbb{C}$, we have*

$$\varphi(\lambda) \sim_{|\lambda|\to\infty} \sum_{(z,k)\in E} \lambda^{-z}(\log\lambda)^k \psi_{(z,k)}.$$

Before continuing, we note that checking the arguments of Part I, it follows that $F(\lambda)$ is holomorphic in λ and $G(\lambda) + H(\lambda)$ is holomorphic in λ.

Lemma 5.2. *As $|\lambda| \to \infty$ in Λ, we have*

$$\operatorname{Tr} G(\lambda) + \operatorname{Tr} H(\lambda) \sim \sum_{k=0}^{\infty} \alpha_k \lambda^{(b-k)/m-N}, \quad \alpha_k \in \mathbb{C}. \tag{5.1}$$

Proof. If $\Delta \cong X$ is the diagonal in X^2, then

$$\operatorname{Tr} G(\lambda) + \operatorname{Tr} H(\lambda) = \int_X K_{G(\lambda)}|_\Delta + \int_X K_{H(\lambda)}|_\Delta.$$

By definition of $x^{Nm-b}\Psi^{-\infty,\mathcal{F}_N(\alpha)}_{\Lambda}(X)$, $\int_X K_{H(\lambda)}|_\Delta$ vanishes to infinite order as $|\lambda| \to \infty$. Hence, $\operatorname{Tr} H(\lambda)$ does not contribute to the trace expansion (5.1).

By definition of $x^{Nm-b}\Psi^{-\infty,m,\mathcal{E}_N(\alpha)}_{c,\Lambda}(X)$, on the interior of Δ, $K_{G(\lambda)}|_\Delta$ vanishes to infinite order as $|\lambda| \to \infty$. Thus, we may assume that $K_{G(\lambda)}|_\Delta$ is supported in a neighborhood $[0,1)_x \times Y$ of X near Y. Let $r = |\lambda|^{-1/m}$

and $\theta = \lambda/|\lambda|$. Then using the definition of $x^{Nm-b}\Psi^{-\infty,m,\mathcal{E}_N(\alpha)}_{c,\Lambda}(X)$ and integrating out the Y factor, we can write

$$\begin{aligned}\int_X K_{G(\lambda)}|_\Delta &= \int_0^1 x^{Nm-b} G(r,\theta,x/r)\frac{dx}{x} \\ &= r^{Nm-b}\int_0^1 x^{Nm-b}G(r,\theta,x)\frac{dx}{x} \quad (x\mapsto rx),\end{aligned}$$

where $G(r,\theta,v)$ is a function smooth at $r=0$, smooth in θ, can be expanded at $v=0$ with index set $E_{N,\mathit{ff}}(\alpha)\geq m-Nm$, and vanishes to infinite order as $v\to\infty$. Since $G(r,\theta,v)$ is smooth at $r=0$ as $r\searrow 0$ we have

$$\operatorname{Tr} G(\lambda) + \operatorname{Tr} H(\lambda) \sim \sum_{k=0}^{\infty} r^{Nm-b+k} g_k(\theta), \quad \text{for some } g_k(\theta) \text{ smooth in } \theta.$$

Since $r = |\lambda|^{-1/m}$, our lemma follows from Proposition 5.1. □

The following lemma completes the proof of the trace expansion (1.1).

Lemma 5.3. *As $|\lambda|\to\infty$ in Λ, we have*

$$\begin{aligned}\operatorname{Tr} F(\lambda) \sim &\sum_{k=0}^{\infty}\Big\{a_k + b_k\log\lambda + c_k(\log\lambda)^2\Big\}\lambda^{(q+n-k)/m-N} \\ &+ \sum_{k=0}^{\infty}\Big\{\beta_k + e_k\log\lambda\Big\}\lambda^{(b-k)/m-N} + \sum_{k=0}^{\infty} f_k\lambda^{-k-N}.\end{aligned} \tag{5.2}$$

Moreover, $b_k=0$ unless $k\in(\mathbb{N}_0+q+n-b)\cup(m\mathbb{N}_0+q+n)$; $c_k=0$ unless $k\in m\mathbb{N}_0\cap(\mathbb{N}_0-b)+q+n$; and $e_k=0$ unless $k\in m\mathbb{N}_0+b$.

Proof. If $\varphi\in C^\infty(X)$ vanishes near the boundary Y, then $\varphi F(\lambda)$ is a type of operator examined in [11]. The results of [11] imply that $\operatorname{Tr}\varphi F(\lambda)$ has an expansion of the form (5.2) with $c_k,\beta_k,e_k=0$.

Thus, it suffices to assume that $F(\lambda)$ is supported in a collar $[0,1)_x\times Y$. By taking a partition of unity of Y, we may assume that $F(\lambda)$ is supported in a coordinate neighborhood in the Y factor. Also, by Proposition 5.1, we need only prove the expansion (5.2) with λ replaced with $r=|\lambda|^{-1/m}$ and with the coefficients functions of $\theta=\lambda/|\lambda|$. Note that θ appears only as a parameter; hence, without loss of generality we may assume that $\Lambda=[0,\infty)$.

Step 1: We reduce the problem to an application of Proposition A.3. Now using the definition of $x^{Nm-b}\Psi^{q-Nm,-Nm,m}_{c,\Lambda}(X)$ and integrating out the Y factor of $[0,1)\times Y$, we can write

$$\operatorname{Tr} F(\lambda) = \int_0^1\int_{\mathbb{R}^n} x^{Nm-b} a(x^m\lambda,x,\xi)\,đ\xi\frac{dx}{x},$$

where $(\lambda,\xi)\mapsto a(\lambda,x,\xi)\in S^{q-Nm,-Nm,m}_{\Lambda,rc\ell}(\mathbb{R}^n)$ and varies smoothly and is compactly supported in x. By assumption, $q-Nm<-n$, so the integral in ξ is absolutely convergent. If $r=1/\lambda^{1/m}$, then

$$\operatorname{Tr}F(\lambda)=\int_0^1 A(r/x,x)\frac{dx}{x},$$

where $A(z,x)=x^{Nm-b}\int_{\mathbb{R}^n}a(z^{-m},x,\xi)\,đ\xi$. Let $\varphi\in C^\infty([0,\infty))$ be such that $\varphi(z)=1$ for $z\le 1$ and $\varphi(z)=0$ for $z\ge 2$. Then,

$$\operatorname{Tr}F(\lambda)=\int_0^1\varphi(r/x)\,A(r/x,x)\frac{dx}{x}+\int_0^1(1-\varphi(r/x))\,A(r/x,x)\frac{dx}{x}. \tag{5.3}$$

We analyze the asymptotics of each integral as $r\searrow 0$. For the second integral, we make the change of variables $x\mapsto rx$, which gives

$$\int_0^1(1-\varphi(r/x))\,A(r/x,x)\frac{dx}{x}=\int_0^1(1-\varphi(1/x))\,A(1/x,rx)\frac{dx}{x}.$$

This integral is absolutely convergent since $Nm-b>0$ and $A(1/x,rx)=r^{Nm-b}x^{Nm-b}\int_{\mathbb{R}^n}a(x^m,rx,\xi)\,đ\xi$. Moreover, since $a(\lambda,x,\xi)$ is smooth at $x=0$, the integral has an expansion at $r=0$ with index set $Nm-b+\mathbb{N}_0$. Thus, the second integral in (5.3) contributes an expansion of the form given by the second sum in (5.2).

It remains to analyze the asymptotics of the first integral in (5.3). Note that $A(z,x)$ has an expansion at $x=0$ with index set $Nm-b+\mathbb{N}_0$ since $a(\lambda,x,\xi)$ is smooth at $x=0$. Thus, as $\varphi(z)\,A(z,x)$ is compactly supported in z and x, Proposition A.3 applies: if $A(z,x)$ has an expansion at $z=0$ with some index set E, then the first integral in (5.3) has an expansion as $r=1/\lambda^{1/m}\searrow 0$ with index set $E\overline{\cup}(Nm-b+\mathbb{N}_0)$ (see equation (A.3) for the definition of $\overline{\cup}$). We show in the following steps that $E=(mN+m\mathbb{N}_0)\overline{\cup}(Nm-q-n+\mathbb{N}_0)$, which, as the reader can verify, completes the proof of (5.2).

Step 2: Since the asymptotics of $A(z,x)$ at $z=0$ do not depend on x, we omit the x variable; thus, it suffices to determine the asymptotics of $A(z)=\int_{\mathbb{R}^n}a(z^{-m},\xi)\,đ\xi$ at $z=0$, where $a(\lambda,\xi)\in S^{q-Nm,-Nm,m}_{\Lambda,rc\ell}(\mathbb{R}^n)$.

Let $\chi(\xi)\in C^\infty(\mathbb{R}^n)$ be such that $\chi(\xi)=0$ near $\xi=0$ and $\chi(\xi)=1$ outside a neighborhood of 0. Then given $M\in\mathbb{N}$, expanding $a(\lambda,\xi)$ in its homogeneous components (see equation (3.6)) we can write

$$A(z)=\sum_{k=0}^{M-1}A_k(z)+R_M(z),$$

where $A_k(z)=\int_{\mathbb{R}^n}\chi(\xi)\,a_k(z^{-m},\xi)\,đ\xi$ with $a_k(\lambda,\xi)$ anisotropic homogeneous of degree $q-Nm-k$, and where $R_M(z)=\int_{\mathbb{R}^n}r_M(z^{-m},\xi)\,đ\xi$ with $r_M(\lambda,\xi)\in S^{q-Nm-M,-Nm,m}_{\Lambda,r}(\mathbb{R}^n)$; thus, $r_M(z^{-m},\xi)=z^{mN}\widetilde{r}_M(z^m,\xi)$, where $\widetilde{r}_M(w,\xi)$ is smooth at $w=0$ and satisfies the estimates (3.4). These properties of $\widetilde{r}_M$

imply that $R_M(z)$ can be expanded to higher and higher order at $z=0$ with index set $mN+m\mathbb{N}_0$ as M is chosen larger and larger. Thus, it suffices to analyze the asymptotics of each $A_k(z)$.

Step 3: To analyze the asymptotics of $A_k(z)$, we use Proposition A.2. Before continuing, we recall to the reader two properties of $a_k(\lambda,\xi)$ (see Section 3.2): $a_k(\delta^m\lambda,\delta\xi)=\delta^{q-Nm-k}a_k(\lambda,\xi)$ for all $\delta>0$, and $a_k(z^{-m},\xi)=z^{mN}\widetilde{a}_k(z^m,\xi)$, where $\widetilde{a}_k(w,\xi)$ is smooth at $w=0$.

Now, making the change of variables $\xi\mapsto z^{-1}\xi$ in the integral defining A_k, then using the homogeneity properties of a_k, we obtain

$$A_k(z)=z^{Nm-q-n+k}\int_{\mathbb{R}^n}\chi(\xi/z)\,a_k(1,\xi)\,đ\xi.$$

Since $(z\partial_z-(Nm-q-n+k))z^{Nm-q-n+k}=0$ and $z\partial_z\chi(\xi/z)=-(\xi\cdot\partial_\xi\chi)(\xi/z)$, where $\xi\cdot\partial_\xi=\sum\xi_j\partial_{\xi_j}$, we have

$$\begin{aligned}\big(z\partial_z-(Nm-q-n+k)\big)A_k(z)&=-z^{Nm-q-n+k}\int_{\mathbb{R}^n}(\xi\cdot\partial_\xi\chi)(\xi/z)\,a_k(1,\xi)\,đ\xi\\&=-\int_{\mathbb{R}^n}(\xi\cdot\partial_\xi\chi)(\xi)\,a_k(z^{-m},\xi)\,đ\xi\\&=-z^{mN}\int_{\mathbb{R}^n}(\xi\cdot\partial_\xi\chi)(\xi)\,\widetilde{a}_k(z^m,\xi)\,đ\xi,\end{aligned}$$

where we changed variables $\xi\mapsto z\xi$ in going from the first to second integral. Since $(\xi\cdot\partial_\xi\chi)(\xi)$ is supported in a compact subset of $\mathbb{R}^n\setminus\{0\}$, the integral $z^{mN}\int_{\mathbb{R}^n}(\xi\cdot\partial_\xi\chi)(\xi)\,\widetilde{a}_k(z^m,\xi)\,đ\xi$ is absolutely convergent, and so can be expanded at $z=0$ with index set $mN+m\mathbb{N}_0$. Hence, by Proposition A.2, $A_k(z)$ can be expanded at $z=0$ with index set $(mN+m\mathbb{N}_0)\overline{\cup}(Nm-q-n+k)$. Thus, as $A(z)$ is an asymptotic sum of the A_k's, $A(z)$ can be expanded at $z=0$ with index set $(mN+m\mathbb{N}_0)\overline{\cup}(Nm-q-n+\mathbb{N}_0)$. □

We remark that a trace expansion comparable to (1.1), but with possibly different exponents, holds for *any* holomorphic operator in

$$x^a\Psi^{m,p,d}_{c,\Lambda}(X)+x^b\Psi^{-\infty,d,\mathcal{E}}_{c,\Lambda}(X)\;+\;x^c\Psi^{-\infty,\mathcal{F}}_{\Lambda}(X),$$

where $m,p,d\in\mathbb{R}$ with $m<-n$, $p/d\in\mathbb{Z}$, and where $a,b,c\in\mathbb{R}$ are sufficiently large so that the operator is of trace class. Similar arguments to those given in Lemmas 5.2 and 5.3 can be used to derive such an expansion.

Appendix A. Two elementary propositions

We present two fundamental propositions that are useful for proving that certain functions have expansions. They both involve the Mellin transform.

Let $f \in C^\infty((0,\infty))$ and suppose that $f = O(x^a)$ at $x = 0$ and $f = O(x^b)$ at $x = \infty$, where $-a < -b$. Then the Mellin transform of f is

$$\mathcal{M}(f)(\tau) = \int_0^\infty x^{-i\tau} f(x) \frac{dx}{x} \;\longleftrightarrow\; f(x) = \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} x^{i\tau} \mathcal{M}(f)(\tau) d\tau. \tag{A.1}$$

The left-hand integral is convergent for all $\tau \in \mathbb{C}$ with $-a < \operatorname{Im}\tau < -b$, and the number $c \in \mathbb{R}$ on the right is any real number such that $-a < c < -b$.

The Mellin transform is related to asymptotic expansions via the following lemma, which is just an application of Cauchy's theorem.

Lemma A.1. *Let $u(\tau)$ be a meromorphic function with a single pole, of order $\ell + 1$, at $\tau = \tau_0$. Then for any closed curve C around τ_0, we have*

$$\int_C x^{i\tau} u(\tau) d\tau = x^{i\tau_0} \sum_{k=0}^{\ell} (\log x)^k u_k, \quad u_k \in \mathbb{C}. \tag{A.2}$$

Moreover, suppose that $v(x) \in C^\infty((0,\infty))$ vanishes to infinite order as $x \to \infty$ and can be expanded at $x = 0$ in the form given by the right-hand side of (A.2) *up to a term that vanishes to infinite order at $x = 0$. Then, $\mathcal{M}(v)(\tau)$ is meromorphic on $\mathbb{C}$ with a single pole at $\tau = \tau_0$ of order $\ell + 1$.*

We now give two applications of this lemma to obtain expansions. Given two index sets E and F, not necessarily C^∞ index sets (cf. Section 2.1), the extended union of these index sets, $E \overline{\cup} F$, is by definition

$$E \overline{\cup} F = E \cup F \cup \{(z, k + \ell + 1) : (z,k) \in E,\ (z,\ell) \in F\}. \tag{A.3}$$

Proposition A.2. *Let $f(x) \in C^\infty((0,\infty))$ vanish to infinite order as $x \to \infty$ and suppose that for some $a \in \mathbb{C}$, we have*

$$(x\partial_x - a) f(x) = g(x), \tag{A.4}$$

where $g(x)$ can be expanded at $x = 0$ with index set E, not necessarily a C^∞ index set. Then f has an expansion at $x = 0$ with index set $E \overline{\cup} \{a\}$.

Proof. We first note that by Lemma A.1, $\mathcal{M}(g)(\tau)$ has a pole at $\tau = -iz$ of order $k + 1$, where $(z,k) \in E$ but $(z, k+1) \notin E$.

Now taking the Mellin transform of (A.4) gives $(i\tau - a)\mathcal{M}(f)(\tau) = \mathcal{M}(g)(\tau)$. Thus, $\mathcal{M}(f)(\tau) = -i\mathcal{M}(g)(\tau)/(\tau + ia)$; hence, $\mathcal{M}(f)(\tau)$ has the same pole structure as $\mathcal{M}(g)(\tau)$ except with a possible extra pole at $\tau = -ia$.

In particular, $f(x)$ is given by the inverse Mellin transform in (A.1) with $c > \max\{-\min E, -a\}$. Moving the contour $\operatorname{Im}\tau = c$ down to $\operatorname{Im}\tau = -\infty$ picks up contour integrals around the poles of $\mathcal{M}(f)(\tau)$, which by Lemma A.1 and the structure of the poles prove our proposition. □

Proposition A.3. *Let $f : [0,1)^2 \to [0,1)$ be the map $f(x,y) = xy$. Suppose that $u = u(x,y)|dxdy/(xy)|$ is a compactly supported b-density on $[0,1)^2$*

with expansions at $x = 0$ *and* $y = 0$ *given by index sets* E_{lb} *and* E_{rb}*, respectively, which are not necessarily* C^∞ *index sets. Then,*

$$(f_* u)(x) = \int_0^1 u(x/y, y)\frac{dy}{y} \cdot \Big|\frac{dx}{x}\Big| = \int_0^1 u(y, x/y)\frac{dy}{y} \cdot \Big|\frac{dx}{x}\Big|, \tag{A.5}$$

and $f_* u(x)$ *can be expanded at* $x = 0$ *with index set* $E_{lb}\overline{\cup}E_{rb}$*.*

Proof. Working out the definition of $f_* u$, a straightforward computation shows that $f_* u$ is given by (A.5).

It remains to prove that $f_* u(x)$ can be expanded at $x = 0$ with index set $E_{lb}\overline{\cup}E_{rb}$. To see this, we take the Mellin transform of $f_* u(x)$. A short computation shows that $\mathcal{M}(f_* u)(\tau) = \mathcal{M}(u)(\tau,\tau)$, where $\mathcal{M}(u)(\tau,\sigma)$ is the two-dimensional Mellin transform of u:

$$\mathcal{M}(u)(\tau,\sigma) = \int_0^\infty\int_0^\infty x^{-i\tau}y^{-i\sigma}u(x,y)\frac{dx}{x}\frac{dy}{y}.$$

Since $u(x, y)$ has expansions at the left and right boundaries of $[0, 1)^2$ given by E_{lb} and E_{rb}, by Lemma A.1 it follows that $\mathcal{M}(u)(\tau,\sigma)$ has poles at $\tau = -iz$ of order $k + 1$, where $(z, k) \in E_{lb}$ but $(z, k + 1) \notin E_{lb}$, and poles at $\sigma = -iw$ of order $\ell + 1$, where $(w, \ell) \in E_{rb}$ but $(w, \ell + 1) \notin E_{rb}$. It follows that $\mathcal{M}(u)(\tau,\tau)$ has poles at $\tau = -iz$ of order $k + 1$, where $(z, k) \in E_{lb}\overline{\cup}E_{rb}$ but $(z, k + 1) \notin E_{lb}\overline{\cup}E_{rb}$, as the reader can verify.

Now that we know the pole structure of $\mathcal{M}(f_* u)(\tau) = \mathcal{M}(u)(\tau,\tau)$, we can proceed as in the final paragraph of Proposition A.2 to show that $f_* u(x)$ can be expanded at $x = 0$ with index set $E_{lb}\overline{\cup}E_{rb}$. □

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Department of Mathematics, MIT, Cambridge, MA 02139, USA
Current address: Binghamton University, Binghamton, NY 13902-6000, USA
E-mail address: `paul@math.binghamton.edu`