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HOW MANY BOOLEAN ALGEBRAS $\mathcal{P}(\mathbb{N})/\mathcal{I}$ ARE THERE?

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Abstract. Which pairs of quotients over ideals on $\mathbb{N}$ can be distinguished without assuming additional set theoretic axioms? Essentially, those that are not isomorphic under the Continuum Hypothesis. A CH-diagonalization method for constructing isomorphisms between certain quotients of countable products of finite structures is developed and used to classify quotients over ideals in a class of generalized density ideals. It is also proved that many analytic ideals give rise to quotients that are countably saturated (and therefore isomorphic under CH).

1. Introduction

The question from the title can be given different interpretations. Taken literally, it has a well-known answer: there are $2^{2^{\aleph_0}}$ isomorphism types of Boolean algebras of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for some ideal $\mathcal{I}$ on $\mathbb{N}$. This follows from the fact that every complete Boolean algebra of size at most continuum is of this form and [19]. In this note we study only quotients over ideals that are 'simply definable.' More precisely, by identifying sets with their characteristic functions we equip $\mathcal{P}(\mathbb{N})$ with the compact metric topology taken from $\{0,1\}^{\mathbb{N}}$. Thus we can speak of Borel, or analytic (a set is analytic if it is a continuous image of a Borel set of reals) ideals on $\mathbb{N}$. Note that there are only $2^{\aleph_0}$ analytic ideals on $\mathbb{N}$, since every analytic set can be coded by a real number.

To avoid trivial considerations, we assume that every ideal includes all finite subsets of $\mathbb{N}$. Since by a classical result of Sierpiński there are no analytic uniform ultrafilters on $\mathbb{N}$, this implies that all algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that we consider are atomless, and therefore elementarily equivalent (see [1]).

Ideals $\mathcal{I}$ and $\mathcal{J}$ are *Rudin–Keisler isomorphic*, $\mathcal{I} \approx_{\mathrm{RK}} \mathcal{J}$, if there are $A \in \mathcal{I}$, $B \in \mathcal{J}$, and a bijection h between $\mathbb{N} \setminus B$ and $\mathbb{N} \setminus A$ such that for all $X \subseteq \mathbb{N} \setminus A$ we have

$$X \in \mathcal{I} \qquad \Leftrightarrow \qquad h^{-1}(X) \in \mathcal{J}.$$

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It is not difficult to see that in this situation the map $[X]_{\mathcal{I}} \mapsto [h^{-1}(X)]_{\mathcal{J}}$ is an isomorphism between $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ (see [8, Lemma 1.2]). Such an isomorphism is said to be *trivial*. There is some evidence that every automorphism between analytic quotients that can be constructed without using the Continuum Hypothesis or some other additional set-theoretic axiom is trivial (see §10 and references thereof).

In this note we consider an extremal situation, when there are as few isomorphism types as possible. Not surprisingly, isomorphisms between analytic quotients are most easily constructed using the Continuum Hypothesis. There is a meta-mathematical explanation of this role of CH. By a result of Woodin [30], a large cardinal assumption implies that every Σ^2_1-statement (in particular, the statement '$\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$') that is true in some forcing extension has to be true in every forcing extension that satisfies the Continuum Hypothesis.

At present we know only of two methods for constructing nontrivial isomorphisms between analytic quotients. One is to prove that the quotients are saturated (in the model-theoretic sense, see [1]), and then conclude that they are isomorphic since they are elementarily equivalent. Clause (1) of the following theorem was first proved by Just and Krawczyk [11].

THEOREM 1. *The quotients over the following ideals are countably saturated, and therefore pairwise isomorphic under CH.*

(1) *All F_σ ideals.*
(2) *All ordinal ideals (see* §2.12*).*
(3) *All CB-ideals (see* §2.13*).*

Proof. This is Corollary 6.4. □

This implies that ideals of different Borel complexities can have isomorphic quotients. Curiously, if $\mathcal{I}$ and $\mathcal{J}$ are analytic P-ideals with isomorphic quotients, then $\mathcal{I}$ and $\mathcal{J}$ have the same Borel complexity (Corollary 6.2).

Another method for constructing isomorphisms was introduced by Just and Krawczyk in [11], where it was used to prove that the ideals of asymptotic zero density and of logarithmic zero density (see §2.6) have isomorphic quotients under CH. By extending their method we prove the following.

THEOREM 2. *Assume CH.*

(1) *There are exactly two isomorphism classes of quotients over dense density ideals (see* §2.8*).*
(2) *Consider the class of all ideals* $\mathrm{Exh}(\sup_n \mu_n)$*, where μ_n are pairwise orthogonal lower-semicontinuous measures on $\mathbb{N}$ such that*

$$\limsup_m \sup_n \mu_n(\{m\}) = 0$$

(see §2.1*). There are exactly six isomorphism classes of quotients over such ideals.*

(3) *All quotients over Louveau–Velickovic (LV) ideals are pairwise isomorphic (see* §2.11*).*

Moreover, the six quotients from (2) *include two quotients from* (1) *and they are nonisomorphic to quotients from* (3)*.*

Proof. Clause (1) is proved in Corollary 5.4, (2) is proved in Theorem 7.3, and (3) is proved in Corollary 5.5.

The moreover part follows from Proposition 3.6 and it does not require CH. □

An earlier version of this note contained a question asking whether there are infinitely many, or even uncountably many, analytic ideals with pairwise nonisomorphic quotients.

THEOREM 3 (Oliver, [20]). *There are uncountably many pairwise nonisomorphic quotients over Borel ideals.* □

THEOREM 4 (Steprāns, [25]). *There are continuum many pairwise nonisomorphic quotients over $F_{\sigma\delta}$ ideals. Moreover, the completions of these distributive lattices are pairwise nonisomorphic.* □

The following question asked in [6, Question 3.14.3] (see §2.7 for the definition) still remains open.[1]

QUESTION 5. *Are there infinitely (or even uncountably) many analytic P-ideals whose quotients are, provably in ZFC, pairwise non-isomorphic?*

PROPOSITION 6. *There are at least 21 pairwise nonisomorphic quotients over analytic P-ideals.*

Proof. This is Proposition 3.6. □

We also consider the effect of CH on the structure of automorphism groups of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$. For example, it implies that the automorphism group of every homogeneous quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is simple.

Organization of this paper. In §2 we review the definitions of various analytic ideals. Proposition 6 is proved in §3. Sections §§4–5 are the longest sections in this paper. In §4 we extend the Just–Krawczyk method for constructing isomorphisms under CH and apply it in §5. In §6 we introduce a class of *layered* ideals and prove that they have countably saturated quotients.

[1]**Added in proof.** M.R. Oliver (PhD thesis, UCLA) gave a complete answer to Question 5 by constructing a family of continuum many analytic P-ideals whose quotients are, provably in ZFC, pairwise non-isomorphic.

In §7 we classify the quotients over dense ideals of the form $\mathrm{Exh}(\phi)$, where ϕ is the supremum of a family of pairwise orthogonal lower semi-continuous measures on $\mathbb{N}$ with an additional property. Homogeneous quotients and automorphism groups are considered in §8 and §9, respectively. The last two sections, §10 and §11, contain some remarks and open problems.

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2. Definitions

This section contains only the basic definitions and examples of objects that will be studied in this paper, most of them appearing in [6], [5] and [8]. We equip $\mathcal{P}(\mathbb{N})$ with a metric,

$$d(x, y) = 2^{-\Delta(x,y)},$$

where $\Delta(x, y)$ is the least integer in the symmetric difference, $x\Delta y$, of x and y. This metric turns $\mathcal{P}(\mathbb{N})$ into a compact space homeomorphic with the Cantor cube. We shall refer to the metric topology of $\mathcal{P}(\mathbb{N})$, but not to the above metric.

2.1. Submeasures on $\mathbb{N}$. A map $\phi\colon \mathcal{P}(I) \to [0,\infty]$ is a *submeasure* if $\phi(\emptyset) = 0$, it is *monotonic* ($\phi(A) \leq \phi(B)$ for all $A \subseteq B$) and *subadditive* ($\phi(A \cup B) \leq \phi(A) + \phi(B)$ for all $A, B \subseteq I$). If ϕ is a submeasure on $\mathbb{N}$ write

$$\phi^\infty(A) = \lim_n \phi(A \setminus n).$$

A submeasure ϕ on $\mathbb{N}$ is *lower semi-continuous* if $\lim_n \phi(A \cap n) = \phi(A)$ for all $A \subseteq \mathbb{N}$. Two submeasures ϕ and ψ are *orthogonal* if we have $\mathbb{N} = A \cup B$ for some A, B such that $\phi(A) = 0$ and $\psi(B) = 0$. If $f\colon \mathbb{N} \to [0,\infty)$ then

$$\nu_f(A) = \sum_{i\in A} f(\{i\})$$

is a lower semi-continuous measure on $\mathbb{N}$. For a submeasure ϕ write

$$\mathrm{at}^+(\phi) = \sup_i \phi(\{i\}).$$

2.2. Ideals on $\mathbb{N}$. An *ideal* on $\mathbb{N}$ is an ideal of a Boolean algebra $\mathcal{P}(\mathbb{N})$ (equivalently, of a Boolean ring $\mathcal{P}(\mathbb{N})$). By $\mathcal{P}(\mathbb{N})/\mathcal{I}$ we denote its quotient algebra. We will consider ideals on $\mathbb{N}$ that are topologically simple with respect to the topology on $\mathcal{P}(\mathbb{N})$, like F_σ, Borel, analytic, and so on. Recall that a subset of a metric space is *analytic* if it is a continuous image of a Borel set of reals.

In order to avoid trivial considerations, we will consider only those ideals

(1) that are *proper*, i.e., distinct from $\mathcal{P}(\mathbb{N})$, and
(2) that include Fin, the ideal of all finite subsets of $\mathbb{N}$.

Since Fin is dense in $\mathcal{P}(\mathbb{N})$, there are no G_δ ideals satisfying these two conditions. Since by (2) all the ideals that we consider in this paper are dense in $\mathcal{P}(\mathbb{N})$ in the topological sense, we will use the adjective 'dense' in an established way.

Definition 2.2.1. An ideal $\mathcal{I}$ on $\mathbb{N}$ is *dense* (or *tall*) if every infinite subset of $\mathbb{N}$ contains an infinite set in $\mathcal{I}$.

Definition 2.2.2. A set $A \subseteq \mathbb{N}$ is *$\mathcal{I}$-positive* if $A \notin \mathcal{I}$. A *restriction* of $\mathcal{I}$ to a positive set, $\mathcal{I} \restriction A$, is an ideal on A defined by

$$\mathcal{I} \restriction A = \mathcal{I} \cap \mathcal{P}(A).$$

2.3. Sums and products of ideals. If $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\mathbb{N}$, define the ideals $\mathcal{I} \oplus \mathcal{J}$ on $\mathbb{N} \times \{0,1\}$ and $\mathcal{I} \times \mathcal{J}$ on $\mathbb{N}^2$ by

$A \in \mathcal{I} \oplus \mathcal{J}$ if $\{n : (n,0) \in A\} \in \mathcal{I}$ and $\{n : (n,1) \in A\} \in \mathcal{J}$
$A \in \mathcal{I} \times \mathcal{J}$ if $\{m : \{n : (m,n) \in A\} \notin \mathcal{J}\} \in \mathcal{I}$.

For example, $\emptyset \times \mathrm{Fin}$ is the ideal of all $A \subseteq \mathbb{N}^2$ such that all vertical sections of A are finite, while $\mathrm{Fin} \times \emptyset$ is the ideal of all $A \subseteq \mathbb{N}^2$ such that at most finitely many vertical sections of A are nonempty.

2.4. Summable ideals. If $f \colon \mathbb{N} \to [0,\infty)$, the *summable ideal* $\mathcal{I}_f$ is defined by (see [6, §1.12])

$$\mathcal{I}_f = \{A : \nu_f(A) < \infty\}.$$

All summable ideals are F_σ. A typical example of a dense summable ideal is

$$\mathcal{I}_{1/n} = \left\{A : \sum_{i \in A} 1/i < \infty\right\}.$$

2.5. F_σ ideals. By [18], $\mathcal{I}$ is an F_σ-ideal if and only if there is a lower semi-continuous submeasure ϕ such that

$$\mathcal{I} = \mathrm{Fin}(\phi) = \{A : \phi(A) < \infty\}.$$

2.6. EU-ideals. If $f \colon \mathbb{N} \to [0,\infty)$ is such that $\limsup_n f(\{n\})/\nu_f(n) = 0$ and $\nu_f(\mathbb{N}) = \infty$, then

$$\mathcal{EU}_f = \left\{A : \limsup_n \frac{\nu_f(A \cap n)}{\nu_f(n)} = 0\right\}$$

is a proper, $F_{\sigma\delta}$ ideal. Following [11], we call these ideals *EU-ideals*. Examples of EU-ideals are the ideal of *asymptotic density zero sets*

$$\mathcal{Z}_0 = \left\{A : \limsup_n \frac{|A \cap n|}{n} = 0\right\}$$

and the ideal of *logarithmic density zero sets* (let $g(n) = 1/n$):

$$\mathcal{Z}_{\log} = \left\{ A : \limsup_n \frac{\nu_g(A \cap n)}{\nu_g(n)} = 0 \right\}.$$

EU-ideals were introduced and studied in [11]. See also [6, §1.13].

2.7. P-ideals. An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence A_n $(n \in \mathbb{N})$ in $\mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n. All summable and all EU-ideals are P-ideals. By a theorem of Solecki ([24, Theorem 3.1]), an analytic ideal $\mathcal{I}$ is a P-ideal if and only if there is a lower semi-continuous submeasure ϕ such that

$$\mathcal{I} = \mathrm{Exh}(\phi) = \{A : \lim_n \phi(A \setminus n) = 0\}.$$

Thus every analytic P-ideal is automatically $F_{\sigma\delta}$. On the other hand, by a result of Zafrany ([31], see §2.12) there are Borel ideals of arbitrarily high Borel complexity.

Note that $\mathrm{Exh}(\phi)$ is a dense ideal if and only if $\limsup_n \phi(\{n\}) = 0$ for some (any) choice of ϕ.

2.8. Density ideals. Assume I_n $(n \in \mathbb{N})$ are pairwise disjoint intervals on $\mathbb{N}$, and μ_n is a measure that concentrates on I_n. Then

$$\mathcal{Z}_\mu = \{A : \limsup_n \mu_n(A) = 0\}$$

is a *density ideal*, as defined in [6, §1.13]. Letting $\phi = \sup_n \mu_n$ we see that $\mathcal{Z}_\mu = \mathrm{Exh}(\phi)$, hence every density ideal is a P-ideal.

It is not difficult to check that (let $I_n = [2^n, 2^{n+1})$)

$$\mathcal{Z}_0 = \{A : \limsup_n 2^{-n} |A \cap I_n| = 0\},$$

hence $\mathcal{Z}_0$ is a density ideal. In fact, the following was proved in [6, Theorem 1.13.3].

Theorem 2.8.1. *The following are equivalent for an ideal $\mathcal{I}$:*

1. *$\mathcal{I}$ is an EU-ideal.*
2. *There are intervals I_n and measures μ_n concentrating on I_n such that $\mu_n(I_n) = 1$ $(n \in \mathbb{N})$ and $\limsup_m \sup_n \mu_n(\{m\}) = 0$.*
3. *$\mathcal{I} = \mathcal{Z}_\mu$ is a dense density ideal such that for every choice of μ_n, I_n $(n \in \mathbb{N})$ we have $\sup_n \mu_n(I_n) < \infty$.* □

In particular, EU-ideals form a proper subclass of dense density ideals. By Theorem 2.8.1, an example of a dense density ideal that is not an EU-ideal is (again $I_n = [2^n, 2^{n+1})$)

$$\mathcal{Z}_\infty = \left\{ A : \limsup_n \frac{|A \cap I_n|}{n} = 0 \right\}.$$

LEMMA 2.8.2. *The restriction of any density (F_σ, summable, P-, EU-) ideal to a positive set is a density (F_σ, summable, P-, EU-, respectively) ideal.*

Proof. This is nontrivial only in the case of EU-ideals, and this case follows from Theorem 2.8.1. □

By the following, all dense density ideals that are not EU-ideals look rather similar (see also Theorem 5.3).

LEMMA 2.8.3.

(a) *If $\mathcal{Z}_\mu$ is a dense density ideal, then it is either an EU-ideal or μ_n, I_n ($n \in \mathbb{N}$) can be chosen so that $\lim_n \mu_n(I_n) = \infty$.*
(b) *We have $\mathcal{Z}_0 \oplus \mathcal{Z}_\infty \approx_{\mathrm{RK}} \mathcal{Z}_\infty$.*

Proof. (a) Assume $\mathcal{Z}_\mu$ is not an EU-ideal. If $\sup_n \mu_n(I_n) < \infty$, then Theorem 2.8.1 implies that $\mathcal{Z}_\mu$ is an EU-ideal. Hence there is an infinite $A \subseteq \mathbb{N}$ such that $\liminf_{n\in A} \mu_n(I_n) = \infty$. We may assume that A is coinfinite, and by re-indexing that $A = \{2n : n \in \mathbb{N}\}$. Let $J_n = I_{2n-1} \cup I_{2n}$ and $\nu_n = \mu_{2n-1} + \mu_{2n}$. Then $\mathcal{Z}_\nu = \mathcal{Z}_\mu$ is as required.

The proof of part (b) is very similar. □

2.9. Ideal $\mathcal{I}_\infty$. For $A \subseteq \mathbb{N}^2$ and $m \in \mathbb{N}$ let $\mu_m(A) = \sum_{(m,n)\in A} 1/mn$, fix any bijection between $\mathbb{N}^2$ and $\mathbb{N}$ and let

$$\mathcal{I}_\infty = \mathrm{Exh}(\sup_m \mu_m).$$

Note that the restriction of $\mathcal{I}_\infty$ to $\{n\} \times \mathbb{N}$ is summable, but there are many $\mathcal{I}_\infty$-positive sets A such that $\mathcal{I}_\infty \restriction A$ is a density ideal. Also note that $\mathcal{I}_\infty$ is dense.

LEMMA 2.9.1. *We have $\mathcal{I}_\infty \approx_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{I}_{1/n} \approx_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{Z}_0 \approx_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{Z}_\infty$.*

Proof. Very similar to the proof of Lemma 2.8.3. □

2.10. Generalized density ideals. Assume $\{I_n\}_{n=1}^\infty$ is a partition of $\mathbb{N}$ into finite intervals and ϕ_n is a submeasure on I_n for every n. Assume moreover that $\limsup_n \mathrm{at}^+(\phi_n) = 0$ (see §2.1 for the definition). Then the ideal

$$\mathcal{Z}_\phi = \{A : \limsup_n \phi_n(A \cap I_n) = 0\}$$

is a *generalized density ideal defined by a sequence of submeasures.*

All these ideals are P-ideals, and the condition $\limsup_n \mathrm{at}^+(\phi_n) = 0$ is equivalent to saying that $\mathcal{Z}_\phi$ is dense. If each ϕ_n is a measure, then $\mathcal{Z}_\phi$ is a *density ideal* as defined in §2.8 above. If $I_n = \emptyset$ for all but one n, and ϕ_n is a measure, then $\mathcal{Z}_\phi$ is a summable ideal, as defined in §2.4 above.

2.11. LV-ideals. A large class of generalized density ideals was introduced by Louveau and Velickovic in [16], where it was proved that the quotients over these ideals are not Borel-isomorphic, even when considered with no algebraic structure. For a rapidly increasing sequence $\{n_i\}$ of natural numbers let I_i be pairwise disjoint intervals such that $|I_i| = 2^{n_i}$ and define $\phi_i(A) = \log_2(|A \cap I_i| + 1)/n_i$. Let

$$\mathcal{LV}_{\{n_i\}} = \mathrm{Exh}(\sup_i \phi_i).$$

If $n_{i+1} = 2^{2^{n_i}}$, we denote $\mathcal{LV}_{\{n_i\}}$ by $\mathcal{LV}$. Each ideal of this kind satisfies the following two conditions:

(LV1) $\phi_i(I_i) \geq 1$ for all i, and
(LV2) $(\forall k)(\forall \varepsilon > 0)(\forall^\infty n)$

$$(\forall a_0, \dots, a_k \subseteq I_n)|\phi_n(a_0 \Delta a_k) - \max_{i<k} \phi_n(a_i \Delta a_{i+1})| < \varepsilon.$$

A generalized density ideal satisfying (LV1) and (LV2) is an *LV-ideal.*

A proof very similar to the proof of [6, Theorem 1.13.3 (b)] gives the following.

LEMMA 2.11.1. *If $\mathcal{I}$ is an LV-ideal, then its restriction to any positive set is an LV-ideal.* □

2.12. Ordinal ideals. Let α be an additively indecomposable countable ordinal, and let P be a countable linear ordering such that α embeds into P, $\alpha \hookrightarrow P$. Then

$$\mathcal{O}_\alpha(P) = \{A \subseteq P : \alpha \not\hookrightarrow A\}$$

is an ideal. Unless it is improper, $\mathcal{O}_\alpha(P)$ is a P-ideal only when $\alpha = \omega$.

Ordinal ideals are of the form $\mathcal{O}_\alpha(\alpha) = \mathcal{I}_\alpha$. These ideals were studied in [31], where it was shown that $\mathcal{I}_{\omega^\alpha}$ is complete $\mathbf{\Pi}^0_{2\alpha}$ for every α.

2.13. CB-ideals. Let α be an additively indecomposable ideal. If X is a countable topological space whose Cantor–Bendixson rank is at least α, then

$$\mathrm{CB}_\alpha(X) = \{Y \subseteq X : \text{Cantor–Bendixson rank of } Y \text{ is } < \alpha\}$$

is, by a topological partition relation due to W. Weiss, an ideal. A special case are *Weiss ideals*, $\mathcal{W}_{\omega^\alpha} = \mathrm{CB}_\alpha(\omega^\alpha)$, the ideal of all subsets of ω^α that do not contain a closed copy of ω^α. It is easily seen that $\mathcal{O}_\alpha(P)$ and $\mathrm{CB}_\alpha(X)$ are P-ideals only when $\alpha = \omega$.

3. Small sets and deep sets

In this section we will prove that a quotient over a density ideal is never isomorphic to a quotient over an LV-ideal, and that there are at least two isomorphism types of dense density ideals.

DEFINITION 3.1. Let $\mathcal{I}$ be an ideal. A set $A \subseteq \mathbb{N}$ is *$\mathcal{I}$-small* if there are sets A_s ($s \in \{0,1\}^{<\mathbb{N}}$) such that for all s we have:

(1) $A_{\langle\rangle} = A$,
(2) $A_s = A_{s^\frown 0} \cup A_{s^\frown 1}$,
(3) $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$, and
(4) For every $b \in \{0,1\}^{\mathbb{N}}$, if $X \setminus A_{b\restriction n} \in \mathcal{I}$ for all n, then $X \in \mathcal{I}$.

A set $A \subseteq \mathbb{N}$ is *$\mathcal{I}$-deep* if $\mathcal{I} \restriction A$ has a countably saturated (in the model-theoretic sense) quotient.

LEMMA 3.2.

(1) *All $\mathcal{I}$-small sets form an ideal $\mathcal{S}_\mathcal{I}$ that includes $\mathcal{I}$.*
(2) *All $\mathcal{I}$-deep sets form an ideal $\mathcal{D}_\mathcal{I}$ that includes $\mathcal{I}$.*
(3) *An isomorphism between $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ sends the equivalence classes of $\mathcal{I}$-small sets into the equivalence classes of $\mathcal{J}$-small sets and the equivalence classes of $\mathcal{I}$-deep sets into the equivalence classes of $\mathcal{J}$-deep sets.*
(4) *$\mathcal{I} \subseteq \mathcal{D}_\mathcal{I}$, $\mathcal{I} \subseteq \mathcal{S}_\mathcal{I}$, and $\mathcal{D}_\mathcal{I} \cap \mathcal{S}_\mathcal{I} \subseteq \mathcal{I}$.* □

PROPOSITION 3.3.

(1) *If $\mathcal{Z}_\mu$ is an EU-ideal, then $\mathcal{S}_{\mathcal{Z}_\mu} = \mathcal{P}(\mathbb{N})$.*
(2) *If $\mathcal{Z}_\mu$ is a density ideal such that $\limsup_n \mu_n(I_n) = \infty$, then $\mathcal{S}_{\mathcal{Z}_\mu}$ is a proper F_σ ideal properly including $\mathcal{Z}_\mu$.*
(3) *If $\mathcal{Z}_\mu$ is a density ideal then every $\mathcal{Z}_\mu$-positive set A contains a positive subset that belongs to $\mathcal{S}_{\mathcal{Z}_\mu}$.*
(4) *If $\mathcal{I}$ is an LV-ideal, then $\mathcal{S}_\mathcal{I} = \mathcal{I}$.*
(5) *If $\mathcal{I}$ is a dense density ideal or an LV-ideal, then $\mathcal{D}_\mathcal{I} = \mathcal{I}$.*

Proof. (1) It suffices to prove that $\mathbb{N} \in \mathcal{S}_{\mathcal{Z}_\mu}$. Recursively define A_s as in Definition 3.1 and so that for every s we have $\limsup_n |\mu_n(A_s \cap I_n) - 2^{-|s|}| = 0$. Then for every $b \in 2^{\mathbb{N}}$ and every X such that $X \setminus A_{b\restriction n} \in \mathcal{Z}_\mu$ for all n we have $X \in \mathcal{Z}_\mu$.

(2) We will prove that

$$\mathcal{S}_{\mathcal{Z}_\mu} = \{A : \limsup_n \mu_n(A) < \infty\}.$$

This ideal is clearly F_σ. We first prove that A such that $\limsup_n \mu_n(A) = \infty$ is not in $\mathcal{S}_{\mathcal{Z}_\mu}$. Let A_s ($s \in \{0,1\}^{<\mathbb{N}}$) be as in Definition 3.1. Pick a branch $\langle\rangle = s_0 \sqsubset s_1 \sqsubset s_2 \sqsubset \cdots$ recursively so that $\limsup_n \mu_n(A_{s_i}) = \infty$ for all i. (This is true for s_0 by the assumption on $\mathcal{Z}_\mu$.) Let n_i ($i \in \mathbb{N}$) be increasing and such that $\mu_{n_i}(A_{s_i}) \geq i$. Then $X = \bigcup_{i=1}^\infty A_{s_i} \cap I_{n_i}$ is included in each A_{s_i} modulo finite, and it does not belong to $\mathcal{Z}_\mu$. Therefore $\mathbb{N} \notin \mathcal{S}_{\mathcal{Z}_\mu}$. To see that $\limsup_n \mu_n(A) < \infty$ implies $A \in \mathcal{S}_{\mathcal{Z}_\mu}$, fix an A such that $\limsup_n \mu_n(A)) < \infty$. We may assume A is $\mathcal{Z}_\mu$-positive, and then $\limsup_n \mu_n(A) < \infty$ implies that $\mathcal{Z}_\mu \restriction A$ is an EU-ideal (Theorem 2.8.1), and therefore $A \in \mathcal{S}_{\mathcal{Z}_\mu}$ by (1).

Clause (3) follows immediately from the characterization of $\mathcal{S}_{\mathcal{Z}_\mu}$ given in (2).

(4) Let ϕ_n and $\phi = \sup_n \phi_n$ be the submeasures defining $\mathcal{I}$. By (LV2), they have the property that

$$\phi^\infty(A \cup B) = \max(\phi^\infty(A), \phi^\infty(B)).$$

Note that $\mathcal{I} = \mathrm{Exh}(\phi) = \{A : \phi^\infty(A) = 0\}$. Hence if $A_{\langle\rangle}$ is positive and A_s are as in Definition 3.1, we can recursively pick a branch b so that $\phi^\infty(A_{b\restriction n}) = \phi^\infty(A_{\langle\rangle}) = \delta > 0$ for all n.

Then $A_{b\restriction n}$ $(n \in \mathbb{N})$ is a $\subseteq$-decreasing sequence such that $\phi^\infty(A_{b\restriction n}) = \delta$ for all n. Hence we can find finite pairwise disjoint sets s_n $(n \in \mathbb{N})$ such that $s_n \subseteq A_{b\restriction m}$ for all $m \leq n$ and $\phi(s_n) \geq \delta/2$ for all n. Then $X = \bigcup_n s_n$ is such that $X \setminus A_{b\restriction m}$ is finite for all m but X is not in $\mathrm{Exh}(\phi)$. This concludes the proof.

(5) Assume $\mathcal{I}$ is a dense density ideal or an LV-ideal. If ϕ is the natural lower semicontinuous submeasure such that $\mathcal{I} = \mathrm{Exh}(\phi)$ and A is a positive set, recursively construct $\mathcal{I}$-positive sets $A = A_1 \supset A_2 \supset A_3 \cdots$ such that $\phi(A_n) < 1/n$ for all n. Then the only lower bound for A_n is $[\emptyset]_{\mathcal{I}}$. Hence the quotient is not countably saturated. $\square$

We now return to the ideal introduced in §2.9.

Lemma 3.4. *If $A \subseteq \mathbb{N}$ then the following are equivalent.*

1. *$\mathcal{I}_\infty \restriction A$ has countably saturated quotient.*
2. *$\mathcal{I}_\infty \restriction A$ is summable.*
3. *$(\exists B \in \mathrm{Fin} \times \emptyset) A \setminus B \in \mathcal{I}_\infty$.*

Proof. All summable ideals are F_σ, so (2) implies (1) by Theorem 6.3 (c). Since (3) implies (1) is obvious, only (1) implies (3) requires a proof.

Assume (3) fails. There is $\varepsilon > 0$ such that the set

$$C = \{n : \mu_n(A) \geq \varepsilon\}$$

is infinite. We may assume $\mathrm{at}^+(\mu_n) < \varepsilon/2$ for all $n \in C$. For $n \in C$ find $B_n \subseteq A \cap I_n$ such that $\mu_n(B_n) \geq \varepsilon/2$, and let $B = \bigcup_{n \in C} B_n$. Then $\mathcal{I}_\infty \restriction B$ is a proper dense density ideal, and (1) fails by Lemma 6.9. $\square$

Lemma 3.5. *Every $C \subseteq \mathbb{N}^2$ set is either in $\mathcal{D}_{\mathcal{I}_\infty}$ or it includes an $\mathcal{I}_\infty$-positive set in $\mathcal{S}_{\mathcal{I}_\infty}$.*

Proof. If $C \notin \mathcal{D}_{\mathcal{I}_\infty}$, then by the proof of Lemma 3.4 there is $B \subseteq \mathbb{N}^2 \setminus C$ such that $\mathcal{I}_\infty \restriction B$ is a proper dense density ideal, so by (3) of Proposition 3.3 it contains a positive set in $\mathcal{S}_{\mathcal{I}_\infty}$. $\square$

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are ideals such that $\mathcal{J}_1 \cap \mathcal{J}_2 \supseteq \mathcal{I}$, we say that $\mathcal{J}_1$ and $\mathcal{J}_2$ form a *pregap* over $\mathcal{I}$. A pregap is *split* by $C \subseteq \mathbb{N}$ if $\mathcal{J}_1 \restriction C \subseteq \mathcal{I}$ and $\mathcal{J}_2 \restriction (\mathbb{N} \setminus C) \subseteq \mathcal{I}$.

If no C splits a pregap, we say that it is a *gap* over $\mathcal{I}$. By (4) of Lemma 3.2, $\mathcal{S}_\mathcal{I}$ and $\mathcal{D}_\mathcal{I}$ always form a pregap over $\mathcal{I}$.

Recall that by Lemma 2.8.3 and Lemma 2.9.1 we have $\mathcal{Z}_0 \oplus \mathcal{Z}_\infty \approx_{\mathrm{RK}} \mathcal{Z}_\infty$ and $\mathcal{I}_\infty \approx_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{I}_{1/n} \approx_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{Z}_0 \approx_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{Z}_\infty$.

Proposition 3.6. *The quotients over the following analytic P-ideals are pairwise nonisomorphic.*

(1) $\mathcal{Z}_\infty$, $\mathcal{Z}_0$, $\mathcal{LV}$, $\mathcal{Z}_\infty \oplus \mathcal{LV}$, $\mathcal{Z}_0 \oplus \mathcal{LV}$,
(2) $\mathcal{Z}_\infty \oplus \mathcal{I}_{1/n}$, $\mathcal{Z}_0 \oplus \mathcal{I}_{1/n}$, $\mathcal{LV} \oplus \mathcal{I}_{1/n}$, $\mathcal{Z}_\infty \oplus \mathcal{LV} \oplus \mathcal{I}_{1/n}$, $\mathcal{Z}_0 \oplus \mathcal{LV} \oplus \mathcal{I}_{1/n}$,
(3) $\mathcal{Z}_\infty \oplus (\emptyset \times \mathrm{Fin})$, $\mathcal{Z}_0 \oplus (\emptyset \times \mathrm{Fin})$, $\mathcal{LV} \oplus (\emptyset \times \mathrm{Fin})$, $\mathcal{Z}_\infty \oplus \mathcal{LV} \oplus (\emptyset \times \mathrm{Fin})$, $\mathcal{Z}_0 \oplus \mathcal{LV} \oplus (\emptyset \times \mathrm{Fin})$,
(4) $\mathcal{I}_{1/n}$, $\emptyset \times \mathrm{Fin}$,
(5) $\mathcal{I}_\infty$, $\mathcal{I}_\infty \oplus \mathcal{LV}$, $\mathcal{I}_\infty \oplus \emptyset \times \mathrm{Fin}$, $\mathcal{I}_\infty \oplus \mathcal{LV} \oplus \emptyset \times \mathrm{Fin}$.

Proof. By Lemma 3.2, we only need to prove that the pairs of ideals $\mathcal{S}_\mathcal{I}$ and $\mathcal{D}_\mathcal{I}$ associated to the fifteen ideals listed above all have different properties. By (1)–(4) of Proposition 3.3, the five ideals in (1) all have different $\mathcal{S}_\mathcal{I}$ and they have $\mathcal{D}_\mathcal{I} = \mathcal{I}$, by (5) of Proposition 3.3. Since $\mathcal{D}_{\mathcal{I}_{1/n}} = \mathcal{P}(\mathbb{N})$, for the ideals $\mathcal{I}$ in (2) the ideal $\mathcal{D}_\mathcal{I}$ is generated by a single set over $\mathcal{I}$. Since $\mathcal{D}_{\emptyset\times\mathrm{Fin}} = \mathrm{Fin} \times \emptyset$, an ideal generated by a countable family of infinite pairwise disjoint sets, for the ideals $\mathcal{I}$ in (3) the ideal $\mathcal{D}_\mathcal{I}$ is generated by a countable family of infinite pairwise disjoint sets.

The only two ideals $\mathcal{I}$ on the list such that $\mathcal{S}_\mathcal{I} = \mathcal{I}$ are Fin and $\emptyset \times \mathrm{Fin}$, hence the quotients over the ideals in (4) are not isomorphic to any of the others. Since one of them is countably saturated and the other is not, they are not isomorphic to each other.

By Lemma 3.5, ideals $\mathcal{S}_{\mathcal{I}_\infty}$ and $\mathcal{D}_{\mathcal{I}_\infty}$ form a gap over $\mathcal{I}_\infty$. Since any ideal $\mathcal{J} \in \{\mathcal{Z}_\infty, \mathcal{Z}_0, \mathcal{LV}, \mathcal{I}_{1/n}, \emptyset \times \mathrm{Fin}\}$ has either $\mathcal{D}_\mathcal{J} = \mathcal{J}$ or $\mathcal{S}_\mathcal{J} = \mathcal{J}$, all ideals $\mathcal{I}$ in (1)–(4) have the property that $\mathcal{S}_\mathcal{I}$ and $\mathcal{D}_\mathcal{I}$ are separated. Therefore quotients over the ideals in (5) are not isomorphic to the quotients over the ideals in (1)–(4).

It remains to distinguish the quotients over the ideals in (5). Clause (4) of Proposition 3.3 implies that any ideal of the form $\mathcal{J} = \mathcal{I} \oplus \mathcal{LV}$ has a positive set A such that $\mathcal{S}_\mathcal{J} \restriction A = \mathcal{D}_\mathcal{J} \restriction A = \mathcal{J} \restriction A$. On the other hand, if $A \notin \mathcal{D}_{\mathcal{I}_\infty}$, then A has a positive subset B such that $\mathcal{I}_\infty \restriction B$ is a density ideal, hence $A \in \mathcal{S}_{\mathcal{I}_\infty}$. The ideal $\mathcal{I}_\infty \oplus \emptyset \times \mathrm{Fin}$ has this property as well. Therefore neither of the quotients over $\mathcal{I}_\infty$ or $\mathcal{I}_\infty \oplus \mathcal{LV}$ is isomorphic to any quotient over an ideal of the form $\mathcal{I} \oplus \mathcal{LV}$.

Finally, if $\mathcal{J} \in \{\mathcal{I}_\infty, \mathcal{I}_\infty \oplus \mathcal{LV}\}$ and A is $\mathcal{D}_\mathcal{J}$-positive, then by Lemma 3.5 it has a $\mathcal{J}$-positive subset B such that $\mathcal{J} \restriction B$ is a dense density ideal, and therefore has a positive subset in $\mathcal{S}_\mathcal{J}$. But any ideal of the form $\mathcal{J} = \mathcal{I} \oplus \emptyset \times \mathrm{Fin}$ clearly has a positive set A such that $\mathcal{J} \restriction A = \emptyset \times \mathrm{Fin}$, hence A has no positive subsets in $\mathcal{S}_{\mathcal{J}}$. Therefore neither of the quotients over $\mathcal{I}_\infty$ or $\mathcal{I}_\infty \oplus \mathcal{LV}$ is isomorphic to the quotients over $\mathcal{I}_\infty \oplus \emptyset \times \mathrm{Fin}$ or $\mathcal{I}_\infty \oplus \mathcal{LV} \oplus \emptyset \times \mathrm{Fin}$. □

4. Strong isometries

In this section we will develop a back-and-forth method for constructing isomorphisms between certain quotients over countable products of finite algebraic structures. It extends the method introduced by Just and Krawczyk in [11]. We will work out the details only in the case of Boolean algebras. However, we are not using any special properties of Boolean algebras. With an appropriate definition of an ε-approximate partial isomorphism (see Definition 4.10), all of the results of this section apply to quotients over any algebraic structures.

Definition 4.1. If $\mathcal{A}$ and $\mathcal{B}$ are models of the same language and $\mathcal{F}$ is a set of partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{F}$ has the *back and forth property* if

(B&F) for every $f \in \mathcal{F}$, for every $a \in \mathcal{A}$ and every $b \in \mathcal{B}$ there is a $g \in \mathcal{F}$ extending f such that $a \in \mathrm{dom}(g)$ and $b \in \mathrm{range}(g)$.

Lemma 4.2. *If X, Y are two models of the same language of cardinality $\aleph_1$ the following are equivalent.*

1. *X and Y are isomorphic.*
2. *There is a family $\mathcal{F}$ of partial maps from X into Y that has back-and-forth property and is closed under taking unions of countable chains.* □

If $\epsilon, K > 0$ and (X, d), (X', d') are metric spaces, then a relation $F \subseteq X \times X'$ is an *ϵ-isometry* if for all (a, b) and (c, d) in F we have

$$|d(a, c) - d'(b, d)| < \epsilon.$$

The reader should note that we do not require F to be a function, and that we even allow F to be empty. For $K \in [0, \infty]$ and $r \geq 0$ let

$$r^K = \min(r, K).$$

For $r, s \geq 0$ define

$$\Delta^K(r, s) = |r^K - s^K|.$$

Thus Δ^K defines a pseudo-metric on $[0, \infty)$ such that $\Delta^K(r, s) \leq |r - s|$ for all r, s.

A relation $F \subseteq X \times X'$ is an *(ϵ, K)-isometry* if for all (a, b) and (c, d) in F we have

$$\Delta^K(d(a, c), d'(b, d)) < \epsilon.$$

A partial function is an ε-isometry if its graph is an ε-isometry. A partial function is an (ε, K)-isometry if its graph is an (ε, K)-isometry.

Assume $(X_n, d_n)_{n=1}^\infty$ and $(X'_n, d'_n)_{n=1}^\infty$ are sequences of metric structures. A mapping $f\colon \prod_{n=1}^\infty X_n \to \prod_{n=1}^\infty X'_n$ is *precise* if for all a and b in its domain we have

$$\limsup_{n\to\infty} |d_n(a(n), b(n)) - d'_n(f(a)(n), f(b)(n))| = 0.$$

If $K < \infty$ then $f\colon \prod_{n=1}^\infty X_n \to \prod_{n=1}^\infty X'_n$ is *K-precise* if for all a and b in $\prod_{n=1}^\infty X_n$ we have

$$\limsup_{n\to\infty}\{\Delta^K(d_n(a(n), b(n)), d'_n(f(a)(n), f(b)(n))) = 0.$$

If $L \leq \infty$ then $f\colon \prod_{n=1}^\infty X_n \to \prod_{n=1}^\infty X'_n$ is *$< L$-precise* if it is K-precise for all $K < L$.

Hence being ∞-precise is the same as being precise, but being $< \infty$-precise is in general weaker.

If $(X_n, d_n)_{n=1}^\infty$ is a sequence of metric structures define an equivalence relation $\sim_{c_0}$ on $\prod_{n=1}^\infty X_n$ as follows:

$$a \sim_{c_0} b \qquad \Leftrightarrow \qquad \limsup\nolimits_{n\to\infty} d_n(a(n), b(n)) = 0$$

Lemma 4.3. *If f is precise, or K-precise for some $K > 0$, then*

$$a \sim_{c_0} b \qquad \Leftrightarrow \qquad f(a) \sim_{c_0} f(b)$$

for all a, b in the domain of f. □

For a partial map $f\colon \prod_{n=1}^\infty X_n \to \prod_{n=1}^\infty X'_n$ and $n \in \mathbb{N}$ define

$$\delta^{K,n}(f) = \sup_{a,b\in \mathrm{dom}(f)} \Delta^K(d_n(a(n), b(n)), d'_n(f(a)(n), f(b)(n))).$$

Lemma 4.4. *If the domain of f is finite, then f is K-precise if and only if $\limsup_n \delta^{K,n}(f) = 0$.*

Proof. The converse direction is easy and it does not need the assumption that $\mathrm{dom}(f)$ is finite. For the direct implication, assume $\limsup_n \delta^{K,n}(f) = \epsilon > 0$. Since $\mathrm{dom}(f)$ is finite, for some fixed $a, b \in \mathrm{dom}(f)$ the distance is at least $\varepsilon/2$ infinitely often, hence f is not K-precise. □

Definition 4.5. If (X_n, d_n) and (X'_n, d'_n) ($n \in \mathbb{N}$) are metric Boolean algebras and $K \leq \infty$, then a *$< K$-precise partial isomorphism* (respectively, *K-precise isomorphism*) is a partial map f from a subset of $\prod_{n=1}^\infty X_n$ into $\prod_{n=1}^\infty X'_n$ such that

(a) f is $< K$-precise (respectively, K-precise), and
(b) Map $[A]_{\sim_{c_0}} \mapsto [f(A)]_{\sim_{c_0}}$ from a subset of $\prod_{n=1}^\infty X_n/\sim_{c_0}$ into $\prod_{n=1}^\infty X'_n/\sim_{c_0}$ is an isomorphism between its domain and its range.

In short, f is a $< K$-precise (or K-precise) map that is a lifting of a partial isomorphism.

If X is a subset of a Boolean algebra $\mathcal{B}$, by $\langle X\rangle_{\mathcal{B}}$ we denote a subalgebra of $\mathcal{B}$ generated by X. The subscript $\mathcal{B}$ will be omitted whenever $\mathcal{B}$ is clear from the context.

Proposition 4.6. *The following are equivalent for every $L \leq \infty$ and all metric Boolean algebras $(X_n, d_n)_{n=1}^\infty$, $(X'_n, d'_n)_{n=1}^\infty$.*

(1) *The family of all finite $< L$-precise partial isomorphisms between $\prod_n X_n$ and $\prod_n X'_n$ has the back-and-forth property.*
(2) *The family of all countable $< L$-precise partial isomorphisms between $\prod_n X_n$ and $\prod_n X'_n$ has the back-and-forth property.*

Proof. It clearly suffices to prove that (1) implies (2). Let us assume (1). Let f be a countable $< L$-precise partial isomorphism from $\prod_{n=1}^\infty X_n$ into $\prod_{n=1}^\infty X'_n$ and fix $a \in \prod_{n=1}^\infty X_n$ and $b \in \prod_{n=1}^\infty X'_n$. We need to find a $< L$-precise partial isomorphism g extending f and such that $a \in \mathrm{dom}(g)$ and $b \in \mathrm{range}(g)$.

Pick a strictly increasing sequence K_i $(i < \mathbb{N})$ converging to L. Write $\mathrm{dom}(f)$ as an increasing union of finite Boolean algebras, $\mathcal{A}_n$ $(n \in \mathbb{N})$, and let $f_n = f \restriction \mathcal{A}_n$. By the assumption, for every n there is a K_n-precise $g_n \supseteq f_n$ such that $a \in \mathrm{dom}(g_n)$ and $b \in \mathrm{range}(g_n)$. We may assume that $\mathrm{dom}(g_n)$ is finite. By Lemma 4.4, for each i we can pick n_i such that for all $m \geq n_i$ we have

$$\delta^{K_i,m}(g_i) < 1/i.$$

This is possible since g_i is K_i-precise. We can assume that $n_i < n_{i+1}$ for all i. If $c \in \langle \mathrm{dom}(f) \cup \{a\}\rangle$ then $c \in \langle \mathcal{A}_i \cup \{a\}\rangle \subseteq \mathrm{dom}(g_i)$ for a large enough $i = i(c)$. Let $g \restriction \mathcal{A}_n = f \restriction \mathcal{A}_n$ for all n and define $g(c)$ by

$$g(c)(j) = g_i(c)(j) \text{ if } j \in [n_i, n_{i+1}) \text{ for } i \geq i(c),$$

and if $j < i(c)$ pick $g(c)(j) \in X'_j$ arbitrarily.

For each $i \in \mathbb{N}$ pick d_i such that $g_i(d_i) = b$. Define d by

(1) $d(j) = d_i(j)$, if $j \in [n_i, n_{i+1})$,

and let $g(d) = b$. We need to define $g(c)$ for $c \in \langle \mathcal{A}_m \cup \{a, d\}\rangle$ for each m. For such c we have $c = (a_1 \cap d) \cup (a_2 \setminus d)$ for some $a_1, a_2 \in \langle \mathcal{A}_i \cup \{a\}\rangle$. Let

(2) $g(c)(j) = g_i((a_1 \cap d_i) \cup (a_2 \setminus d_i))(j)$, if $j \in [n_i, n_{i+1})$ for some i such that $a_1, a_2 \in \mathcal{A}_i$,

and if $j \in [n_i, n_{i+1})$ for an i such that either a_1 or a_2 is not in $\mathcal{A}_i$, pick $g(c)(j) \in X'_j$ arbitrarily.

Then g extends f, $\mathrm{dom}(g)$ is a countable subalgebra of $\prod_{i=1}^\infty X_i$, $a \in \mathrm{dom}(g)$ and $b \in \mathrm{range}(g)$. It only remains to check that g is $< L$-precise and a partial isomorphism. Pick $b_1, b_2 \in \mathrm{dom}(g)$, and find m so that $b_1, b_2 \in \langle \mathcal{A}_m \cup \{a, d\}\rangle$.

Let us for a moment assume that $b_1, b_2 \in \langle \mathcal{A}_m \cup \{a\}\rangle$. Then $g(b_1)(j) = g_i(b_1)(j)$ and $g(b_2)(j) = g_i(b_2)(j)$ if $j \in [n_i, n_{i+1})$ for $i \geq m$ and

$d'_i(g(b_1)(i), g(b_2)(i)) = d'_i(g_i(b_1(i)), g_i(b_2(i))$. This implies that

$$\limsup_{i\to\infty} \Delta^{K_j}(d_i(b_1(i), b_2(i)), d'_i(g(b_1)(i), g(b_2)(i))) = 0$$

for all $j \geq m$.

Now assume that one (or both) of b_i is not in $\langle \mathcal{A}_m \cup \{a\}\rangle$ for infinitely many m. For all large enough m we have

$$b_i = (c_{i1} \cap d) \cup (c_{i2} \setminus d)$$

for some $c_{i1}, c_{i2} \in \langle \mathcal{A}_m \cup \{a\}\rangle$. The conclusion that

$$\limsup_{i\to\infty} \Delta^{K_j}(d_i(b_1(i), b_2(i)), d'_i(g(b_1)(i), g(b_2)(i))) = 0$$

now follows by the definition of $g(b_i)$. This proves that g is $< L$-precise and concludes the proof. □

The following variation of Proposition 4.6 will also be useful.

PROPOSITION 4.7. *The following are equivalent for every $K < \infty$ and all metric Boolean algebras $(X_n, d_n)_{n=1}^{\infty}$, $(X'_n, d'_n)_{n=1}^{\infty}$.*

1. *The family of all finite K-precise partial isomorphisms between $\prod_n X_n$ and $\prod_n X'_n$ has the back-and-forth property.*
2. *The family of all countable K-precise partial isomorphisms between $\prod_n X_n$ and $\prod_n X'_n$ has the back-and-forth property.*

Proof. Like the proof of Proposition 4.6, but taking $K_i = K$ for all i. □

THEOREM 4.8 (CH). *Assume X_n, X'_n ($n \in \mathbb{N}$) are finite or countable metric Boolean algebras. If there is an $L \leq \infty$ such that the family of all finite $< L$-precise partial isomorphisms between $\prod_{n=1}^{\infty} X_n$ and $\prod_{n=1}^{\infty} X'_n$ has the back-and-forth property, then $\prod_{n=1}^{\infty} X_n/\sim_{c_0}$ and $\prod_{n=1}^{\infty} X'_n/\sim_{c_0}$ are isomorphic.*

Moreover, the isomorphism can be chosen to be a $< L$-isometry with respect to the sup metric.

Proof. By Proposition 4.6, the family of all countable $< L$-precise partial isomorphisms has the back-and-forth property. Since a map is $< L$-precise if and only if its restriction to every two-element set is $< L$-precise, the family of $< L$-precise partial isomorphisms is closed under taking unions of increasing chains. Therefore by Lemma 4.2 the conclusion follows. □

A similar argument using Proposition 4.7 gives the following.

THEOREM 4.9 (CH). *Assume X_n, X'_n ($n \in \mathbb{N}$) are finite or countable metric Boolean algebras. If there is an $L \leq \infty$ such that the family of all finite L-precise partial isomorphisms between $\prod_{n=1}^{\infty} X_n$ and $\prod_{n=1}^{\infty} X'_n$ has the back-and-forth property, then $\prod_{n=1}^{\infty} X_n/\sim_{c_0}$ and $\prod_{n=1}^{\infty} X'_n/\sim_{c_0}$ are isomorphic.*

Moreover, the isomorphism can be chosen to be an L-isometry with respect to the sup metric. □

Definition 4.10. Assume that $\mathcal{B}$ and $\mathcal{B}'$ are Boolean algebras, $\mathcal{B}'$ is equipped with a metric d and $G \subseteq \mathcal{B} \times \mathcal{B}'$. Then F is an *ε-approximate partial homomorphism* (with respect to d) if for all (a, a'), (b, b'), and (c, c') in F such that

$$d(a \cup b, c) \leq \varepsilon$$

we have

$$d(a' \cup b', c') \leq \varepsilon.$$

The point here is that $a \cup b$ need not be in the domain of F. F is an *ε-approximate partial isomorphism* if both $\mathcal{B}$ and $\mathcal{B}'$ are equipped with a metric and both F and its inverse are ε-approximate partial homomorphisms.

The following technical lemma will be a useful tool for assembling precise partial isomorphisms.

Lemma 4.11. *Assume X_n, X'_n $(n \in \mathbb{N})$ are finite or countable metric Boolean algebras, $G_n \subseteq X_n \times X'_n$ is an (ε_n, K_n)-isometry for each n, $\lim_n \varepsilon_n = 0$ and $\lim_n K_n = L$ for a non-decreasing sequence K_n. Also assume that $A \subseteq \prod_{n=1}^{\infty} X_n$ is such that*

$$(\forall a \in A)(\forall^{\infty} n) a(n) \in \operatorname{dom}(G_n).$$

Finally, assume that each G_n is an ε_n-approximate partial isomorphism.

(a) *Then any map $f \colon A \to \prod_{n=1}^{\infty} X'_n$ such that for all n $a(n) \in \operatorname{dom}(G_n)$ implies $(a(n), f(a)(n)) \in G_n$ is an $< L$-precise partial isomorphism.*

(b) *If $K_n = L$ for all n, then f as in* (a) *is L-precise.*

Proof. We will prove only (a), since the proof of (b) is similar. Fix $a, b \in A$. For all but finitely many n we have $\{a(n), b(n)\} \subseteq \operatorname{dom}(G_n)$, hence

$$\{(a(n), f(a)(n)), (b(n), f(b)(n))\} \subseteq G_n.$$

Since for every $\varepsilon > 0$ and every $K < L$, for all but finitely many n we have that G_n is an (ε, K)-isometry, f is $< L$-precise.

The fact that $\lim_n \varepsilon_n = 0$ and that G_n is an ε_n-approximate partial isomorphism implies that f is a partial isomorphism. □

5. Isomorphic quotients

The method developed in §4 will now be applied to quotients over some density-like ideals. In this section, we assume that each submeasure ϕ is strictly positive. If ϕ is a submeasure on a set I, define a metric d_ϕ on $\mathcal{P}(I)$ by

$$d_\phi(A, B) = \phi(A \Delta B).$$

LEMMA 5.1. *If I_n are pairwise disjoint, ϕ_n is a submeasure on I_n, and $\mathcal{Z}_\phi = \mathrm{Exh}(\sup_n \phi_n)$, then $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\phi$ is isomorphic to $(\prod_{n=1}^\infty \mathcal{P}(I_n))/\sim_{c_0}$, and the isomorphism can be chosen to be a strong isometry.*

Proof. The map $a \mapsto \langle a \cap I_n \rangle_{n=1}^\infty$ is an isomorphism and a strong isometry. $\square$

By using Lemma 5.1, we can identify $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\phi$ with $(\prod_{n=1}^\infty \mathcal{P}(I_n))/\sim_{c_0}$, and in particular we can talk about K-precise or $< L$-precise maps between quotients $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\phi$. Note that a precise map between two quotients gives an isometry between the corresponding metric spaces.

THEOREM 5.2 (Just–Krawczyk). *Assume CH. Then all EU-ideals have isomorphic quotients.*

Proof. Consider EU-ideals $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$. By Theorem 2.8.1, we may assume that $\phi_n(I_n) = \psi_n(J_n) = 1$ for all n. By Lemma 5.1, it suffices to prove that the quotients $(\prod_{n=1}^\infty \mathcal{P}(I_n))/\sim_{c_0}$ and $(\prod_{n=1}^\infty \mathcal{P}(J_n))/\sim_{c_0}$ are isomorphic.

By Theorem 4.9, it will suffice to prove that the family $\mathcal{F}$ of all finite ∞-precise partial isomorphisms has the back-and-forth property. Pick $f \in \mathcal{F}$ and $a, b \subseteq \mathbb{N}$. We need to find g extending f such that $[a]_{\mathcal{Z}_\mu} \in \mathrm{dom}(g)$ and $[b]_{\mathcal{Z}_\nu} \in \mathrm{range}(g)$. We shall first describe how to get $[a]_{\mathcal{Z}_\mu} \in \mathrm{dom}(g)$. Let $a_1, \dots, a_k$ be pairwise disjoint subsets of $\mathbb{N}$ whose union is equal to $\mathbb{N}$ such that $[a_1]_{\mathcal{Z}_\mu}, \dots, [a_k]_{\mathcal{Z}_\mu}$ are the atoms of $\mathrm{dom}(f)$. Let $b_1, \dots, b_k$ be such that $f(a_i) = b_i$ for all $i \leq k$. By making small changes to b_i's, we may assume that they form a disjoint partition of $\mathbb{N}$. Let

$$F_m = \{(c(m), f(c)(m)) : c \in \mathrm{dom}(f)\}.$$

Fix $i \in \mathbb{N}$ and find n_i such that for all $m \geq n_i$ we have that F_m is a $1/(2ki)$-isometry and $\max(\mathrm{at}^+(\mu_m), \mathrm{at}^+(\nu_m)) < 1/(2ki)$. The former condition can be assured since f is ∞-precise, while the latter condition can be assured since both ideals are, by the assumption, dense. We may assume that the sequence n_i is strictly increasing. Fix $m \in [n_i, n_{i+1})$. Since $\mathrm{at}^+(\nu_m) < 1/(2ki)$, we can find $c(m) \subseteq J_m$ such that

(1) $|\nu_m(c(m) \cap b_j(m)) - \mu_m(a_j(m) \cap a(m))| < 1/(2ki)$

for all $j \leq k$. Since both μ_m and ν_m are measures and $|\mu_m(a_j(m)) - \nu_m(b_j(m))| < 1/(2ki)$, we have

(2) $|\mu_m(a_j(m) \setminus a(m)) - \nu_m(b_j(m) \setminus c(m))| < 1/ki$.

Every $d \in \langle \mathrm{dom}(F_m) \cup \{a(m)\} \rangle$ is of the form

$$d = \bigcup_{j \in Z_1(d)} (a_j(m) \cap a(m)) \cup \bigcup_{j \in Z_2(d)} (a_j(m) \setminus a(m))$$

for some disjoint subsets $Z_1(d)$ and $Z_2(d)$ of $\{1,\ldots,k\}$. Let

$$G_m = F_m \cup \Bigg\{ \Bigg(d, \bigcup_{j\in Z_1(d)} (b_j(m)\cap c(m)) \cup \bigcup_{j\in Z_2(d)} (b_j(m)\setminus c(m)) \Bigg) : \\ d \in \langle \mathrm{dom}(F_m) \cup \{a(m)\}\rangle \Bigg\}.$$

We claim that G_m is $1/i$-isometry. Assume $(d,e)\in G_m$ and $(d',e')\in G_m$. Then for each $j\leq k$ we have

$$|\mu_m((d\Delta d')\cap a_j(m)) - \nu_m((e\Delta e')\cap b_j(m))| \leq \frac{1}{ki},$$

hence $|\mu_m(d\Delta d') - \nu_m(e\Delta e')| \leq 1/i$, and G_m is $1/i$ isometry.

Let g be a function whose domain is the subalgebra generated by $\mathrm{dom}(f)$ and a, and such that for every $d\in \mathrm{dom}(g)$ we have $(d(m), g(d)(m))\in G_m$ for all $m\geq n_i$. The conditions of Lemma 4.11 are easily checked, hence g is precise and a partial isomorphism. It remains to extend g so that $b\in \mathrm{range}(g)$, but assuring this condition is very similar to assuring $a\in\mathrm{dom}(g)$. This proves that the family $\mathcal{F}$ has the back-and-forth property, and by Theorem 4.9 this concludes the proof. □

THEOREM 5.3 (CH). *If $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$ are dense density ideals and neither of them is an EU-ideal, then their quotients are isomorphic.*

Proof. By Lemma 2.8.3 we may assume that $\lim_n \mu_n(I_n) = \lim_n \nu(J_n) = \infty$. By Lemma 5.1, it suffices to prove that $(\prod_{n=1}^\infty \mathcal{P}(I_n))/\sim_{c_0}$ and $(\prod_{n=1}^\infty \mathcal{P}(J_n))/\sim_{c_0}$ are isomorphic. Let $\mathcal{F}$ be the family of all finite $<\infty$-precise partial isomorphisms. We claim that $\mathcal{F}$ has the back-and-forth property.

The proof is very similar to the proof of Theorem 5.2. Fix an $f\in\mathcal{F}$, let $\{a_1,\ldots,a_k\}$ enumerate all atoms of $\mathrm{dom}(f)$ and let $\{b_1,\ldots,b_k\}$ be atoms of $\mathrm{range}(f)$ such that $f(a_i)=b_i$ for each $i\leq k$. Let

$$F_m = \{(c(m), f(c)(m) : c\in \mathrm{dom}(f)\}.$$

For $i\in\mathbb{N}$ find n_i such that for all $m\geq n_i$ we have that $\delta^{4i,m}(f) < 1/(2ki)$ and $\max(\mathrm{at}^+(\mu_m), \mathrm{at}^+(\nu_m)) < 1/(2ki)$. We may assume $n_i < n_{i+1}$ for all i. For $m\in[n_i,n_{i+1})$ there is a partition $\{1,\ldots,k\} = X_0^m \dot\cup X_1^m$ such that

$$\mu_m(a_j(m)) < 3i \text{ if and only if } j\in X_0^m.$$

Note that $|\mu_m(a_j(m)) - \nu_m(b_j(m))| < 1/(2ki)$ for all $j\in X_0^m$. We will now describe how to choose $c(m)\in\mathcal{P}(J_m)$, by imposing a condition on the choice of $c(m)\cap b_j(m)$ for $j\leq k$.

For $j\in X_0^m$ make sure that

$$(*)\quad |\mu_m(c(m)\cap b_j(m)) - \nu_m(a(m)\cap a_j(m))| < 1/(2ki),$$

and note that, by the additivity of μ_m and ν_m, this implies

(**) $|\mu_m(b_j(m) \setminus c(m)) - \nu_m(a_j(m) \setminus a(m)))| < 1/(ki)$.

For $j \in X_1^m$ such that $\nu_m(a(m) \cap a_j(m)) \leq i$, choose $c(m) \cap b_j(m)$ as in (*). Then $\nu_m(a_j(m) \setminus a(m)) > i$ and therefore $\mu_m(b_j(m) \setminus c(m)) > i$.

For $j \in X_1^m$ such that $\nu_m(a_j(m) \setminus a(m)) \leq i$, choose $c(m) \cap b_j(m)$ so that (**) holds. Note that in this case $\mu_m(b_j(m) \cap c(m)) > i$.

Finally, assume $j \in X_1^m$ is such that

$$\min(\nu_m(a_j(m) \cap a(m)), \nu_m(a_j(m) \setminus a(m))) > i.$$

Since $\delta^{4i,m}(f) < 1/(2ki)$ and $\nu_m(a_j(m)) \geq 3i$, we have $\mu_m(b_j(m)) \geq 3i - 1/(2ki)$. Since $\mathrm{at}^+(\mu_m) < 1/(2ki)$, we can choose $c(m) \cap b_j(m)$ so that

(***) $\mu_m(b_j(m) \setminus c(m)) \geq i$ and $\mu_m(b_j(m) \cap c(m)) \geq i$.

This describes the choice of $c(m)$. For all $j \leq k$ we have

$$\Delta^i(\mu_m(b_j(m) \cap c(m)), \nu_m(a_j(m) \cap a(m))) < 1/ki$$

and

$$\Delta^i(\mu_m(b_j(m) \setminus c(m)), \nu_m(a_j(m) \setminus a(m))) < 1/ki.$$

Therefore G_m defined in the same fashion as in the proof of Theorem 5.2 is an $(1/i, i)$-isometry. Also, g formed from G_m's so that $\mathrm{dom}(g) = \langle \mathrm{dom}(f) \cup \{a\} \rangle$ and for each $d \in \mathrm{dom}(g)$ we have $(d(m), g(d)(m)) \in G_m$ for all large enough m is a $< \infty$-precise partial isomorphism by Lemma 4.3.

The proof that for any b we can further extend g so that b is in the range of g is similar. Thus $\mathcal{F}$ has the back-and-forth property and by Theorem 4.8 this concludes the proof. $\square$

A dense density ideal $\mathcal{Z}_\infty$ was defined in §2.8, and in Proposition 3.6 it was proved that its quotient is not isomorphic to a quotient over any EU-ideal.

Corollary 5.4 (CH). *There are exactly two isomorphism types of quotients over dense density ideals.*

Proof. By Lemma 2.8.3 and Theorem 5.3, if $\mathcal{Z}_\mu$ is a dense density ideal, then its quotient is isomorphic either to the quotient over $\mathcal{Z}_0$ or to the quotient over $\mathcal{Z}_\infty$. $\square$

LV-ideals were defined in §2.11.

Theorem 5.5 (CH). *Every two quotients over LV ideals are isomorphic.*

Proof. Let ϕ_n $(n \in \mathbb{N})$ and ψ_n $(n \in \mathbb{N})$ be submeasures such that if $\phi = \sup_n \phi_n$ and $\psi = \sup_n \psi_n$ then $\mathrm{Exh}(\phi_n)$ and $\mathrm{Exh}(\psi_n)$ are LV ideals. By Lemma 5.1, it suffices to prove that $(\prod_{n=1}^\infty \mathcal{P}(I_n))/\sim_{c_0}$ and $(\prod_{n=1}^\infty \mathcal{P}(J_n))/\sim_{c_0}$ are isomorphic.

We claim that the family $\mathcal{F}$ of all finite 1-precise isomorphisms from $(\prod_{n=1}^\infty \mathcal{P}(I_n))$ into $(\prod_{n=1}^\infty \mathcal{P}(J_n))$ has the back-and-forth property. The proof is similar to the proofs of Theorem 5.2 and Theorem 5.3.

Fix an $f \in \mathcal{F}$, let $\{a_1, \dots, a_k\}$ enumerate all atoms of $\mathrm{dom}(f)$ and let $\{b_1, \dots, b_k\}$ enumerate atoms of $\mathrm{range}(f)$ such that $f(a_i) = b_i$ for all $i \leq k$. For $m \in \mathbb{N}$, let

$$F_m = \{(c(m), f(c)(m)) : c \in \mathrm{dom}(f)\}.$$

Let $\varepsilon = 1/i$. Using (LV2), for $i \in \mathbb{N}$ find n_i large enough so that for all $m \geq n_i$ we have

(3) $(\forall p_0, \dots, p_{k+2} \subseteq I_m)|\phi_m(p_0 \Delta p_{k+2}) - \max_{i<k+2} \phi_m(p_i \Delta p_{i+1})| < \varepsilon$,
(4) $(\forall p_0, \dots, p_{k+2} \subseteq J_m)|\psi_m(p_0 \Delta p_{k+2}) - \max_{i<k+2} \psi_m(p_i \Delta p_{i+1})| < \varepsilon$,
(5) $\delta^{1,m}(f) < \varepsilon$, and
(6) $\max(\mathrm{at}^+(\phi_m), \mathrm{at}^+(\psi_m)) < \varepsilon$.

We may assume $n_i < n_{i+1}$ for all i. We need to describe how to choose $f(a)(m) = c(m)$ for each $m \geq n_1$. Fix m and let i be such that $m \in [n_i, n_{i+1})$. For $j \leq k$ let

$$\alpha_j = \phi_m(a_j(m)),$$

and note that by (3) we have

$$\alpha_j \geq \max(\phi_m(a_j(m) \cap a(m)), \phi(a_j(m) \setminus a(m)) - \varepsilon.$$

We choose $c(m) \cap b_j(m)$ for each $j \leq k$ according to the following cases.

If $\phi_m(a_j(m) \cap a(m)) < \alpha_j - 2\varepsilon$, use $\mathrm{at}^+(\psi_m) < \varepsilon$ to pick $c(m) \cap b_j(m)$ so that

(*1) $\Delta^1(\phi_m(a_j(m) \cap a(m)), \psi_m(b_j(m) \cap c(m))) < \varepsilon$.

Then (3) implies $\phi_m(a_j(m) \setminus a(m)) \geq \alpha_j - \varepsilon$. Also, (5) and (4) imply and $\psi_m(b_j(m) \setminus c(m)) \geq \psi_m(b_j(m)) - \varepsilon$, and therefore

(*2) $\Delta^1(\phi_m(a_j(m) \setminus a(m)), \psi_m(b_j(m) \setminus c(m))) < \Delta^1(\phi_m(a_j(m)), \psi_m(b_j(m))) + 2\varepsilon \leq 3\varepsilon$.

In the case when $\phi_m(a_j(m) \setminus a(m)) < \alpha_j - 2\varepsilon$, choose $c(m) \cap b_j(m)$ so that

(*3) $\Delta^1(\phi_m(a_j(m) \setminus a(m)), \psi_m(b_j(m) \setminus c(m))) < \varepsilon$.

By the above argument, in this case we have

(*4) $\Delta^1(\phi_m(a_j(m) \cap a(m)), \psi_m(b_j(m) \cap c(m))) < 3\varepsilon$.

The remaining case is when

$$\min(\phi_m(a_j(m) \setminus a(m)), \phi_m(a_j(m) \cap a(m)) \geq \alpha_j - 2\varepsilon,$$

and we will find $c(m)$ so that

(*5) $\psi_m(b_j(m) \setminus c(m)) \geq \psi_m(b_j(m)) - 2\varepsilon$ and
(*6) $\psi_m(b_j(m) \cap c(m)) \geq \psi_m(b_j(m)) - 2\varepsilon$.

Since $\Delta^1(\phi_m(a_j(m)), \psi_m(b_j(m))) < \varepsilon$, this will imply

(*7) $\Delta^1(\phi_m(a_j(m) \cap a(m)), \psi_m(b_j(m) \cap c(m))) < 4\varepsilon$, and
(*8) $\Delta^1(\phi_m(a_j(m) \setminus a(m)), \psi_m(b_j(m) \setminus c(m))) < 4\varepsilon$.

Let

$$\mathcal{U} = \{d \subseteq b_j : \psi_m(d) \geq \psi_m(b_j(m)) - \varepsilon\}.$$

If $\mathcal{U}$ contains two pairwise disjoint sets, let $c(m) \cap b_j(m)$ be one of them. In this case (*5) and (*6) are clearly satisfied.

Otherwise, let d be any minimal element of $\mathcal{U}$. If d is a singleton, then since $\mathrm{at}^+(\psi_m) < \varepsilon$, we have $\psi_m(d) < \varepsilon$ and $\psi_m(b_j(m)) \leq \psi_m(d) + \varepsilon = 2\varepsilon$. Therefore $c(m) \cap b_j(m) = \emptyset$ satisfies (*5) and (*6).

Now assume d is not a singleton. Write $d = d_0 \dot\cup d_1$ for some nonempty d_0 and d_1. Since $d_i \notin \mathcal{U}$, we have $\psi_m(b_j(m)) - \varepsilon > \psi_m(d_i)$ for both $i < 2$. But $\psi_m(d) \leq \max(\psi_m(d_0), \psi_m(d_1)) + \varepsilon$, hence there is an $i < 2$ such that

$$\psi_m(d_i) \geq \psi_m(d) - \varepsilon \geq \psi_m(b_j) - 2\varepsilon.$$

Without a loss of generality, $i = 0$. Let $c(m) = d_0$. Then (*6) holds. Since $c(m) \cap b_j(m) \notin \mathcal{U}$, we have $\psi_m(b_j(m)) > \psi_m(c(m) \cap b_j(m)) + \varepsilon$. But

$$\psi_m(b_j(m)) \leq \max(\psi_m(c(m) \cap b_j(m)), \psi_m(b_j(m) \setminus c(m))) + \varepsilon,$$

and therefore $\psi_m(b_j(m) \setminus c(m)) \geq \psi_m(b_j(m)) - \varepsilon$, and (*5) is satisfied.

Now we define G_m as in the proof of Theorem 5.2. Every $d \in \langle \mathrm{dom}(F_m) \cup \{a(m)\}\rangle$ is of the form

$$d = \bigcup_{j \in Z_1(d)} (a_j(m) \cap a(m)) \cup \bigcup_{j \in Z_2(d)} (a_j(m) \setminus a(m))$$

for some disjoint subsets $Z_1(d)$ and $Z_2(d)$ of $\{1, \dots, k\}$. Let

$$G_m = F_m \cup \left\{ \left(d, \bigcup_{j \in Z_1(d)} (b_j(m) \cap c(m)) \cup \bigcup_{j \in Z_2(d)} (b_j(m) \setminus c(m)\right) : \\ d \in \langle \mathrm{dom}(F_m) \cup \{a(m)\}\rangle \right\}.$$

Then G_m is a $(1, 4/i)$-isometry (recall that $\varepsilon = 1/i$). This follows by (*1)–(*8) and the fact that by (3) and (4) if $d \in \mathrm{dom}(G_m)$, then

$$|\phi_m(d) - \max_{j \leq k} \phi_m(d \cap a_j(m))| < \varepsilon$$

and if $e \in \mathrm{range}(G_m)$ then

$$|\psi_m(e) - \max_{j \leq k} \phi_m(e \cap b_j(m))| < \varepsilon.$$

Like in the proof of Theorem 5.2, g defined from the G_m's is as required by Lemma 4.3. Thus $\mathcal{F}$ has the back-and-forth property and by Theorem 4.8 this concludes the proof. □

THEOREM 5.6 (CH). *Consider the class of all ideals of the form* $\mathrm{Exh}(\sup_m \mu_m)$*, where* μ_m *are lower semicontinuous measures concentrating on pairwise orthogonal sets* I_n *and such that*

(1) $\mu_n(I_n) = \infty$ *for all* n,
(2) $\limsup_m \sup_n \mu_n(\{m\}) = 0$.

All quotients over ideals in this class are pairwise isomorphic.

Proof. Fix ideals $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$ in this class and let I_n (respectively, J_n) denote the pairwise disjoint sets on which μ_n (respectively, ν_n) concentrates. We will prove that there is a countably closed family of partial isomorphisms with the back-and-forth property and apply Lemma 4.2. Let $\mathcal{F}$ be the family of all countable partial isomorphisms f from a subset of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ such that

(3) $[A]_{\mathcal{Z}_\mu} \mapsto [f(A)]_{\mathcal{Z}_\nu}$ is a partial isomorphism.
(4) $\mathbb{N} \in \mathrm{dom}(f)$.
(5) $a \in \mathrm{dom}(f)$ implies $a \cap I_n \in \mathrm{dom}(f)$ for all n.
(6) $a \subseteq \bigcup_{i \leq k} I_i$ for some k implies $f(a) \subseteq \bigcup_{i \leq k} J_i$.
(7) For all $a, b \in \mathrm{dom}(f)$ and all $K < \infty$ we have

$$\limsup_{n\to\infty} \Delta^K(\mu_n(a\Delta b), \nu_n(f(a)\Delta f(b)) = 0.$$

In the situation when (3) applies we say that f is a *lifting* of a partial isomorphism (note that we do not require f to have any algebraic properties).

Lemma 5.7. *The family $\mathcal{F}$ has the back-and-forth property.*

Proof. Fix $f \in \mathcal{F}$, and $a \subseteq \mathbb{N}$. We will describe how to find g in $\mathcal{F}$ extending f that includes a in its domain. Let $f_n = f \restriction \mathcal{P}(I_n)$. Since $\mathcal{Z}_\mu \restriction I_n$ and $\mathcal{Z}_\nu \restriction J_n$ are both F_σ ideals, by Corollary 6.4 we may extend f_n to f'_n so that $\mathrm{dom}(f'_n) = \langle \mathrm{dom}(f_n) \cup \{a \cap I_n\}\rangle$ and f'_n is a lifting of a partial isomorphism between countable subalgebras of $\mathcal{P}(I_n)/\mathcal{Z}_\mu$ and $\mathcal{P}(J_n)/\mathcal{Z}_\nu$. Now we canonically extend f to f' such that $\mathrm{dom}(f') = \langle \mathrm{dom}(f) \cup \{a \cap I_n : n \in \mathbb{N}\}\rangle$ and f' extends all f_n. If $d \in \langle \mathrm{dom}(f) \cup \{a \cap I_n : n \in \mathbb{N}\}\rangle$, then $d = c \cap t(a \cap I_1, \dots, a \cap I_n)$, for some $c \in \mathrm{dom}(f)$, Boolean term t and $n \in \mathbb{N}$. Let

$$f'(d) = f(c) \cap t(f_1(a \cap I_1), \dots, f_n(a \cap I_n)).$$

Then f' still satisfies (3)–(6), and since $f'(d)\Delta f'(c) \subseteq \bigcup_{j \leq n} J_j$, it satisfies (7) as well. Hence $f' \in \mathcal{F}$. Write $\mathrm{dom}(f') = \bigcup_{j=1}^\infty \mathcal{A}_j$, where $\mathcal{A}_j$ is an increasing chain of finite Boolean algebras. For each $j \in \mathbb{N}$ find n_j such that for all $m \geq n_j$ and all $c, d \in \mathcal{A}_j$ we have

$$\Delta^j(\mu_m(c\Delta d), \nu_m(f'(c)\Delta f'(d))) < \varepsilon.$$

We may assume that the sequence n_j is strictly increasing. For each $m \in [n_j, n_{j+1})$ find a finite $s_m \subseteq I_m$ and a finite $t_m \subseteq J_m$ such that for all $c \in \langle A_j \cup \{a \cap I_m\}\rangle$ we have:

(8) $\mu_m((c \cap I_m) \setminus s_m) \geq \varepsilon$ implies $\mu_m(c \cap I_m) = \infty$ and $\mu_m((c \cap I_m) \cap s_m) \geq j$.

(9) $\nu_m((f'(c)\cap J_m)\setminus t_m)\geq \varepsilon$ implies $\nu_m(f'(c)\cap J_m)=\infty$ and $\nu_m((f'(c)\cap J_m)\cap t_m)\geq j$.

Let

$$X=\bigcup_{m=1}^{\infty} I_m\cap s_m \qquad \text{and} \qquad Y=\bigcup_{m=1}^{\infty} J_m\cap t_m.$$

If necessary, increase some of the s_m and t_m so that the sets X and Y satisfy the following condition for all $c,d\in \operatorname{dom}(f')$:

(10) $(\forall n)((c\Delta d)\setminus \bigcup_{i\leq n} I_i)\cap X\notin \mathcal{Z}_\mu$ if and only if

$$(\forall n)((f'(c)\Delta f'(d))\setminus \bigcup_{i\leq n} J_i)\cap Y\notin \mathcal{Z}_\nu.$$

Since $\operatorname{dom}(f')$ is countable, this can be done by a simple diagonalization argument.

Fix a well-ordering $<_w$ of $\operatorname{dom}(f')$ and let f'' be defined on the set $\{c\cap X : c\in \operatorname{dom}(f')\}$ by

$$f''(d)=f'(c)\cap Y,$$

where c is the $<_w$-minimal element of $\operatorname{dom}(f')$ such that $c\cap X=d$. Note that (10) implies that $(c\Delta d)\cap X\in \mathcal{Z}_\mu$ if and only if $(f''(c)\Delta f''(d))\cap Y\in\mathcal{Z}_\nu$.

We may think of f'' as a map from $\prod_{n=1}^\infty \mathcal{P}(s_n)$ to $\prod_{n=1}^\infty \mathcal{P}(t_n)$. Using the restriction of μ_n to s_n and the restriction of ν_n to t_n, we can talk about f'' being $<\infty$-precise.

Claim 1. *The function f'' is $<\infty$ precise.*

Proof. Fix $c,d\in\operatorname{dom}(f'')$ and $K<\infty$. There is $j\geq K$ large enough so that $c,d\in\mathcal{A}_j$ and by (7)

$$\sup_{n\geq n_j}\Delta^K(\mu_n(c\Delta d),\nu_n(f(c)\Delta f(d))<\varepsilon.$$

If $m\geq n_j$, then by (8) we have

$$\Delta^K(\mu_m(c),\mu_m(c\cap s_m))<\varepsilon$$

and by (9) we have

$$\Delta^K(\nu_m(f''(c)\cap t_m),\nu_m(f''(c)))<\varepsilon.$$

These conditions, together with analogous conditions for d, imply

$$\Delta^K(\mu_m((c\Delta d)\cap s_m),\nu_m((f'(c)\Delta f'(d))\cap t_m))<3\varepsilon.$$

Since c,d and K were arbitrary, this proves the claim. □

By Claim 1 and the proof of Theorem 5.3 we can extend f'' to a $<\infty$-precise map $f'''\colon \prod_{n=1}^\infty\mathcal{P}(s_n)\to\prod_{n=1}^\infty\mathcal{P}(t_n)$ such that $a\cap X\in\operatorname{dom}(f''')$.

Finally define g as follows. If $d \in \langle \mathrm{dom}(f') \cup \{a\} \rangle$, then $d = (c_1 \cap a) \cup (c_2 \setminus a)$ for some $c_1, c_2 \in \mathrm{dom}(f')$. Let

$$g(c_1 \cap a) = \left(\bigcup_{j=1}^{\infty} f'(c_1 \cap a \cap I_n) \setminus Y \right) \cup f'''(c_1 \cap a \cap X)$$

and

$$g(c_2 \setminus a) = \left(\bigcup_{j=1}^{\infty} f'((c \setminus a) \cap I_n) \setminus Y \right) \cup f'''((c_2 \setminus a) \cap X),$$

and $g(d) = g(c_1 \cap a) \cup g(c_2 \setminus a)$.

Since f''' is $< \infty$-precise, by Claim 1, (8) and (9), g satisfies (7).

An analogous argument proves that g can be extended so that its range contains an arbitrary $b \subseteq \mathbb{N}$. This concludes the proof that $\mathcal{F}$ has the back-and-forth property. $\square$

By Lemma 4.2, this concludes the proof. $\square$

6. Countable saturatedness of analytic quotients

The results of this and the following section apply to arbitrary ideals on $\mathbb{N}$. By '$\mathcal{A}$ is countably saturated' we mean '$\mathcal{A}$ is $\aleph_1$-saturated,' i.e., that every consistent countable type with parameters from $\mathcal{A}$ is satisfied in $\mathcal{A}$ (see, e.g., [1]). As pointed out before, any two atomless Boolean algebras are elementarily equivalent, therefore all countably saturated quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are isomorphic under the Continuum Hypothesis.

An *ω-limit* in a Boolean algebra is an increasing sequence A_n $(n \in \mathbb{N})$ that has the lowest upper bound.

Proposition 6.1. *For an ideal $\mathcal{I}$ on $\mathbb{N}$ that includes* Fin *the following are equivalent:*

(1) *The quotient over $\mathcal{I}$ is not countably saturated.*
(2) *There is an ω-limit in $\mathcal{P}(\mathbb{N})/\mathcal{I}$.*
(3) *There is a partition of $\mathbb{N}$ into pairwise disjoint, $\mathcal{I}$-positive sets B_n $(n \in \mathbb{N})$ such that for all $A \subseteq \mathbb{N}$ we have*

$$A \in \mathcal{I} \qquad \Leftrightarrow \qquad (\forall n) A \cap B_n \in \mathcal{I}.$$

(4) *There are ideals $\mathcal{I}_n$ $(n \in \mathbb{N})$ on $\mathbb{N}$ such that $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \prod_{n=1}^{\infty}(\mathcal{P}(\mathbb{N})/\mathcal{I}_n)$.*

If $\mathcal{I}$ is an analytic P-ideal, then the above conditions are equivalent to

(5) *$\mathcal{I}$ is not F_σ.*

Proof. In [12, Corollary 2.4] it was proved that (1) is equivalent to

(2′) There is a sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ of $\mathcal{I}$-positive sets such that for every $\mathcal{I}$-positive set A we have $A \setminus A_n \notin \mathcal{I}$ for some n.

Clearly, (2′) is equivalent to (2). Obviously, (2) implies (3) implies (4). Finally, a product of countably many Boolean algebras cannot be countably saturated, therefore (4) implies (1).

It remains to prove that if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is countably saturated and $\mathcal{I}$ is an analytic P-ideal, then $\mathcal{I}$ is F_σ. Assume $\mathcal{I}$ is an analytic P-ideal and that it is not F_σ. By [24, Theorem 3.1], $\mathcal{I} = \mathrm{Exh}(\phi)$ for a lower semicontinuous submeasure ϕ. Then Case 2 of the proof of [24, Theorem 3.3] applies, hence there are $\mathcal{I}$-positive sets X_n such that $\phi(X_n) \leq 2^{-n}$. So the sets $Y_n = \bigcup_{i=n}^{\infty} X_n$ form a strictly decreasing sequence of $\mathcal{I}$-positive sets whose only lower bound is $[\emptyset]_{\mathcal{I}}$. □

Corollary 6.2. *If $\mathcal{I}$ and $\mathcal{J}$ are analytic P-ideals and their quotients are isomorphic, then $\mathcal{I}$ and $\mathcal{J}$ have the same Borel complexity.*

Proof. By [24], every analytic P-ideal is $F_{\sigma\delta}$. By the equivalence of (1) and (5) in Proposition 6.1, ideals $\mathcal{I}$ and $\mathcal{J}$ are either both F_σ or both $F_{\sigma\delta} \setminus F_\sigma$. □

Case (c) of the following theorem was proved in [11].

Theorem 6.3. *If α is an indecomposable countable ordinal, then the quotients over*

(a) *all ideals $\mathcal{O}_\alpha(P)$ such that P is well-founded,*
(b) *all Cantor–Bendixson ideals $\mathrm{CB}_\alpha(X)$,*
(c) *all F_σ ideals,*
(d) *all ideals of the form $\mathcal{I} \times \mathcal{J}$ where $\mathcal{I}$ is as in* (a), (b) *or* (c),

are countably saturated.

Proof. By Lemma 6.7 and Proposition 6.6 below. □

Corollary 6.4 (CH). *The quotients over all*

(i) *F_σ ideals,*
(ii) *ideals $\mathcal{O}_\alpha(P)$ for indecomposable countable ordinal α and well-ordered P,*
(iii) *Cantor–Bendixson ideals, and*
(iv) *ideals of the form $\mathcal{I} \times \mathcal{J}$, where $\mathcal{I}$ is as in* (i)–(iv) *and $\mathcal{J}$ is arbitrary,*

are pairwise isomorphic.

In particular, all quotients over ordinal ideals, all Weiss ideals and all F_σ ideals are pairwise isomorphic.

Proof. By Theorem 6.3, all of these quotients are countably saturated. The family of all countable partial isomorphisms between two countably saturated models has the back-and-forth property and it is σ-closed. Therefore the conclusion follows by Lemma 4.2. □

In Lemma 6.7 we will show that the following definition gives a sufficient condition for a quotient to be countably saturated.

Definition 6.5. An ideal $\mathcal{I}$ is *layered* if there is $f\colon \mathcal{P}(\mathbb{N}) \to [0,\infty]$ such that

(L1) $A \subseteq B$ implies $f(A) \leq f(B)$,
(L2) $\mathcal{I} = \{A : f(A) < \infty\}$,
(L3) $f(A) = \infty$ implies $f(A) = \sup\{f(B) : B \subseteq A \text{ and } f(B) < \infty\}$.

Proposition 6.6.

1. *Every F_σ-ideal is layered.*
2. *If P is well-ordered and α is an indecomposable ordinal, then $\mathcal{O}_\alpha(P)$ is layered.*
3. *If X is a countable topological space whose Cantor-Bendixson rank is at least an indecomposable ordinal α, then $\mathrm{CB}_\alpha(X)$ is layered.*
4. *If $\mathcal{J}$ is a layered ideal and $\mathcal{I}$ is an arbitrary ideal on $\mathbb{N}$, then $\mathcal{J} \times \mathcal{I}$ is layered.*

Proof. (1) This is because by a result of K. Mazur stated in §2.5 for every F_σ ideal $\mathcal{I}$ there is a lower semicontinuous submeasure ϕ on $\mathbb{N}$ such that

$$\mathcal{I} = \{A : \phi(A) < \infty\}.$$

Then $f = \phi$ satisfies conditions (L1)–(L3) from Definition 6.5.

(2) Take a strictly increasing sequence α_n $(n \in \mathbb{N})$ of ordinals converging to α and let

$$f(A) = \min\{n : \alpha_n \text{ does not embed into } A\}.$$

Since P is well-ordered, conditions (L1)–(L3) are easily checked.

(3) Let α_n $(n \in \mathbb{N})$ be an increasing sequence of ordinals converging to α and let

$$f(A) = \min\{n : \text{ Cantor–Bendixson rank of } A \text{ is less than } \alpha_n\}.$$

The conditions (L1)–(L3) are easily checked.

(4) Let $f_{\mathcal{J}}$ be a function satisfying (L1)–(L3) for $\mathcal{J}$, and define f by (for $A \subseteq \mathbb{N}^2$ let $A_n = \{m : (n,m) \in A\}$)

$$f(A) = f_{\mathcal{J}}\{n : A_n \notin \mathcal{I}\}.$$

Then (L1) and (L2) are clearly satisfied. To prove (L3), fix A such that $f(A) = \infty$. If $B = \{n : A_n \notin \mathcal{I}\}$, for each n find $B_n \subseteq B$ such that $f_{\mathcal{J}}(B_n) \geq n$. Then $f(A \cap (B_n \times \mathbb{N})) = f_{\mathcal{J}}(B_n) \geq n$ for each n, therefore (L3) is satisfied. $\square$

Lemma 6.7. *If $\mathcal{I}$ is layered, then the quotient over $\mathcal{I}$ is countably saturated.*

Proof. We need only to check that (2) of Proposition 6.1 fails. Let f be a witness that $\mathcal{I}$ is layered. Let A_i ($i \in \mathbb{N}$) be a decreasing sequence of $\mathcal{I}$-positive sets. For each i pick $B_i \subseteq A_i$ in $\mathcal{I}$ such that $f(B_i) \geq i$. Then $A = \bigcup_n B_n$ satisfies $f(A) \geq i$ for all i, hence it is $\mathcal{I}$-positive. Also, $A \setminus A_i \subseteq \bigcup_{j=1}^{i-1} B_j \in \mathcal{I}$, and A is as required. □

Definition 6.8. A *factor* of a Boolean algebra of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is a Boolean algebra of the form $\mathcal{P}(A)/(\mathcal{I} \restriction A)$ for some positive set A. A quotient is *nowhere countably saturated* if none of its factors is countably saturated.

Lemma 6.9. *If $\mathcal{Z}_\mu$ is a density ideal, then its quotient has a countably saturated factor if and only if $\mathcal{Z}_\mu$ is not dense.*

Proof. Recall that for a lower semi-continuous ϕ the ideal $\mathrm{Exh}(\phi)$ is dense if and only if $\limsup_n \phi(\{n\}) = 0$. Therefore a density ideal $\mathcal{Z}_\mu$ is dense if and only if $\limsup_n \mathrm{at}^+(\mu_n) = 0$.

By (5) of Proposition 3.3, if $\mathcal{Z}_\mu$ is a dense density ideal then its quotient is not countably saturated. Since the restriction of a dense density ideal to any positive set is a dense density ideal, its quotient has no countable factors.

Now assume $\mathcal{Z}_\mu$ is not dense. There is $\varepsilon > 0$ is such that $\mathrm{at}^+(\mu_n) \geq \varepsilon$ for infinitely many n. The set $A = \{i : \mu(\{i\}) \geq \varepsilon\}$ is then infinite, and $\mathcal{Z}_\mu \restriction A$ is isomorphic to Fin and its quotient is therefore countably saturated. The other direction is a consequence of (a). □

7. A classification result for a class of quotients

In Corollary 5.4 we have shown that under CH there are only finitely many (namely, two) isomorphism classes of quotients over dense density ideals. We shall extend this result to a larger class of ideals.

Definition 7.1. Let $\mathbb{D}$ denote the class of all ideals $\mathcal{Z}_\mu$ of the following form. Assume that μ_n ($n \in \mathbb{N}$) are measures on $\mathbb{N}$ concentrating on pairwise disjoint sets, I_n ($n \in \mathbb{N}$), and that $\limsup_i \sup_n \mu_n(\{i\}) = 0$. We require that μ_n of each finite set is finite, but we allow $\mu_n(I_n) = \infty$. Let

$$\mathcal{Z}_\mu = \mathrm{Exh}(\sup_n \mu_n) = \{A : \limsup_k \sup_n \mu_n(A \setminus k) = 0\}.$$

Class $\mathbb{D}$ includes all dense density ideals (the case when all I_n are finite), all dense summable ideals (the case when only one I_n is nonempty), $\mathcal{I}_\infty$, and it is closed under $\oplus$. All ideals occurring in Proposition 3.6 except those that have $\emptyset \times \mathrm{Fin}$ or $\mathcal{LV}$ as a summand belong to $\mathbb{D}$.

Lemma 7.2. *Class $\mathbb{D}$ coincides with the class of all dense ideals of the form $\mathrm{Exh}(\phi)$, where ϕ is the pointwise supremum of a family of pairwise orthogonal lower semicontinuous measures on $\mathbb{N}$.*

Proof. This is because every family of pairwise orthogonal lower semicontinuous nonvanishing measures on $\mathbb{N}$ has to be countable, and the ideal $\mathrm{Exh}(\sup_n \mu_n)$ is dense if and only if $\limsup_i \sup_n \mu_n(\{i\}) = 0$. □

THEOREM 7.3. *Let $\mathbb{D}$ be the class of all ideals as in Lemma 7.2.*

(a) *There are six ideals in $\mathbb{D}$ with pairwise nonisomorphic quotients.*
(b) *Assume CH. Then every quotient over an ideal in $\mathbb{D}$ is isomorphic to one of the six quotients from* (a).

Proof. (a) Consider the following six ideals (for definitions see §2.2 and the paragraph before Theorem 7.3).

(1) $\mathcal{Z}_0$, the asymptotic density zero ideal (see §2.6).
(2) $\mathcal{Z}_\infty$, a dense density ideal that is not an EU-ideal (see §2.8).
(3) $\mathcal{I}_{1/n}$, a summable ideal (see §2.4).
(4) $\mathcal{I}_{1/n} \oplus \mathcal{Z}_0$.
(5) $\mathcal{I}_{1/n} \oplus \mathcal{Z}_\infty$.
(6) $\mathcal{I}_\infty$ (see §2.9).

In Proposition 3.6 we have proved that quotients over these ideals are pairwise nonisomorphic.

(b) Consider an ideal $\mathcal{Z}_\mu$ in class $\mathbb{D}$. If $\mu_n(I_n) < \infty$ for all n, then we can find $B \subseteq \mathbb{N}$ such that $B \cap I_n$ is finite for all n and $\mathbb{N} \setminus B \in \mathcal{Z}_\mu$. Thus we can assume that all I_n are finite, so the quotient over $\mathcal{Z}_\mu$ is isomorphic to a quotient over $\mathcal{Z}_0$ or $\mathcal{Z}_\infty$, by Corollary 5.4. We can therefore assume that $\mu_n(I_n) = \infty$ for some n.

Now assume $\mu_n(I_n) = \infty$ for finitely many n and let k be such that $n \geq k$ implies $\mu_n(I_n) < \infty$. Let $A = \bigcup_{n<k} I_n$, $B = \bigcup_{n\geq k} I_n$, and $\nu = \sum_{n\leq k} \mu_n$. Note that $\mathcal{Z}_\mu \restriction A$ is equal to the summable ideal $\mathrm{Exh}(\nu)$. Depending on whether $\limsup_n(\mu_n(I_n))$ is equal to 0 or not we conclude that $B \in \mathcal{Z}_\mu$ or $\mathcal{Z}_\mu \restriction B$ is a dense density ideal. Therefore by Corollary 5.4 and Corollary 6.4 the quotient over $\mathcal{Z}_\mu$ is isomorphic to the quotient over $\mathcal{I}_{1/n}$, $\mathcal{I}_{1/n} \oplus \mathcal{Z}_0$ or $\mathcal{I}_{1/n} \oplus \mathcal{Z}_\infty$.

The remaining case is when $\mu_n(I_n) = \infty$ for infinitely many n. By using the proof of Lemma 2.8.3, we may assume that $\mu_n(I_n) = \infty$ for all n, and the conclusion therefore follows from Theorem 5.6. □

It should be noted that in the situation when the conclusion of the Rigidity Conjecture holds (see Conjecture 10.1), each of the six classes of quotients from (a) of Theorem 7.3 contains continuum many pairwise nonisomorphic quotients. For the summable and density ideals this was proved in [6], and the result for the other classes can be easily deduced from this fact.

QUESTION 7.4. *Consider the class of all ideals of the form* $\mathrm{Exh}(\sup_n \mu_n)$, *where $\{\mu_n : n \in \mathbb{N}\}$ are lower-semicontinuous measures concentrating on* *pairwise disjoint subsets of* $\mathbb{N}$. *Are there infinitely many isomorphism classes of quotients over ideals in this class?*

8. Homogeneous quotients

A Boolean algebra $\mathcal{B}$ is *homogeneous* if it is isomorphic to each one of its factors, $\mathcal{B}_a = \{b \in \mathcal{B} : b \leq a\}$ for $a \neq 0_{\mathcal{B}}$. The quotient $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is clearly homogeneous, because Fin is Rudin–Keisler isomorphic to its restriction to any positive set. In the situation when the conclusion of the Rigidity Conjecture holds, $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is the only homogeneous quotient over a non-pathological analytic P-ideal (see [6, Proposition 3.7.4]). On the other hand, CH implies that every quotient over an EU-ideal is homogeneous ([6, Corollary 1.13.7]). The following was essentially proved in [6, Corollary 1.13.8].

PROPOSITION 8.1. *If* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *is homogeneous and not countably saturated, then it is isomorphic to its countably infinite power.*

Proof. Since $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not countably saturated, by Proposition 6.1 there are pairwise disjoint positive sets A_n $(n \in \mathbb{N})$ such that $B \in \mathcal{I}$ if and only if $B \cap A_n \in \mathcal{I}$ for all n. Thus $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \prod_{n=1}^{\infty}(\mathcal{P}(A_n)/\mathcal{I} \restriction A_n) \approx (\mathcal{P}(\mathbb{N})/\mathcal{I})^{\mathbb{N}}$. □

By Lemma 2.11.1, Lemma 2.8.2, Theorem 5.2, Theorem 5.5 and Corollary 6.2 we have the following.

COROLLARY 8.2 (CH). *The quotients over all LV-ideals, all EU-ideals and all* F_σ *ideals are homogeneous.* □

How many nonisomorphic homogeneous analytic quotients are there? The method of §3 clearly cannot distinguish more than three. Note that certain quotients are homogeneous under CH but not homogeneous when the conclusion of Rigidity Conjecture holds. For example, this is true for any dense summable ideal, any EU-ideal, or any LV-ideal (see [6, §3.7]). This may be true for all analytic P-ideals except Fin (this is [6, Conjecture 3.7.5]). All of the ordinal and the Weiss ideals have provably homogeneous quotients, but all of their quotients are isomorphic under CH, by Corollary 6.4. The following result was proved in [9].

THEOREM 8.3. *The ideals*

$$\begin{aligned}\operatorname{NWD}(\mathbb{Q}) &= \{A \subseteq \mathbb{Q} \cap [0,1] : A \textit{ is nowhere dense}\},\\ \operatorname{NULL}(\mathbb{Q}) &= \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A} \textit{ is of Lebesgue measure } 0\}.\end{aligned}$$

have homogeneous, but not isomorphic quotients. Moreover, neither of these two quotients is isomorphic to a quotient over an analytic P-ideal. □

9. Automorphism groups

In a situation when the conclusion of the Rigidity Conjecture holds, every automorphism of an analytic quotient is induced by a Rudin–Keisler automorphism of the ideal, or shortly *trivial*. This fact was exploited in [6, §3.6]. On the other hand, CH implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has the maximal number, $2^{2^{\aleph_0}}$, of nontrivial automorphisms ([21]). Therefore the statement 'all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial' is independent from the usual axioms of set theory. (It should be pointed out that Shelah's [22] consistency proof of this assumption was the first instance of the Rigidity Conjecture known to be consistent, long before the Rigidity Conjecture was formulated.) The results of [6] imply that the quotients over density ideals, LV-ideals, and all other 'nonpathological' analytic P-ideals consistently have only trivial automorphisms.

Proposition 9.1 (CH). *Every quotient over a layered ideal, a density ideal, or an LV-ideal has $2^{2^{\aleph_0}}$ automorphisms.*

Proof. A quotient over a layered ideal is saturated, and therefore isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Therefore it has $2^{2^{\aleph_0}}$ automorphisms by [21]. Also, the proofs of §5 can be easily modified to show that all dense density ideals and all LV-ideals have $2^{2^{\aleph_0}}$ automorphisms. The point is that if f is a countable strong isometry that is a partial automorphism, and a is not in $\operatorname{dom}(f)$, then f can be extended to countable strong isometries g_1 and g_2 that are partial automorphisms, and such that $g_1(a)\Delta g_2(a)$ is positive. Therefore we may construct $2^{\aleph_1} = 2^{2^{\aleph_0}}$ distinct automorphisms.

If an ideal $\mathcal{I}$ is not dense, then some factor of the algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, and therefore $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has at least as many automorphisms as $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. □

We do not know whether there is an analytic ideal such that in every model of ZFC all automorphisms of its quotient are trivial, but this seems rather unlikely. Let us prove a simple yet amusing fact about automorphism groups of quotient algebras.

Proposition 9.2. *If $\mathcal{I}$ is an arbitrary ideal on $\mathbb{N}$ such that its quotient is homogeneous and not countably saturated, then the automorphism group of its quotient is simple.*

Proof. By Proposition 8.1, $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is isomorphic to its countably infinite power. But by ([29, Corollary 5.9a]), if a homogeneous Boolean algebra satisfies this condition then its automorphism group is simple. □

Since CH implies that the automorphism group of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is simple, we have the following (first pointed out to me by David Fremlin in the case of $\mathcal{I} = \mathcal{Z}_0$).

Corollary 9.3 (CH). *If $\mathcal{I}$ is an arbitrary ideal on $\mathbb{N}$ such that its quotient algebra is homogeneous, then the automorphism group of its quotient is simple.* □

By a result of van Douwen ([2]) the automorphism group of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is simple if all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial. By a result of Koppelberg ([15]), CH implies that there is a homogeneous Boolean algebra whose automorphism groups is not simple. It is unknown whether it is consistent that every homogeneous Boolean algebra has a simple automorphism group.

10. The other side—Rigidity Conjecture

When considering simply definable quotient structures, one often restricts the attention to only those connecting maps that are definable themselves. In our situation, it is natural to consider isomorphisms with a Borel-measurable lifting. If $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ is a homomorphism, then $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a *lifting* of Φ if the diagram ($\pi_{\mathcal{I}}$ and $\pi_{\mathcal{J}}$ are the natural projections)

$$\begin{array}{ccc}
\mathcal{P}(\mathbb{N}) & \xrightarrow{F} & \mathcal{P}(\mathbb{N}) \\
\downarrow{\scriptstyle \pi_{\mathcal{I}}} & & \downarrow{\scriptstyle \pi_{\mathcal{J}}} \\
\mathcal{P}(\mathbb{N})/\mathcal{I} & \xrightarrow[\Phi]{} & \mathcal{P}(\mathbb{N})/\mathcal{J}
\end{array}$$

commutes. (We should remark that sometimes it is customary to require a lifting to be additive, while in our terminology a *lifting* is any map between the underlying structures which induces the given homomorphism of quotients.)

If an isomorphism between two analytic quotients has a Borel-measurable lifting, we say that these quotients are *Borel isomorphic*. It is curious that the existence of a lifting that is Borel-measurable (or even merely Baire-measurable or Lebesgue-measurable) is equivalent to the existence of a continuous lifting (see [28, p. 132], [27, Theorem 3], [13], [10, Proposition 1C]). The statement '$\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/J$ are Borel isomorphic' is $\boldsymbol{\Sigma}^1_2$, and therefore absolute for transitive models of set theory that contain all countable ordinals (by Shoenfield's absoluteness theorem). On the other hand, the statement '$\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are isomorphic' is Σ^2_1, and therefore not necessarily absolute. Therefore the question whether two given analytic quotients are isomorphic can be sensitive to the choice of set-theoretic axioms that one assumes. However, two extremal situations emerge in this study. One of them, when there are as few isomorphism types as possible, was studied in the previous sections of this paper.

Conjecture 10.1 (Rigidity Conjecture, [8]). *Assume Martin's Maximum.*

(a) *If $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals and Φ is an isomorphism between their quotients, then Φ has a continuous lifting.*

(b) *Moreover, Φ is induced by a Rudin–Keisler isomorphism between the ideals $\mathcal{I}$ and $\mathcal{J}$.*

The Rigidity Conjecture (or RC) says, among other things, that a model of MM is 'minimal' in the sense that every isomorphism between two analytic quotients is witnessed by a Rudin–Keisler isomorphism, and therefore exists in any transitive model of set theory containing all countable ordinals and codes for the ideals in question. The following was proved in [6] and [7] (see also §10 and [8]).

THEOREM 10.2. *The Rigidity Conjecture is true for*

(1) *all summable ideals,*
(2) *all density (and therefore all EU-) ideals,*
(3) *all LV-ideals,*
(4) *ideals* NWD($\mathbb{Q}$) *and* NULL($\mathbb{Q}$). □

Part (a), or the 'Borel part,' of the Rigidity Conjecture for the ordinal ideals and the CB-ideals was proved by Kanovei and Reeken in [14] (see also [13]). Although it is not known whether the Rigidity Conjecture is true for the ordinal ideals and the CB- ideals, it is known that if there is a weakly compact cardinal then there is a forcing extension in which all ordinal ideals and all Weiss ideals have pairwise non-isomorphic quotients (see [8]). Theorem 10.2, together with relatively straightforward computations shows that Martin's Maximum (and in fact a bit weaker assumption) implies that there are $2^{\aleph_0}$ pairwise non-isomorphic quotients in any of these classes of ideals (see [6, §1.11, §1.12] and [8, §2.1]).

On the other hand, all ideals for which part (a) of Conjecture 10.1 has been proved to date are $F_{\sigma\delta}$. The current state of knowledge on Conjecture 10.1 is presented in [8] and [7].

11. Concluding remarks

Every known proof that CH implies that two analytic quotients are isomorphic uses Lemma 4.2, and the back-and-forth property $\mathcal{F}$ always turns out to be analytic.

PROBLEM 11.1. Are the following equivalent for every pair of analytic ideals $\mathcal{I}$ and $\mathcal{J}$?

(1) There is an analytic family of partial isomorphisms between $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ that is σ-closed and has the back-and-forth property (see Definition 4.1).
(2) ZFC does not imply that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are not isomorphic.
(3) CH implies that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are isomorphic.

Note that (1) implies (3) implies (2) is easy. A positive answer to the above problem would imply that CH provides the optimal setting for constructing isomorphisms between analytic quotients. It would also imply that the relation 'the quotients over $\mathcal{I}$ and $\mathcal{J}$ are consistently isomorphic' is an analytic equivalence relation.

Since (2) of Proposition 6.1 is a $\mathbf{\Sigma}^1_2$-statement, if $\mathcal{I}$ is an analytic ideal then the statement '$\mathcal{P}(\mathbb{N})/\mathcal{I}$ is countably saturated' is absolute for transitive models of set theory containing all countable ordinals.

QUESTION 11.2. *Assume $\mathcal{I}$ is an analytic ideal whose quotient is countably saturated. Is $\mathcal{I}$ necessarily layered?*

In [23] it was proved that after adding $\aleph_2$ Cohen reals to a model of CH the quotient $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ still has $2^{2^{\aleph_0}}$ automorphisms. However, the methods of [23] cannot be used to prove that two countably saturated quotients are isomorphic in this model. For example, [4, Proposition 6.2] implies that in this model the quotient over $\mathcal{I}_{\omega^2}$ is not isomorphic to the quotient over Fin. Moreover, J. Steprāns [26] showed that after adding $\aleph_2$ Cohen reals to a model of CH the quotient over $\mathcal{I}_{1/n}$ is not isomorphic to the quotient over Fin. This raises many questions, for example the following.

QUESTION 11.3. *Assume that CH fails. Can the quotients over* Fin *and* $\mathcal{I}_{\omega^2}$ *(aka* $\operatorname{Fin}\times\operatorname{Fin}$*) still be isomorphic?*

Similarly, could the quotients over all ideals of the form $\operatorname{Fin}\times\mathcal{I}$, for $\mathcal{I}$ analytic ideal, be isomorphic to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ even when CH fails (cf. Corollary 6.4)? A more general question also seems to be open (but not an even more general one—see [3]).

QUESTION 11.4. *Assume that the Čech–Stone remainders of all locally compact, zero-dimensional, countably compact, non-compact spaces of weight at most continuum are pairwise homeomorphic. Does this imply CH?*

A problem closely related to counting the number of equivalence classes of analytic quotients is describing which quotients can be embedded into a given quotient. The Rigidity Conjecture has a natural formulation that applies to this situation and that is known to be true in many cases (see [6], [8]). If CH is assumed the situation is much simpler.

PROPOSITION 11.5 (CH). *Every analytic quotient embeds into every other analytic quotient.*

Proof. By a result of Mathias ([17]), $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ embeds into every other analytic quotient. But by a result of Olin, $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is saturated under CH (see, e.g., §6), and therefore every Boolean algebra of size $2^{\aleph_0}$, in particular every analytic quotient, embeds into it. $\square$

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