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ON THE GEOMETRY OF HIGHER DUALS OF A BANACH SPACE

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Abstract. In this paper we study the geometry of higher duals of a Banach space using techniques from the theory of M-ideals. We show that any Banach space that is an M-ideal in its bidual is an M-ideal in all duals of even order. As a consequence of this result, we show that continuous linear functionals on such spaces have unique norm preserving extensions to all duals of even order.

1. Introduction

In this paper we study the geometry of higher duals of a Banach space using techniques from M-structure theory; a standard reference for all matters related to M-structures is the monograph [1]. Our work is motivated by the classical work of Sullivan [2] who undertook the first detailed study of the geometric properties of higher duals of a Banach space.

Our notation and terminology is standard and can be found in [1] and [2]. We consider only non-reflexive Banach spaces. We always consider a Banach space as canonically embedded in its bidual. We denote by i_X the canonical injection. We recall that for any closed subspace $Y \subset X$, i_X is an extension of i_Y and the range of i_Y is identified as $Y^{\perp\perp} \subset X^{**}$. We will be using this implicitly throughout this paper. For a Banach space X and for $n \geq 4$, we denote by $X^{(n)}$ the n-th dual of X. We let X_1 denote the closed unit ball of X.

We recall that a closed subspace $Y \subset X$ is said to have the *property U* in X if every continuous linear functional on Y has a unique norm preserving extension to X. We also recall from Chapter III of [1] that a Banach space X is said to be an M-embedded space if $i_X(X)$ is an M-ideal in X^{**}, i.e., if the natural projection $\pi_X = i_{X^*} \circ (i_X)^* \colon X^{***} \to X^{***}$ satisfies the condition $\|\Lambda\| = \|\pi_X(\Lambda)\| + \|\Lambda - \pi_X(\Lambda)\|$, for all $\Lambda \in X^{***}$; see Proposition III.1.2 of [1]. We refer to Chapters III and VI of [1] for a large collection of examples of M-embedded spaces, from among function spaces and spaces of operators.

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It follows from Corollary III.2.15 of [1] that such a space has the *property* U (HB-smooth in the terminology of [2]) in its bidual. Our first theorem establishes a stability result by showing that when X is M-embedded, then X is an M-ideal under the composition of appropriate canonical embeddings, in all duals of even order. We note that for a non-reflexive M-embedded space, the bidual is not an M-embedded space. Since M-ideals enjoy several nice geometric properties (see Chapter I of [1]), our result allows us to make several interesting observations on the geometry of higher duals. As a main consequence we conclude that X has the *property* U, when considered as a subspace in the above canonical sense, in all duals of even order. We also show that if X is an L-ideal in its bidual then it is an L-ideal under the canonical embeddings in all duals of even order.

2. Main results

In the proof of our first result we need a lemma on the transitivity of M-ideals.

Lemma 1. *Let X, V, W be closed subspaces of a Banach space Z with $X \subset V \subset Z$ and $X \subset W \subset Z$. Suppose X is an M-ideal in V and W is an M-ideal in Z. Assume further that there is an onto isometry $\Phi\colon V \to W$ such that $\Phi(x) = x$ for all $x \in X$. Then X is an M-ideal in Z.*

Proof. Since Φ is an onto isometry with $\Phi(X) = X$, it is easy to see using the characterization of M-ideals in terms of the 3-ball property (see [1, p. 18]) that $X = \Phi(X)$ is an M-ideal in $\Phi(V) = W$. Now since W is an M-ideal in Z, it follows from Proposition I.1.17 of [1] that X is an M-ideal in Z.

Theorem 2. *Let X be an M-embedded space. Then X is an M-ideal under the appropriate composition of the canonical embeddings, in all duals of even order. Hence X has the property U in all duals of even order.*

Proof. The proof proceeds by induction on the order of the dual. With a view to keep the notation simple we shall indicate the complete details only for the first two steps of this inductive procedure. Accordingly we start with $n = IV$. From the definition of the canonical embeddings it follows that $(i_{X^{**}} \circ i_X)(X) \subset (i_X(X))^{\perp\perp}$. We will first show that there is an onto isometry $\Phi\colon X^{**} \to (i_X(X))^{\perp\perp}$ such that $\Phi(i_X(X)) = (i_{X^{**}} \circ i_X)(X)$. Since $i_X(X)$ is an M-ideal in X^{**} it would then follow from the above lemma that $(i_{X^{**}} \circ i_X)(X)$ is an M-ideal in $(i_X(X))^{\perp\perp}$. Now again, since X is an M-embedded space, we have $X^{(IV)} = (i_X(X))^{\perp\perp} \oplus_\infty (i_{X^*}(X^*))^\perp$ (where $\oplus_\infty$ denotes the ℓ^∞-direct sum). Therefore, since $(i_X(X))^{\perp\perp}$ is an M-summand, we conclude that $(i_{X^{**}} \circ i_X)(X)$ is an M-ideal in $X^{(IV)}$.

Let $\Psi_1\colon X^{***}/(i_X(X))^\perp \to X^*$ and $\Psi\colon (i_X(X))^{\perp\perp} \to (X^{***}/i_X(X)^\perp)^*$ be the canonical (via the Hahn-Banach theorem) isometries. For any $\tau \in X^{**}$

define $\Phi(\tau) = \tau'$ where $\Psi(\tau')([\Lambda]) = \tau(\Lambda | i_X(X))$. Here $[\Lambda] \in X^{***}/(i_X(X))^\perp$. It can be seen that $\Phi(i_X(X)) = (i_{X^{**}} \circ i_X)(X)$. It is also easy to see that Φ is an onto isometry. Therefore $(i_{X^{**}} \circ i_X)(X)$ is an M-ideal in $X^{(IV)}$.

Let $\pi' \colon X^{(V)} \to X^{(V)}$ be the contractive projection $\pi' = (i_{X^{**}} \circ i_X)^*$. Since $\ker \pi' = ((i_{X^{**}} \circ i_X(X))^\perp$, by the uniqueness of projections (Proposition I.1.2 in [2]) we conclude that π' is an L-projection. Thus its range $(i_{X^{***}} \circ i_{X^*})(X^*)$ is an L-summand in $X^{(V)}$. We also note that π' is the natural extension of the L-projection π_X on X^{***}. These crucial arguments allow us to carry forward the proof to the next step of the induction. This completes the proof of the first step of the induction.

It is now easy to see that since $X^{(VI)} = (i_{X^{**}} \circ i_X)(X)^{\perp\perp} \oplus_\infty (i_{X^{***}} \circ i_{X^*}(X^*))^\perp$, we obtain as before an isometry Φ' of X^{**} onto $(i_{X^{**}} \circ i_X(X))^{\perp\perp}$ such that $\Phi'(i_X(X)) = (i_{X^{(IV)}} \circ i_{X^{**}} \circ i_X)(X)$. Applying the Lemma once again we get that $(i_{X^{(IV)}} \circ i_{X^{**}} \circ i_X)(X)$ is an M-ideal in $X^{(VI)}$. Continuing in this way, we obtain by induction that for any n the image of X under the appropriate canonical embedding is an M-ideal in $X^{(2n)}$.

Remark 3. Let X be a Banach space. It is easy to see that a unit vector $x^* \in X^*$ has a unique norm preserving extension to $X^{(IV)}$ if and only if it is a point of weak*-weak continuity of the identity map on X_1^* and $i_{X^*}(x^*)$ is a point of weak*-weak continuity of the identity map on X_1^{***}. We also note that since $i_{X^*}(X_1^*)$ is weak* dense in X_1^{***}, any such point of continuity of the identity map on X_1^{***} comes from $i_{X^*}(X_1^*)$. Thus in an M-embedded space all unit vectors of X^* are, under the appropriate canonical embedding, points of weak*-weak continuity for the identity map on all unit balls of duals of odd order.

Remark 4. As a further application we point out that if X is an M-embedded space then X is, under the appropriate canonical embedding, a proximinal subspace (i.e., it admits a best approximation) of all duals of even order.

Remark 5. It is well-known that in a C^*-algebra M-ideals and closed two-sided ideals coincide. Also the bidual of a C^*-algebra is again a C^*-algebra. Thus for the C^*-algebra $\mathcal{K}(\ell^2)$ we obtain a geometric proof of the algebraic fact that, under the appropriate canonical embeddings, it is a two-sided ideal of all of its duals of even order.

Let X be an L-embedded space, that is, $i_X(X)$ is the range of an L-projection in X^{**}; see Chapter IV of [1] for several examples and properties of these spaces. Clearly the dual of an M-embedded space X has this property. As a part of the proof of the above theorem we have seen that X^*, under the appropriate canonical embeddings, is an L-ideal of all duals of odd order of

X. The bidual of an L-embedded space need not be an L-embedded space (see [1, p. 162]).

THEOREM 6. *Let X be an L-embedded space. Then X is an L-ideal, under the appropriate canonical embedding, in all duals of even order.*

Proof. As in the proof of the previous theorem we will give the details only for the first two steps of the inductive procedure. In this proof we will be using the transitivity of L-ideals.

From the hypothesis we have $X^{**} = i_X \oplus_1 X'$ (where $\oplus_1$ denotes the ℓ^1-direct sum), where X' is a closed subspace of X^{**}. Thus by duality we have $X^{(IV)} = (i_X)^{\perp\perp} \oplus_1 (X')^{\perp\perp}$. In view of our remark made in the Introduction about the embedding map, we get that $(i_{X^{**}} \circ i_X)(X)$ is an L-ideal in $X^{(IV)}$.

Thus we now have $X^{(IV)} = (i_{X^{**}} \circ i_X)(X) \oplus_1 X''$, where X'' is a closed subspace of $X^{(IV)}$. Now taking the double adjoint once more and proceeding as before we get that $(i_{X^{(IV)}} \circ i_{X^{**}} \circ i_X)(X)$ is an L-ideal in $X^{(VI)}$.

One can thus continue the inductive procedure to get the desired conclusion.

REMARK 7. Since an L-ideal is a Chebyshev subspace (i.e., it admits unique best approximations), it follows from the above theorem that any L-embedded space is, under the appropriate embeddings, a Chebyshev subspace of all duals of even order.

REFERENCES

[1] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math., no. 1547, Springer-Verlag, Berlin, 1993.

[2] F. Sullivan, *Geometric properties determined by the higher duals of a Banach space*, Illinois J. Math. **21** (1977), 315–331.

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