CONTINOUSLY SPLITTABLE DISTRIBUTIONS IN HILBERT SPACE

BY

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1. Introduction

1.1. This paper is concerned with a class of weak distributions on Hilbert space. Let H be a real Hilbert space. A weak distribution is a linear mapping on H which takes each linear function $(x, \cdot)$ on H into a random variable $m(x)$ on a probability measure space. It is supposed that σ-algebra of measurable sets is the smallest such that all the $m(x)$ are measurable. See [2], [4] and [5].

The normal distribution n is characterized, up to a variance parameter c, by the property that orthogonal vectors x and y correspond to stochastically independent random variables $n(x)$ and $n(y)$. Then each $n(x)$ is normally distributed with variance $c\| x \|^2$ and mean zero. See [5, Theorem 3].

1.2. By a spectral measure $\mathcal{E}$ we mean a completely additive Boolean algebra of commuting projections. We say that $\mathcal{E}$ splits a weak distribution m if, for each x in H and each P in $\mathcal{E}$, $m(Px)$ and $m((I - P)x)$ are stochastically independent. Every spectral measure splits the normal distribution.

One way splittable distributions arise is from suitably smooth stochastic processes with independent increments. For example let X_t, $0 \leq t \leq 1$, be such a process. Let $H = L_2(0, 1)$. Let $m(f) = \int f(t)\, dX_t$. Then m is split by the natural spectral measure on $L_2(0, 1)$.

1.3. A non-atomic spectral measure is one without any non-zero minimal projections. Our main result says if $\mathcal{E}$ is a non-atomic spectral measure which splits a weak distribution m, and if m is absolutely continuous with regard to the normal distribution n, then m is equivalent to n and is actually a translate of n by an element of H. Our proof makes use of two properties of the normal distribution both due to I. E. Segal. They are the duality transform [4, Theorem 3], and the ergodicity theorem [3, Theorem 1].

1.4. Let $x_1, \cdots, x_n$ be orthogonal vectors in H. Let

$$\varphi(t_1, \cdots, t_n)$$

be a bounded Baire function. Then $f(x) = \varphi(t_1, \cdots, t_n)$ with $t_1 = (x_1, x), \cdots t_n = (x_n, x)$ is called a tame function on H. It clearly corresponds to a random variable with regard to the normal distribution. Given a trans-

Received April 9, 1965.

[1] Most of this work was done in 1964 when the first named author was a summer visitor at Argonne National Laboratory, Argonne, Illinois.

formation T on H, which is not necessarily linear, it is natural to ask whether the map $f(x) \to f(T^{-1}x)$ sends the normal distribution into one absolutely continuous to it. A complete answer for linear T has been given by Segal in [2, Theorem 3] and a sufficient condition in the non-linear case by Gross [1, Theorem 4]. Our result lets us extend Segal's result as follows:

We will say that a not necessarily linear transformation T on H is split by a spectral measure $\mathcal{E}$ if $Tx = PTPx + (I - P)T(I - P)x$ for each P in $\mathcal{E}$ and each x in H. (When $\mathcal{E}$ is the set of projections of a maximal abelian ring, this is essentially the class of transformations on L_2 of a measure space such that $(Tf)(x) = F(x, f(x))$ for a suitable $F(x, t)$.) As an immediate consequence of our main result we have:

COROLLARY. *Let T be a not necessarily linear transformation on a real Hilbert space H which is split by a non-atomic spectral measure $\mathcal{E}$ on H. Suppose that T maps the normal distribution into a distribution absolutely continuous to it; then T is just the translation $x \to x + a$ for some fixed a in H.*

2. The details

2.1. Let n be the normal distribution on the real Hilbert space H. Let $\Gamma[H]$ be the probability measure space on which the random variables $n(x)$ act. A random variable over $\Gamma[H]$ will be referred to as a random variable over H. Since, if x and y are orthogonal then $n(x)$ and $n(y)$ are stochastically independent; it follows that, for any projection P,

$$\Gamma[H] \cong \Gamma[PH] \times \Gamma[(I - P)H].$$

DEFINITION. Let P be a projection on H. Let f be a random variable over H. Then P splits f additively (respectively multiplicatively) if $f = f_1 + f_2$ (respectively $f = f_1 \cdot f_2$) where f_1 is a random variable over PH and f_2 is a random variable over $(I - P)H$. We shall say that a spectral measure $\mathcal{E}$ splits f if each P in $\mathcal{E}$ splits f. If f splits with respect to a non-atomic spectral measure, we shall say that f splits continuously.

2.2. A random variable over H of the form $a + n(x)$ with x in H will be called an affine functional.

PROPOSITION 1. *Let f be a square integrable random variable relative to the normal distribution on a real Hilbert space H. If f is split additively by a non-atomic spectral measure on H then f is an affine functional.*

The proof depends on the following:

LEMMA 1. *Let $\mathcal{E}$ be a non-atomic spectral measure on a real Hilbert space H. Let K be a second real Hilbert space and let t be in $H \otimes K$. If for each P in $\mathcal{E}$ there are orthogonal projections Q and R on K so that*

$$[P \otimes Q + (I - P) \otimes R]t = t,$$

then $t = 0$.

Proof. Let T be the Hilbert-Schmidt transformation from H to K corresponding to t. That is $(Tx, y) = (t, x \otimes y)$ for all x in H and y in K. Then

$$QTP + RT(I - P) = T \quad \text{and} \quad PT^*Q + (I - P)T^*R = T^*.$$

A computation shows that T^*T commutes with each P in $\mathcal{E}$. But T^*T is of trace-class. Since it commutes with a non-atomic spectral measure it must be zero.

Proof of Proposition 1. By the duality transform [4, Theorem 3], f may be considered as a symmetric tensor over H. Therefore $f = \sum_{r=0}^{\infty} f_r$ where f_r is a symmetric tensor of rank r. Denoting the space of symmetric tensors over H by $S[H]$, we have for any projection P,

$$S[H] \cong S[PH] \otimes S[(I - P)H].$$

It follows readily that for P in $\mathcal{E}$ and $r \geq 1$, P splits f_r in the sense that

$$[P \otimes \cdots \otimes P + (I - P) \otimes \cdots \otimes (I - P)]f_r = f_r .$$

Hence by Lemma 1, $f_r = 0$ for $r \geq 2$. This is equivalent to the stated result.

2.3. The map $x \to -x$ on H induces an automorphism of the measurable functions over H which preserves expectations. We denote this by $f(x) \to f(-x)$ although strictly speaking f is not a function of the variable x in H but of a variable in $\Gamma[H]$. We say that f is even if $f(x) = f(-x)$ almost everywhere.

PROPOSITION 2. *Suppose that f is a random variable over the real Hilbert space H relative to the normal distribution. Suppose further that f is even and non-negative. Suppose finally that f splits multiplicatively with regard to a non-atomic spectral measure $\mathcal{E}$ on H. Then f is a constant.*

Proof. For P in $\mathcal{E}$ suppose $f = f_1 \cdot f_2$ where f_1 is a random variable over PH and f_2 is a random variable over $(I - P)H$. Then

$$f(x) = f(-x) = f_1(-Px) \cdot f_2(-(I - P)x.$$

And so

$$f^2(x) = \{f_1(Px) \cdot f_1(-Px)\} \cdot \{f_2((I - P)x) \cdot f_2(-(I - P)x)\}.$$

It follows that $f^2(x) = f^2(Ux)$ where U is the orthogonal operator $-P + I - P$. Since $\mathcal{E}$ is non-atomic the set of all U cannot leave invariant any subspace having finite positive dimension. Segal's ergodicity theorem, (Theorem 1 of [3]) says that any square integrable random variable invariant under such a set must be constant. The requirement of square integrability is not essential since g is invariant if and only if all $\varphi(g)$ are where φ ranges over the bounded continuous functions. We conclude that f^2 is constant and hence f is also.

2.4. COROLLARY 1. *Let f be a random variable over a real Hilbert space H relative to the normal distribution. Suppose that f splits additively with regard to a non-atomic spectral measure $\mathcal{E}$. We have the following:*

(a) *If f is even then f is constant.*

(b) *The random variable f is a constant plus an odd-random variable.*

(c) *If the non-negative part of f is square integrable, then f is an affine functional.*

Proof. Part (a) follows on applying Proposition 2 to exp (f). Part (b) follows immediately from (a). To see part (c) we have $f = \lambda + g$ with λ constant and g is an odd random variable. Denoting the non-negative part of h by h^+ we have $g(x) = g(x)^+ - g(-x)^+$. Now g^+ is square integrable. It follows that f itself is square integrable. The result follows from Proposition 1.

2.5. We refer to a measurable subset of $\Gamma[H]$ as an event. An event A splits if the characteristic function $\chi(A)$ splits multiplicatively.

PROPOSITION 3. *Let A be an event over a real Hilbert space H relative to the normal distribution. Suppose A splits relative to a non-atomic spectral measure $\mathcal{E}$. Then A has probability* 0 *or* 1.

Proof. Suppose prob $(A) < 1$. Let $-A$ denote the event with characteristic function $\chi(A)(-x)$. Then $A \cap -A$ is the event with characteristic function $\chi(A)\cdot\chi(-A)$. It has probability less than 1. Since $\chi(A)\cdot\chi(-A)$ is even and splits relative to $\mathcal{E}$, we have prob $(A \cap - A) = 0$ by Proposition 2. It follows that prob (A) + prob $(-A) \leq 1$. But prob $(-A)$ = prob (A). Therefore prob $(A) \leq \frac{1}{2}$. We have shown that if A is a continuously splittable event then prob $(A) = 1$ or prob $(A) \leq \frac{1}{2}$.

For P in $\mathcal{E}$ let $A(P)$ denote the event over PH determined by A. If $P_1, P_2 \cdots$ are orthogonal projections in $\mathcal{E}$ and $P = \sum P_i$, it is easy to see that

$$\Gamma[PH] \cong \prod \Gamma[P_i H].$$

Hence, if for each i, prob $(A(P_i)) = 1$, then prob $(A(P)) = 1$. It follows that we can pick a maximal P in $\mathcal{E}$ such that prob $(A(P)) = 1$. By maximality prob $(A(P')) \leq \frac{1}{2}$ for any P' orthogonal to P.

Suppose $P \neq I$. Since $\mathcal{E}$ is non-atomic, given k, we can write

$$I - P = Q_1 + \cdots + Q_k$$

with the Q_i in $\mathcal{E}$. Then

$$\text{prob } (A(I - P)) = \text{prob } (A(Q_1)) \cdots \text{prob } (A(Q_k)) \leq (\tfrac{1}{2})^k.$$

Therefore prob $(A(I - P)) = 0$ and

$$\text{prob } (A) = \text{prob } (A(P))\cdot\text{prob } (A(I - P)) = 0.$$

2.6. PROPOSITION 4. *Let H be a real Hilbert space. Let f be a non-negative random variable relative to the normal distribution on H. Suppose f splits multiplicatively relative to a non-atomic spectral measure $\mathcal{E}$ on H. Suppose further that the non-negative part of* $\log(f)$ *is square integrable. Then f is a constant times the exponential of a continuous linear functional on H.*

Proof. The event $A = \{x \mid f(x) > 0\}$ splits continuously. Hence by Proposition 3 either $f = 0$ or f is positive. In the latter case part (c) of Corollary 1 applies to $\log(f)$.

2.7. THEOREM. *Let m be a weak distribution over a real Hilbert space H which splits relative to a non-atomic spectral measure and is absolutely continuous with regard to the normal distribution. Then m is the translate of the normal distribution by a vector in H.*

Proof. Let f be the Radon-Nikodym derivative of m relative to n. Then f splits multiplicatively. By Proposition 3, f is positive; consequently the distributions are equivalent. Now, denoting the non-negative part of $\log(g)$ by $[\log(g)]^+$, $g \to [\log(g)]^+$ maps the non-negative functions in L_1 to functions in L_2. Consequently $[\log(f)]^+$ is square integrable and the result follows from Proposition 4.

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