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A FIXED POINT THEOREM FOR BOUNDED DYNAMICAL SYSTEMS

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ABSTRACT. We show that a continuous map or a continuous flow on $\mathbb{R}^n$ with a certain recurrence relation must have a fixed point. Specifically, if there is a compact set W with the property that the forward orbit of every point in $\mathbb{R}^n$ intersects W, then there is a fixed point in W. Consequently, if the omega limit set of every point is nonempty and uniformly bounded, then there is a fixed point.

In this note we will prove a fixed point theorem that holds for both discrete dynamical systems (f is a continuous map) and continuous dynamical systems (φ^t is a continuous flow). We will show that if every point in $\mathbb{R}^n$ returns to a compact set, then there must be a fixed point. This investigation began in an attempt to answer a related question about smooth flows posed by Richard Schwartz.

We will prove the theorem for maps, and derive the theorem for flows as a consequence. In fact, many of the definitions and proofs follow analogously for both cases. When that is the case, we will just refer to the "dynamical system," with the recognition that the statement applies to both flows and maps. Where necessary, separate definitions and proofs will be included.

We now introduce the notion of a window; a compact set W is a *window* for a dynamical system on X if the forward orbit of every point $x \in X$ intersects W. If a dynamical system has a window, then we will say that it is *bounded*.

We will prove the following fixed point theorem.

THEOREM 1. *Every bounded dynamical system on $\mathbb{R}^n$ has a fixed point.*

The following corollaries are elementary applications of Theorem 1.

COROLLARY 2. *If W is a window for a dynamical system on $\mathbb{R}^n$, then there is a fixed point in W.*

COROLLARY 3. *If there is a compact set K such that $\emptyset \neq \omega(x) \subset K$ for all $x \in \mathbb{R}^n$, then the dynamical system has a fixed point.*

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We will need the following theorem.

THEOREM 4 (Lefschetz Fixed Point Theorem). *Let $f\colon M \to M$ be a continuous map of an n-dimensional manifold (with or without boundary), and let $f_k\colon H_k(M;\mathbb{R}) \to H_k(M;\mathbb{R})$ be the induced map on homology. If $\sum_{k=0}^{n}(-1)^k \operatorname{tr}(f_k) \neq 0$, then f has a fixed point.*

The main result in this paper concerns dynamical systems on $\mathbb{R}^n$, but some of the results hold more generally. Unless otherwise stated, our dynamical system is defined on a locally compact topological space.

We have the following result. The proof is similar to one by Conley [2, §II.5].

LEMMA 5. *Every bounded dynamical system has a forward invariant window.*

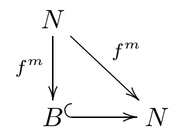
Proof. Suppose our dynamical system is a continuous map f. Because f is bounded, there is a window W. Without loss of generality, we may assume that the forward orbit of each point visits the interior of W. If not, then we may replace W by any compact neighborhood of W. For each $x \in W$ there exists an $n_x > 0$ for which $f^{n_x}(x) \in \operatorname{Int} W$. Clearly there is an open neighborhood U_x of x such that $f^{n_x}(y) \in \operatorname{Int} W$ for all $y \in U_x$. The sets $\{U_x : x \in W\}$ form an open cover of W. Since W is compact, there is a finite subcover, $\{U_{x_1}, \ldots, U_{x_m}\}$. Let $n = \max\{n_{x_k} : k = 1, \ldots, m\}$. It follows that $W_0 = \bigcup_{k=0}^{n} f^k(W)$ is a forward invariant window.

The proof for a flow φ^t is similar. $\square$

Proof of Theorem 1 for maps. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a bounded, continuous map with a forward invariant window W_0. Let B be a closed ball containing W_0 in its interior. We begin by constructing a manifold with boundary N containing B such that $f(N) \subset \operatorname{Int} N$.

Arguing as in the proof of Lemma 5, there is a positive integer n such that for each x in B the set $x \cup f(x) \cup \cdots \cup f^n(x)$ intersects W_0. Since W_0 is forward invariant, it follows that $f^n(B) \subset W_0$. The set $\bigcup_{k=0}^{n} f^k(B)$ is forward invariant; following [1, Thm. 3.3(a)] we will enlarge this set slightly so that it maps into its interior. Define the set-valued map $V_r\colon \mathbb{R}^n \to \mathbb{R}^n$ sending a point x to the closed ball of radius r centered at x. Since $f^n(B) \subset \operatorname{Int} B$, Lemma 3.2 of [1] tells us that for $\delta > 0$ sufficiently small $(V_\delta \circ f)^n(B) \subset \operatorname{Int} B$. (The lemma is actually stated for compact spaces, but the same proof remains valid in the present case.) Thus, the set $B_0 = B \cup (V_\delta \circ f)(B) \cup \cdots \cup (V_\delta \circ f)^{n-1}(B)$ has the property $f(B_0) \subset \operatorname{Int} B_0$. Finally, there exists a submanifold with boundary N sufficiently close (in the Hausdorff topology) to B_0 containing B such that $f(N) \subset \operatorname{Int} N$.

We claim that $f_k \colon H_k(N;\mathbb{R}) \to H_k(N;\mathbb{R})$ is nilpotent for $k \neq 0$. Again, there is a positive integer m such that $f^m(N) \subset W_0 \subset B$. Thus the map $f^m \colon N \to N$ factors through B, i.e., the following diagram commutes:



Therefore the map $(f^m)_* \colon H_*(N;\mathbb{R}) \to H_*(N;\mathbb{R})$ factors through $H_*(B;\mathbb{R})$. Since B is contractible, its homology consists of an $\mathbb{R}$ in dimension zero and zeroes elsewhere, so $(f^m)_*$ is the identity in dimension zero and the zero map elsewhere. That is, for $k \neq 0$, we have $0 = (f^m)_k = (f_k)^m$, i.e., f_k is nilpotent. Since the trace of a nilpotent matrix is zero, the alternating sum of the traces of f_* is 1. Thus, N must contain a fixed point. □

Proof of Theorem 1 for flows. For $s \in [0,1]$, define $f_s \colon \mathbb{R}^n \to \mathbb{R}^n$ to be the time-s map of φ^t (i.e., $f_s(x) = \varphi^s(x)$). If W is a window for the flow, then the set $W' = \bigcup_{t=0}^{1} \varphi^t(W)$ is a window for each f_s. Thus Corollary 2 for maps tells us that every f_s has a fixed point in W'.

Take a sequence of parameters s tending to zero and let x be a limit point of the corresponding sequence of fixed points of the maps f_s; we claim that x is a fixed point of φ^t. If this is not the case, then there exist a neighborhood U of x and a time t_0 such that $\varphi^{t_0}(U) \cap U = \emptyset$. But every neighborhood of x must contain periodic orbits, which is a contradiction. □

Remark 6. The same proof can apply in other spaces. What we need is that the forward invariant window W_0 have a compact acyclic neighborhood B (i.e., $\widetilde{H}_*(B;\mathbb{R}) = 0$). In the case of a nonsingular flow on S^1, for example, $W_0 = S^1$ and the proof fails.

Remark 7. There is another proof of Theorem 1 in the case of a smooth flow φ^t, the outline of which we will present now. We may add a point p at infinity to $\mathbb{R}^n$ to obtain S^n; φ^t induces a smooth flow $\widetilde{\varphi}^t$ on S^n with a fixed point at p. Because there exists a smooth Lyapunov function for $\widetilde{\varphi}^t$, we can find a submanifold with boundary L such that $\widetilde{\varphi}^t(L) \subset \operatorname{Int} L$ for all $t \geq 0$ and $\bigcap_{t\geq 0} \widetilde{\varphi}^{-t}(\mathrm{cl}(S^n \backslash L)) = \{p\}$. The set $\mathrm{cl}(S^n \backslash L)$ is contractible (the homotopy is given by running the flow backwards, so that every point goes to p). Thus $\mathrm{cl}(S^n \backslash L)$ is diffeomorphic to a closed n-disk; by the Schoenflies theorem, the same is true of L. Therefore L is acyclic, and we can proceed as above.

A closed subset $C \subset X$ is an *attracting neighborhood* for f if $f(C) \subset \operatorname{Int} C$. A set A is an *attractor* provided there is an attracting neighborhood C such that $A = \bigcap_{k\geq 0} f^k(C)$. (See [2] or [1, Ch. 3] for details. Attractors for flows are defined similarly.) An attractor A is a *global attractor* if $\emptyset \neq \omega(x) \subset A$ for all $x \in X$.

Corollary 8. *Every bounded dynamical system has a unique global attractor, A. If the dynamical system is a homeomorphism or a flow on $\mathbb{R}^n$, then the reduced Čech cohomology of A, $\tilde{\check{H}}^*(A)$, is zero.*

Proof. For a map f, the set $A = \bigcap_{n\geq 0} f^n(W_0)$ is clearly a global attractor. Suppose A' is another global attractor and N and M are attracting neighborhoods for A and A', respectively. Since N and M are both windows, there exists $n \geq 0$ such that $f^n(N) \subset M$ and $f^n(M) \subset N$. Thus, $A = A'$. We argue analogously for a flow φ^t.

If the dynamical system is invertible, then A is equal to the intersection of a nested sequence of open, contractible sets. For a map f, $A = \bigcap_{k\geq 0} f^{km}(\operatorname{Int} B)$ with m, B as above. In the flow case, for any ball B' containing W_0 in its interior, there is a $t_0 > 0$ such that $\varphi^{t_0}(B') \subset W_0 \subset \operatorname{Int} B'$. So, $A = \bigcap_{k\geq 0} \varphi^{kt_0}(\operatorname{Int} B')$. Thus $\tilde{\check{H}}^*(A) = 0$. □

Of course, if the dynamical system is a homeomorphism or a flow, then one has analogous results for f^{-1} or φ^{-t}.

Corollary 9. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (respectively φ^t is a flow). If f^{-1} (resp. φ^{-t}) is bounded, then there is a fixed point. In particular, if there is a compact set K such that $\emptyset \neq \alpha(x, f) \subset K$ (resp. $\emptyset \neq \alpha(x, \varphi^t) \subset K$) for all $x \in \mathbb{R}^n$, then there is a fixed point.*

Finally, we have the following interesting corollary.

Corollary 10. *Every dynamical system on a non-compact, locally compact space has a point whose forward orbit is not dense.*

Proof. If the forward orbit of every point is dense, then any compact set W with nonempty interior is a window. But Lemma 5 shows that the existence of a window implies the existence of a compact, forward invariant set, which is impossible. □

References

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