THE COHOMOLOGY OF STABLE TWO STAGE POSTNIKOV SYSTEMS[1]

BY
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In this note we will study the cohomology algebra of a certain class of two stage Postnikov system. This question was considered originally in [4] [5] [6] [7]. We begin with a few definitions.

DEFINITION. By a generalized Eilenberg-MacLane space (GEM) we shall mean a Cartesian product of $K(\pi, n)$ spaces where π is a finitely generated abelian group and $n \geq 1$.

DEFINITION. A two stage Postinikov system $\mathcal{E}$ is a diagram

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & E_F \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow{\varphi} & B_F \end{array}$$

where

(i) F and B are GEM's.

(ii) $F \to E_F \xrightarrow{p} B_F$ is the path space fibration over B_F. B_F is of course a simply connected GEM.

(iii) $F \to E \to B$ is the fibre space induced from $F \to E_F \to B_F$ by the map $\varphi : B \to B_F$.

Now it is well known that B and B_F have H-space structures, unique up to homotopy, derived from the product in π. (In fact well chosen models are actually topological abelian groups.)

DEFINITION. $\mathcal{E}$ is called stable if B and B_F have H-space structures, multiplicatively homotopy equivalent to the standard ones, in which $\varphi : B \to B_F$ is a map of H-spaces.

Associated with a two stage Postnikov system we have an Eilenberg-Moore spectral sequence (see [12]) $\{E_r, d_r\}$ such that

$$E_r \Rightarrow H^*(E; k), \qquad E_2 = \mathrm{Tor}_{H^*(B_F; k)}(H^*(B; k), k)$$

where k is a field.

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This spectral sequence is part of an extensive research conducted by Eilenberg and Moore. Their work is just now beginning to appear in Commentarii Mathematici Helvetici and would of course serve as an encyclopedic reference for the properties of this spectral sequence used in our work.

Our study of stable two stage Postnikov systems centers around the result established in §2 and §4.

THEOREM. *If $\mathcal{E}$ is a stable two stage Postnikov system, $k = Z_p$ p a prime, and B contains no factors of the form $K(Z, 1)$, $K(Z_{2^r}, 1)$, $r > 1$, when $p = 2$, then $E_p = E_\infty$.*

Our methods actually allow us to compute E_p. Further we are able to determine the structure of $H^*(E; Z_p)$ as an algebra over Z_p.

1. Hopf algebras

We will begin by developing a few simple techniques for computing $\mathrm{Tor}_\Gamma(A, k)$ when A and Γ are Hopf algebras over a field k. These will prove useful in the sequel. We refer to [8] for the basic definitions.

All algebras will be assumed graded, connected, commutative and locally finite. $\otimes$ always means $\otimes_k$.

THEOREM 1.1. *Suppose that Γ is an algebra, $\Lambda \subset \Gamma$ a sub-algebra with Γ a projective Λ-module. Let $\Omega = \Gamma /\!/ \Lambda$ and suppose given modules $(A_{\Omega,\Gamma}\, C)$; then there exists a spectral sequence $\{E_r, d_r\}$ such that*

(i) $E_r \Rightarrow \mathrm{Tor}_\Gamma(A, C)$

(ii) $E_2^{p,q} = \mathrm{Tor}_\Omega^p(A, \mathrm{Tor}_\Lambda^q(k, C))$.

Proof. See [3, p. 349]. □

We wish to employ this spectral sequence when Γ is a Hopf algebra and Λ is a sub-Hopf algebra of Γ. To do this we shall need

THEOREM 1.2 (Milnor-Moore). *If Γ is a Hopf algebra, $\Lambda \subset \Gamma$ a sub-Hopf algebra and $\Omega = \Gamma /\!/ \Lambda$ then $\Gamma \cong \Lambda \otimes \Omega$ as a left Λ-module and a right Ω-co-module.*

Proof. See [8, Prop. 4.4]. □

COROLLARY 1.3. *If Γ is a Hopf algebra and $\Lambda \subset \Gamma$ is a sub-Hopf algebra then Γ is a free Λ-module.* □

DEFINITION. An ideal $I \subset \Gamma$ is called a Hopf ideal if

$$\psi(I) \subset \Gamma \otimes I + I \otimes \Gamma$$

where $\psi : \Gamma \to \Gamma \otimes \Gamma$ is the co-product in Γ.

If I is a Hopf ideal then Γ/I is a Hopf algebra with the induced co-product.

PROPOSITION 1.4. *If Γ is a co-commutative Hopf algebra and $I \subset \Gamma$ is a Hopf* *ideal then there exists a unique sub-Hopf algebra* $\Lambda \subset \Gamma$ *so that*

$$I = \bar{\Lambda}\cdot\Gamma = (\bar{\Lambda})$$

Proof. Let $\Omega = \Gamma/I$. Then there is a natural epimorphism of Hopf algebras $\nu : \Gamma \to \Omega$ with kernel I. Passing to duals we obtain a monomorphism $\nu^* : \Omega^* \to \Gamma^*$.

Set $\Lambda^* = \Gamma^* /\!\!/ \Omega^*$

Passing to duals again and identifying Γ and Ω with their double duals we obtain a sub-Hopf algebra $\Lambda^{**} \subset \Gamma$. Set $\Lambda = \Lambda^{**}$.

Now by Theorem 1.2, $\Gamma^* \cong \Lambda^* \otimes \Omega^*$ as a Λ^*-comodule and an Ω^*-module. The map $\nu^* : \Omega^* \to \Gamma^*$ is given by $y^* \to 1 \otimes y^*$. Passing to duals we obtain $\Gamma \cong \Lambda \otimes \Omega$ as a Λ-module and $\nu : \Gamma \to \Omega$ is given by

$$\begin{aligned} x \otimes y &\to xy \quad \text{if} \quad \deg x = 0 \\ &\to 0 \quad \text{if} \quad \deg x > 0; \end{aligned}$$

thus $\ker \nu = \{\sum x_i \cdot y_i \in \Gamma \mid x_i \in \Lambda \text{ and } \deg x_i > 0\}$, i.e. $\ker \nu = \bar{\Lambda}\cdot\Gamma$.

Uniqueness follows from $\Lambda^* = \Gamma^* /\!\!/ \Omega^*$. □

Notation. Let Γ and A be co-commutative Hopf algebras, $\varphi : \Gamma \to A$ a map of Hopf algebras. Then $\ker \varphi \subset \Gamma$ is a Hopf ideal and we can apply Proposition 1.4 to obtain a sub-Hopf algebra generating $\ker \varphi$. We will denote this sub-Hopf algebra by sub-ker φ.

PROPOSITION 1.5. *Suppose that* Γ, A *are co-commutative Hopf algebras and* $\varphi : \Gamma \to A$ *is a map of Hopf algebras. Let* $\Lambda =$ sub-ker φ*; then as Hopf algebras*

$$\operatorname{Tor}_\Gamma(A, k) \cong A /\!\!/ \varphi \otimes \operatorname{Tor}_\Lambda(k, k).$$

Proof. Since Λ is a sub-Hopf algebra of Γ it follows by Corollary 1.3 that Γ is a free Λ-module. Thus by Theorem 1.1 we have a spectral sequence $\{E_r, d_r\}$ such that

(i) $E_r \Rightarrow \operatorname{Tor}_\Gamma(A, k)$
(ii) $E_2 = \operatorname{Tor}_\Omega(A, \operatorname{Tor}_\Lambda(k, k))$

where $\Omega = \Gamma /\!\!/ \Lambda = \operatorname{im} \varphi \subset A$.

Now $\operatorname{im} \varphi \subset A$ is a sub-Hopf algebra and therefore by Corollary 1.3, A is a free $\operatorname{im} \varphi$-module. Thus

$$E_2^{0,*} \cong A \otimes_\Omega \operatorname{Tor}_\Lambda(k, k), \qquad E_2^{p,*} = 0 \qquad p \neq 0,$$

and so the edge homomorphism gives an isomorphism

$$A \otimes_\Omega \operatorname{Tor}_\Lambda(k, k) \cong \operatorname{Tor}_\Gamma(A, k).$$

Finally observe that $\operatorname{Tor}_\Lambda(k, k)$ is a trivial Ω-module and hence

$$A \otimes_\Omega \operatorname{Tor}_\Lambda(k, k) \cong A /\!\!/ \Omega \otimes \operatorname{Tor}_\Lambda(k, k) \cong A /\!\!/ \varphi \otimes \operatorname{Tor}_\Lambda(k, k). \qquad \square$$

2. Two stage Postnikov systems. Z_2-coefficients

For convenience we recall a few things from the introduction. Throughout this section $\mathcal{E}$ will be a fixed *stable* two stage Postnikov system

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \\ E \arrow[r] \arrow[d, "p"'] & E_F \arrow[d] \\ B \arrow[r, "\varphi"] & B_F \end{tikzcd}$$

We will let $\{E_r, d_r\}$ denote its Eilenberg-Moore spectral sequence and assume that the ground field k is Z_2. All cohomology will be taken with Z_2 as coefficients and we leave it out of our notation. Finally we assume B contains no factors of the form $K(Z, 1)$ or $K(Z_{2^r}, 1)$, $r > 1$.

PROPOSITION 2.1. $E_2 \cong H^*(B) /\!/ \operatorname{im} \varphi^* \otimes \operatorname{Tor}_{\text{sub-ker } \varphi^*}(Z_2, Z_2)$.

Proof. Since $\mathcal{E}$ is stable $\varphi^* : H^*(B_F) \to H^*(B)$ is a map of Hopf algebras. Further by the results of Serre, $H^*(B_F)$ and $H^*(B)$ are co-commutative. Now apply the results of §1. $\square$

PROPOSITION 2.2. $E_2 = E_\infty$.

Proof. By the results of Serre, $H^*(B_F) = P[V]$, i.e. a polynomial algebra on a certain vector space V. Now sub-ker $\varphi^* \subset P[V]$. It therefore follows by Borel's structure theorem for Hopf algebras over Z_2 [8, Theorem 7.11] that sub-ker φ^* is also a polynomial algebra, say sub-ker $\varphi^* = P[x_1, \cdots x_n, \cdots]$.

Therefore using a Koszul complex [3, p. 151] or [12, §§2.1 and 2.2] we see that

$$\operatorname{Tor}_{\text{sub-ker } \varphi^*}(Z_2, Z_2) \cong E[u_1, \cdots u_n, \cdots] \qquad \deg u_i = (-1, \deg x_i).$$

Thus $E_2 \cong H^*(B) /\!/ \operatorname{im} \varphi^* \otimes E[u_1, \cdots u_n, \cdots]$ as an algebra. Hence as an algebra E_2 is generated by $E_2^{0,*}$ and $E_2^{-1,*}$. But recall that

$$d_r : E_2^{-p,*} \to E_2^{-p+r,*} = 0$$

if $p = 0, 1$ and $r \geq 2$. Therefore $d_r = 0, r \geq 2$, and $E_2 = E_\infty$. $\square$

COROLLARY 2.3. Ker $\{p^* : H^*(B) \to H^*(E)\} = (\overline{\operatorname{im} \varphi^*})$.

Proof. Recall that we have a commutative diagram

$$\begin{tikzcd} H^*(B) \arrow[r, "\theta"] \arrow[rr, bend right, "p^*"'] & E_2^{0,*} \subset & H^*(E). \end{tikzcd}$$

Since clearly $\theta : H^*(B) \to H^*(B) /\!\!/ \operatorname{im} \varphi^* \otimes E[u_1, \cdots]$ has kernel $(\overline{\operatorname{im} \varphi^*})$ the result follows. □

The above result for $p = 2$ is deceptively easy. We turn next to a consideration of the situation when p is an odd prime. We will return to the case $p = 2$ in §5 to study the structure of $H^*(E)$ as an algebra and as an algebra over $\mathfrak{A}(2)$.

3. Z_p-coefficients, p an odd prime. Preliminaries

One of the most basic of all stable two stage Postnikov systems is the diagram

$$\begin{array}{ccc} K(Z_p, n-1) & = & K(Z_p, n-1) \\ \| & & \downarrow \\ K(Z_p, n-1) & \to & L(Z_p, n) \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(Z_p, n) \end{array}$$

where $L(Z_p, n)$ is the path space on $K(Z_p, n)$. More directly we could consider this system as simply the fibration

$$K(Z_p, n-1) \to L(Z_p, n) \to K(Z_p, n).$$

Throughout this section p is an odd prime. The necessary modifications to accommodate the case $p = 2$ are left to the reader. We will denote the mod p Eilenberg-Moore spectral sequence of the above system by $\{\bar{E}_r, \bar{d}_r\}$. In this section we will determine completely the behavior of the spectral sequence $\{\bar{E}_r, \bar{d}_r\}$. In the next section we will see how this can be used to study the case of an arbitrary stable two stage Postnikov system.

We begin by recalling the results of Cartan [2] on $H^*(Z_p, n, Z_p)$. We denote by $\mathfrak{A}(p)$ the Steenrod algebra mod p [11]. By a monomial in $\mathfrak{A}(p)$ we mean an element (see [11])

$$\beta^{\varepsilon_0} P^{s_1} \beta^{\varepsilon_1} \cdots P^{s_n} \beta^{\varepsilon_n}$$

where $\varepsilon_i = 0, 1$, and s_i are positive integers. We will write

$$\beta^{\varepsilon_0} P^{s_1} \beta^{\varepsilon_1} \cdots P^{s_n} \beta^{\varepsilon_n} = P^I, \qquad I = (\varepsilon_0, s_1, \varepsilon_1, \cdots, s_n, \varepsilon_n).$$

DEFINITION. A monomial P^I is called admissible if

$$s_i \geq p s_{i+1} + \varepsilon_i .$$

DEFINITION. If P^I is admissible we define the excess of P^I, denoted by $e(P^I)$ by

$$e(P^I) = \textstyle\sum_{i=1}^n 2(s_i - p s_{i+1} - \varepsilon_i) + \sum_{i=0}^n \varepsilon_i .$$

If we write $P^I = \beta^{\varepsilon_0} P^{s_1} P^J$ and P^I is admissible then it is easy to see that

(i) $e(P^I) = \varepsilon_0 + 2s_1 - \deg P^J$.

We also have

(ii) $e(P^I) + \deg P^I = 2ps_1 + \varepsilon_0$.

THEOREM 3.1 (Cartan). *$H^*(Z_p, n, Z_p) \cong S[\{\beta^\varepsilon P^I \iota_n\}]$ where $\varepsilon = 0, 1$ and P^I is admissible with $e(P^I) < n$.* □

Here $S[x_1, \cdots]$ denotes the free commutative algebra on generators $x_1, \cdots$. The above isomorphism is an isomorphism of Hopf algebras over $\mathcal{A}(p)$.

Let $\sigma^* : QH^*(Z_p, n, Z_p) \to PH^*(Z_p, n - 1, Z_p)$ (see [1], [2]) denote the cohomology suspension of the fibration

$$K(Z_p, n - 1) \to L(Z_p, n) \to K(Z_p, n).$$

COROLLARY 3.2. *ker σ^* is the vector space spanned by $\beta^\varepsilon P^I \iota_n$ where $e(I) = n - 1$ in $QH^*(Z_p, n, Z_p)$.*

Proof. Merely observe that

$$\sigma^*(\beta^\varepsilon P^J \iota_n) = \beta^\varepsilon P^J \sigma^*(\iota_n) = \beta^\varepsilon P^J \iota_{n-1}$$

and apply Cartan's Theorem. □

LEMMA 3.3. *$P^s P^J$ is admissible and $e(P^s P^J) = n - 1$ iff P^J is admissible, $e(P^J) \leq n - 1$ and $2s = \deg P^J + n - 1$.*

Proof. This follows by manipulating with the formula

$$e(P^s P^J) = 2s - \deg P^J. \ \square$$

LEMMA 3.4. *The correspondence*

$$P^J \iota_n \to \beta P^s P^J \iota_n$$

where J is admissible and

(i) *$\deg (P^J \iota_n)$ is odd*

(ii) *$2s = \deg P^J + n - 1$*

defines a vector space isomorphism between the indecomposable elements of odd degree in $H^*(Z_p, n, Z_p)$ and ker σ^*.

Proof. Follows from Theorem 3.1, Corollary 3.2 and Lemma 3.3. □

DEFINITION. If V is a graded vector space then sV is the graded vector space defined by $(sV)^i = V^{i+1}$. If $x_1, \cdots x_{n_1}, \cdots$ is a vector space base for V then $sx_1, \cdots$ is the corresponding basis for sV.

DEFINITION. If V is a graded vector space then we define V^+ and V^- by

$$\begin{aligned} (V^+)^i &= V^i \quad \text{if } i \text{ is even} \\ &= 0 \quad \text{if } i \text{ is odd} \\ (V^-)^i &= V^i \quad \text{if } i \text{ is odd} \\ &= 0 \quad \text{if } i \text{ is even.} \end{aligned}$$

(If $p = 2$ set $V^+ = V$, $V^- = (0)$.)

DEFINITION. If V is a graded vector space then

$$S^*(V) = \Gamma[V^+] \otimes E[V^-]$$
$$T(V) = P[V^+]/(V^+)^p \otimes E[V^-].$$

DEFINITION. If V is a graded vector space, we define the Poincaré series of V to be the formal power series

$$P(V, t) = \sum_{n=0}^{\infty} (\dim V^n) t^n.$$

DEFINITION. If W is a bigraded vector space, we define the Poincaré series of W to be the formal power series

$$P(W, t) = \sum_{n=0}^{\infty} c_n t^n, \qquad c_n = \sum_{i+j=n} \dim W^{i,j}.$$

If V is a graded vector space we recall that

$$\mathrm{Tor}_{S[V]}(Z_p, Z_p) \cong S^*[sV]$$

as a Hopf algebra.

If W is another vector space $f : V \to W$ a linear map then there is an induced map

$$f : S[V] \to S[W]$$

and hence an induced map

$$f^* : \mathrm{Tor}_{S[V]}(Z_p, Z_p) \to \mathrm{Tor}_{S[W]}(Z_p, Z_p)$$

which is given by

$$f^* = sf : S^*[sV] \to S^*[sW]$$

where $(sf)(sv) = sf(v)$ and requiring sf to be a map of algebras with divided powers, i.e. $sf(\gamma_n(sv) = \gamma_n(sf(v))$ for all $v \in V^+$.

LEMMA 3.5. *$\bar{E}_2 \cong S^*[\{s\beta^\varepsilon P^I \iota_n\}]$, $\varepsilon = 0, 1$, $e(P^I) < n$, as a Hopf algebra.*

Proof. Apply Cartan's Theorem and the above remarks. □

If $x \in \bar{E}_2^{-1,*}$ then $d_r(x) \in \bar{E}_2^{-1+r,*} = 0$, $r \geq 2$ and hence x determines an element $x_\infty \in \bar{E}_\infty^{-1,*}$. Since $\bar{E}_\infty^{0,q} = 0$, $q > 0$, $\bar{E}_\infty^{0,0} = Z_p$, we see that if x has positive complementary degree x determines a unique element, possibly zero,

$$]x[\in F^{-1}H^*(Z_p, n-1, Z_p) \subset H^*(Z_p, n-1, Z_p).$$

In [12] we showed that

$$\sigma^*(\beta^\varepsilon P^I \iota_n) =]s\beta^\varepsilon P^I \iota_n[$$

where σ^* is the cohomology suspension. (See also [1] for a very similar situation.) Thus we see

LEMMA 3.6. *The elements in $\bar{E}_2^{-1,*}$ that are boundaries under some d_r are all linear combinations of elements of the form $s\beta P^I \iota_n$ where $e(I) = n - 1$.*

Proof. Apply Corollary 3.2 and the observation above concerning the cohomology suspension. □

Before continuing we make the important observation that $\{\bar{E}_r, \bar{d}_r\}$ is a spectral sequence of Hopf algebras, i.e. $\bar{E}_r$ is a differential Hopf algebra with differential $\bar{d}_r$ and $\bar{E}_{r+1} \cong H(\bar{E}_r, \bar{d}_r)$ as a Hopf algebra. This can be seen either by diagram chasing or direct from the definition of $\{\bar{E}_r, \bar{d}_r\}$ in [12].

LEMMA 3.7 $\bar{E}_\infty \cong T[\{s\beta^\varepsilon P^I \iota_n\}]$ *where* $\varepsilon = 0, 1$; $e(P^I) < n - \varepsilon$.

Proof. We know by Lemma 3.6 that

$$\bar{E}_\infty^{-1,*} = \{\text{vector space spanned by all elements of the form } s\beta^\varepsilon P^I \iota_n \text{ where } \varepsilon = 0, 1,\ e(I) < n - \varepsilon\}$$

and hence by Borel's structure theorem for Hopf algebras over Z_p and Lemma 3.5 we see that

$$\bar{E}_\infty \supset T[\{s\beta^\varepsilon P^I \iota_n\}], \qquad \varepsilon = 0, 1; e(P^I) < n - \varepsilon.$$

The lemma will be complete when we show that

$$P(T[\{s\beta^\varepsilon P^I \iota_n\}], t) = P(H^*(Z_p, n-1, Z_p), t).$$

To see this we recall that by Cartan's Theorem

$$H^*(Z_p, n-1, Z_p) \cong S[\{\beta^\varepsilon P^J \iota_{n-1}\}], \qquad \varepsilon = 0, 1; e(P^J) < n - 1$$

as a Hopf algebra. If we now filter $H^*(Z_p, n-1, Z_p)$ by the primitive filtration [8] and denote the associated graded by G we see

$$G = S^*[\{\beta^\varepsilon P^J \iota_{n-1}\}].$$

We now recall that for any $x \in H^*(Z_p, n-1, Z_p)^+$ then

$$x^p = P^r x, \qquad 2r = \deg x.$$

and so if we define a correspondence

$$\gamma_p(\beta^\varepsilon P^J \iota_{n-1}) \xrightarrow{\theta} P^t \beta^\varepsilon P^J \iota_n \qquad 2t = \varepsilon + \deg J + n - 1$$

and more generally

$$\gamma_{p^q}(\beta^\varepsilon P^J \iota_{n-1}) \xrightarrow{\theta} P^t \theta(\gamma_{p^{q-1}}(\beta^\varepsilon P^J \iota_{n-1}) \qquad 2t = \deg \theta(\gamma_{p^{q-1}}(\beta^\varepsilon P^J \iota_{n-1})$$

we can define a map

$$\begin{array}{ll} S^*(\{\beta^\varepsilon P^J \iota_{n-1}\}) \xrightarrow{\omega} & T[\{s\beta^\varepsilon P^I \iota_n\}] \\ \varepsilon = 0, 1 & \varepsilon = 0, 1 \\ e(P^J) < n - 1 & e(P^I) < n - \varepsilon \end{array}$$

by

$$\beta^\varepsilon P^J \iota_{n-1} \xrightarrow{\omega} s\beta^\varepsilon P^J \iota_n$$

$$\gamma_{p^q}(\beta^\varepsilon P^J \iota_{n-1}) \xrightarrow{\omega} s\theta\gamma_{p^q}(\beta^\varepsilon P^J \iota_{n-1})$$

and requiring that ω be a map of algebras, where of course we have to apply

Lemma 3.3 to show that $P^t\beta^\varepsilon P^J$ is admissible of excess $<n$. It is straightforward to verify that ω is an isomorphism. Hence

$$P(T[\{s\beta^\varepsilon P^I\iota_n\}], t) = P(G, t) = P(H^*(Z_p, n-1; Z_p), t).$$
$$\varepsilon = 0, 1, e(P^I) < n - \varepsilon$$

and hence the lemma is established. □

LEMMA 3.8. *If P^I is admissible and does not begin in a β and*
(i) $\deg P^I = 2t(p-1)$
(ii) $e(P^I) = 2t$
then $P^I = P^t$.

Proof. Write $P^I = P^{t_1}P^J$; then we have

$$e(P^I) + \deg(P^I) = 2pt_1$$

and

$$e(P^I) + \deg(P^I) = 2t + 2t(p-1) = 2pt.$$

Therefore $t_1 = t$ and from

$$2t = 2t_1 - \deg P^J = e(P^I) = 2t$$

we see $\deg P^J = 0$ and hence $P^J = 1$. □

LEMMA 3.9. *Suppose that $n = 2t + 1$ and $\{\bar{E}_r, \bar{d}_r\}$ is the Eilenberg-Moore spectral sequence of*

$$K(Z_p, n-1) \to L(Z_p, n) \to K(Z_p, n)$$

(*i.e. as above*), *then*
(i) $\bar{d}_r(\gamma_p(s\iota_{2t+1})) = 0, r < p - 1$
(ii) $\bar{d}_{p-1}(\gamma_p(s\iota_{2t+1})) = \lambda s\beta P^t\iota_{2t+1}, \lambda \neq 0 \in Z_p$.

Proof. If we apply the argument of [1, Theorem 4.1] to the spectral sequence $\{\bar{E}_r, \bar{d}_r\}$ we see that $\bar{d}_r = 0, r < p-1$, and so (i) follows.

To see (ii) we note that be Lemma 3.7 $\gamma_p(s\iota_{2t+1})$ cannot survive to $\bar{E}_\infty$ and by the argument of [1, Theorem 4.1] it can never be a boundary under any $\bar{d}_r$. If $\bar{d}_{p-1}(\gamma_p s\iota_{2t+1}) = 0$ then since $\bar{d}_r(\gamma_p s\iota_{2t+1})$, $r \geq p$, would vanish for dimensional reasons we would contradict the fact that $\gamma_p s\iota_{2t+1}$ does not survive to $\bar{E}_\infty$.

Therefore $\bar{d}_{p-1}(\gamma_p(s\iota_{2t+1})) \neq 0$.

Now note

$$\deg \bar{d}_{p-1}(\gamma_p(s\iota_{2t+1})) = (-1, 2tp + 2)$$
$$= (-1, 2t(p-1) + 1 + 2t + 1).$$

Therefore applying Lemma 3.6 we see that $\bar{d}_{p-1}(\gamma_p(s\iota_{2t+1})$ is a linear combination of elements of the form $s\beta P^I\iota_{2t+1}$ where

(i) $e(P^I) = 2t$
(ii) $\deg P^I = 2t(p-1)$.

But by Lemma 3.8 the only admissible monomial satisfying this condition is βP^t. Therefore

$$\bar{d}_{p-1}(\gamma_p(s\iota_{2t+1})) = \lambda s\beta P^t\iota_{2t+1}, \qquad \lambda \neq 0 \in Z_p. \ \square$$

THEOREM 3.10. *Let $\{\bar{E}_r, d_r\}$ be the Eilenberg-Moore spectral sequence of the fibration*

$$K(Z_p, n-1) \to L(Z_p, n) \to K(Z_p, n)$$

(as it has been throughout this section); then

(i) $\bar{d}_r = 0, r < p-1$

(ii) $\bar{d}_{p-1}$ *is determined by*

$$\bar{d}_{p-1}(\gamma_p(sP^J\iota_n)) = \lambda s\beta P^tP^J\iota_n, \qquad \lambda \neq 0 \in Z_p \text{ and } 2t+1 = \deg J + n,$$

and requiring that it be a derivation of algebras

(iii) $\bar{E}_p = \bar{E}_\infty$.

Proof. We obtain (i) by applying the argument of [1, Theorem 4.1] to the present case.

To prove (ii) let

$$\deg P^J + n = 2t+1$$

and choose a map

$$g : K(Z_p, n) \to K(Z_p, 2t+1)$$

such that

$$g^*(\iota_{2t+1}) = P^J\iota_n.$$

We then have a diagram

$$\begin{array}{ccc} K(Z_p, n-1) & \xrightarrow{\Omega g} & K(Z_p, 2t) \\ \downarrow & & \downarrow \\ L(Z_p, n) & \xrightarrow{Pg} & L(Z_p, 2t+1) \\ \downarrow & & \downarrow \\ K(Z_p, n) & \xrightarrow{g} & K(Z_p, 2t+1). \end{array}$$

Denote the Eilenberg-Moore spectral sequence of the right hand fibration by $\{\tilde{E}_r, \tilde{d}_r\}$. By naturality of the Eilenberg-Moore spectral sequence we have a map of spectral sequences

$$\{g_r^*\} : \{\tilde{E}_r, \tilde{d}_r\} \to \{\bar{E}_r, \bar{d}_r\}.$$

Then

$$g^*\gamma_p(s\iota_{2t+1}) = \gamma_p(sP^J\iota_n)$$

and so applying Lemma 3.9 at the crucial point we see

$$\begin{aligned} \bar{d}_{p-1}(\gamma_p(sP^J\iota_n)) &= \bar{d}_{p-1}\, g^*\gamma_p(s\iota_{2t+1}) \\ &= g^*\tilde{d}_{p-1}\, \gamma_p(s\iota_{2t+1}) \\ &= g^*(\lambda s\beta P^t\iota_{2t+1}) \\ &= \lambda s\beta P^tP^J\iota_n \end{aligned}$$

(by remark preceding Lemma 3.5).

If we now recall that a complex of the form $\Gamma[u] \otimes E[v]$ with $\partial(u) = v$ is acyclic we see that $\bar{E}_p = T[\{s\beta^\varepsilon P^I\iota_n\}]$, $\varepsilon = 0, 1$; $e(P^I) < n - \varepsilon$ and applying Lemma 3.7 we see that $\bar{E}_p = \bar{E}_\infty$. □

It is of course very likely that the constant λ above is actually ± 1 but we see no way to prove this. Actually we have been somewhat sloppy, the constant λ depends on $P^J\iota_n$ and we have failed to indicate this dependence, but this is of no consequence.

We will close this section with some remarks on $H^*(\pi, n, Z_p)$.

PROPOSITION 3.11. *If* $x \in Q^{2t+1}H^*(\pi, n, Z_p)$ *then* $\beta P^t x \neq 0 \in QH^*(\pi, n, Z_p)$.

Proof. This follows from [2, Exposé 16, §2, Theorem 1]. □

COROLLARY 3.12. *The map* $\rho : Q^{2t+1}H^*(\pi, n, Z_p) \to Q^{2tp+2}H^*(\pi, n, Z_p)$ *by* $x \to \beta P^t x$ *is a monomorphism of vector spaces.* □

It is interesting to note that Rothenberg-Steenrod (unpublished) have recently obtained new proofs of Cartan's theorems using the "Milnor-Moore" spectral sequence derived from Milnor's construction of a classifying space for a topological group.

4. Two stage Postnikov systems. Z_p-coefficients

Throughout this section $\mathcal{E}$ will be a fixed *stable* two stage Postnikov system

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \\ E \arrow[r] \arrow[d, "p"'] & E_F \arrow[d, "p_F"] \\ B \arrow[r, "\varphi"] & B_F . \end{tikzcd}$$

All cohomology will be taken with Z_p-coefficients p an odd prime (the necessary modifications to accomodate the case $p = 2$ will be left to the reader). The mod p Eilenberg-Moore spectral sequence of $\mathcal{E}$ will be denoted by $\{E_r, d_r\}$.

PROPOSITION 4.1. $E_2 \cong H^*(B) /\!/ \varphi^* \otimes \mathrm{Tor}_{\text{sub-ker } \varphi^*}(Z_p, Z_p)$.

Proof. Apply the material of §1 to the case at hand. □

LEMMA 4.2. *Suppose given a diagram of spaces;*

$$\begin{tikzcd} & Y \arrow[rr] \arrow[d] & & Y' \arrow[d] \\ & B \arrow[rr] & & B' \\ X \arrow[ur] \arrow[rr] & & X' \arrow[ur] & \end{tikzcd}$$

then we can fill it in to a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[rr] \arrow[dd] & & Y' \arrow[dd] \\
X \times_B Y \arrow[ur] \arrow[rr] \arrow[dd] & & X' \times_{B'} Y' \arrow[ur] \arrow[dd] & \\
 & B \arrow[rr] & & B' \\
X \arrow[ur] \arrow[rr] & & X' \arrow[ur] &
\end{tikzcd} .$$

Proof. Direct from the definitions. □

Let $x_i \in Q$ sub-ker φ^* and choose a map

$$\zeta_i : B_F \to K(Z_p\,,\,\deg x_i) = K$$

representing x_i. Consider the diagram

$$\begin{tikzcd}
\Omega K \arrow[r, equal] \arrow[d] & \Omega K \arrow[d] \\
B \times_K PK \arrow[r] \arrow[d] & PK \arrow[d] \\
B \arrow[r, "\zeta_i \circ \varphi"] & K.
\end{tikzcd}$$

One readily sees that $(\zeta_i \circ \varphi)^* : \tilde{H}^*(K) \to \tilde{H}^*(B)$ is the zero map. Since $K = K(Z_p\,,\,\deg x)$ it follows that $\zeta_i \circ \varphi : B \to K$ is null homotopic. Hence $B \times_K PK$ has the same homotopy type as $B \times \Omega K$.

Let $\{\hat{E}_r\,,\,\hat{d}_r\}$ denote the Eilenberg-Moore spectral sequence of the above diagram. The Eilenberg-Moore spectral sequence of the fibration

$$\Omega K \to PK \to K$$

will be denoted by $\{\bar{E}_r\,,\,\bar{d}_r\}$. We then have

PROPOSITION 4.3. *$\hat{E}_r \cong H^*(B) \otimes \bar{E}_r$ and $\hat{d}_r = 1 \otimes \bar{d}_r$.* □

Applying Lemma 4.2 we obtain a diagram

$$\begin{tikzcd}
 & F \arrow[rr] \arrow[dd] & & \Omega K \arrow[dd] \\
F \arrow[ur, equal] \arrow[rr] \arrow[dd] & & \Omega K \arrow[ur, equal] \arrow[dd] & \\
 & E_F \arrow[rr] \arrow[dd] & & PK \arrow[dd] \\
E \arrow[ur] \arrow[rr] \arrow[dd] & & B \times \Omega K \arrow[ur] \arrow[dd] & \\
 & B_F \arrow[rr] & & K \\
B \arrow[ur] \arrow[rr] & & B \arrow[ur] &
\end{tikzcd} .$$

By the naturality of the Eilenberg-Moore spectral sequence we obtain a map of spectral sequences

$$\{g_r^*\} : \{\hat{E}_r\,;\,\hat{d}_r\} \to \{E_r\,,\,d_r\}.$$

Suppose now that we choose a basis $x_1, \cdots x_n, \cdots$ for Q sub-ker φ^{*-}.

LEMMA 4.4. *$\beta P^{t_1} x_1, \cdots \beta P^{t_n} x_n, \cdots, 2t_i + 1 = \deg x_i$, are linearly independent elements of Q* sub-ker φ^{*+}.

Proof. By applying Theorem 1.2 we see that $x_1, \cdots x_n, \cdots$ are indecomposable in $H^*(B_F)$. Hence by Proposition 3.11, $\beta P^{t_i} x_i$ is an indecomposable element of $H^*(B_F)$ and since

$$\varphi^* \beta P^{t_i} x_i = \beta P^{t_i} \varphi^*(x_i) = \beta P^{t_i} 0 = 0$$

it follows that $\beta P^{t_i} x_i \in \ker \varphi^*$. Since $\beta P^{t_i} x_i$ is also indecomposable it follows that $\beta P^{t_i} x_i \in$ sub-ker φ^*. The result now follows by Corollary 3.12. □

Thus we can choose a basis

$$\{x_1, \cdots x_n, \cdots\} \cup \{\beta P^{t_1} x_1, \cdots\} \cup \{y_1, \cdots y_n, \cdots\}$$

for Q sub-ker φ^* and thus we have

LEMMA 4.5. Sub-ker $\varphi^* = S[\{x_i\} \cup \{\beta P^{t_i} x_i\} \cup \{y_i\}]$.

Proof. Clearly it suffices to show that sub-ker φ^* is a free commutative algebra. Recall that sub-ker φ^* is a sub-Hopf algebra of $H^*(B_F)$. By Cartan's Theorem $H^*(B_F)$ is a free commutative algebra. Hence by Borel's structure theorem for Hopf algebras over Z_p, [8, Theorem 7.11] we have sub-ker φ^* is a free commutative algebra. □

COROLLARY 4.6.

$$E_2 \cong H^*(B) /\!/ \varphi^* \otimes \Gamma[sx_1, \cdots] \otimes E[s\beta P^{t_1} x_1, \cdots] \otimes E[sy_1, \cdots].$$

Proof. Since $\deg y_i$ is always even this follows from Proposition 4.1 and Lemma 4.5. □

THEOREM 4.7. *The differentials d_r satisfy*

(i) $d_r = 0, 2 \leq s < p - 1$

(ii) $d_{p-1}(\gamma_p(sx_i)) = \lambda s\beta P^{t_i} x_i,\ 2t_i + 1 = \deg x_i,\ \lambda \neq 0 \in Z_p$.

Proof. Let $d_r, r \geq 2$, be the first non-zero differential and let $z \in E_2$ be an element of minimal degree with $d_r(z) \neq 0$. Then z is indecomposable. By Corollary 4.6 the indecomposable elements of E_2 have filtration degree 0, -1 or $-p^q$. Since $r \geq 2$ we can assume that the filtration degree of z is $-p^q$. Thus without loss of generality we can assume that $z = \gamma_{p^q}(sx_i)$.

Now consider the map $\{g_r^*\} : \{\hat{E}_r, \hat{d}_r\} \to \{E_r, d_r\}$. We have

$$g^*(s_{2t_i+1}) = sx_i$$

and hence

$$g^*(\gamma_{p^q}(sx_i)) = \gamma_{p^q}(s\iota_{2t_i+1}).$$

Now applying Theorem 3.10 we see that

$$d_r(\gamma_{p^q}(sx_i)) = g^*\bar{d}_r\gamma_{p^q}(s\iota_{2t_i+1})$$
$$= 0 \quad \text{if } 2 \leq r < p - 1.$$

Since

$$d_{p-1}(\gamma_p(sx_i)) = g^*\bar{d}_{p-1}(\gamma_p s\iota_{2t_i+1})$$
$$= g^*\lambda s\beta P^{t_i}{}_{2t_i+1} = \lambda\beta P^{t_i}x_i$$

the result is established. □

THEOREM 4.8. *$E_p = H^*(B) /\!/ \varphi^* \otimes T[\{sx_1, \cdots\} \cup \{y_1, \cdots\}]$ and hence $E_p = E_\infty$.*

Proof. By Theorem 4.7 $E_2 = E_{p-1}$ and

$$E_p = H[H^*(B) /\!/ \varphi^* \otimes \Gamma[sx_1, \cdots] \otimes E[s\beta P^{t_1}x_1, \cdots] \otimes E[sy_1, \cdots]]$$

where

$$d_{p-1}(\gamma_p(sx_i)) = \lambda s\beta P^{t_i}x_i$$

and so

$$E_p = H^*(B) /\!/ \varphi^* \otimes P[sx_1, \cdots]/(sx_1^p, \cdots) \otimes E[sy_1, \cdots] \otimes H[\Gamma[\{\gamma_p sx_i\}] \otimes E[\{\beta P^{t_i}x_i\}]].$$

Now recalling that

$$\Gamma[\gamma_p sx_r] \otimes E[\beta P^{t_i}x_i]$$

is acyclic and that

$$P[sx_1, \cdots]/(sx_1^p, \cdots) \otimes E[sy_1, \cdots] = T[\{sx_1, \cdots\} \cup \{y_1, \cdots\}],$$

we see that

$$E_p \cong H^*(B) /\!/ \varphi^* \otimes T[\{sx_1, \cdots\} \cup \{y_1, \cdots\}].$$

Therefore as an algebra, E_p is generated by $E_p^{0,*}$ and $E_p^{-1,*}$. Since

$$d_r : E_p^{\eta,*} \to E_p^{r+\eta,*} = 0, \qquad r \geq p, \eta = 0, 1$$

we see that $d_r = 0$, $r \geq p$, on a set of algebra generates for E_p. Since d_r is a derivation it follows that $d_r = 0$, $r \geq p$, and hence $E_p = E_\infty$. □

If the reader has been making the necessary modifications to accomodate the case $p = 2$ then yet another, although unusually complicated proof of Proposition 2.2 is obtained.

COROLLARY 4.9. $\ker p^* \cong (\overline{\operatorname{im} \varphi^*})$.

Proof. As in Corollary 2.3. □

5. The algebra structure

As in the last few sections $\mathcal{E}$ will denote a fixed *stable* two stage Postnikov system

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, dashed, "i"'] & F \arrow[d, "i_F"] \\ E \arrow[r] \arrow[d, "p"'] & E_F \arrow[d, "p_F"] \\ B \arrow[r, "\varphi"'] & B_F . \end{tikzcd}$$

All cohomology will be taken with Z_p-coefficients, p a prime ($p = 2$ is not excluded). The mod p Eilenberg-Moore spectral sequence of $\mathcal{E}$ is denoted by $\{E_r, d_r\}$.

In the last section we established that there is a filtration $\{F^{-n}H^*(E)\}$ with associated graded object, denoted by $\mathcal{G}H^*(E)$, given by

$$\mathcal{G}H^*(E) \cong E_p$$

as Hopf algebras.

To determine the structure of $H^*(E)$ we recall come results of [11, Chapter 1]. Consider the maps

$$\bar{p}_F, \bar{\varphi} : B \times E_F \to B \times B_F \times E$$

given by

$$\bar{p}_F(x, y) = (x, p_F(y), y), \qquad \bar{\varphi}(x, y) = (x, \varphi(y), y)$$

Note that $E \subset B \times E_F$ and that $\bar{p}_F \mid E = \bar{\varphi} \mid E$. Therefore we have defined a difference homomorphism [11]

$$(\bar{p}_F - \bar{\varphi})^* : H^*(B \times B_F \times E_F) \to H^*(B \times E_F, E).$$

Consider the diagram

$$H^*(B \times B_F \times E_F) \xrightarrow{(p_F - \varphi)^*} H^*(B \times E_F, E) \xleftarrow{\delta} H^*(E).$$

PROPOSITION 5.1.

$$F^{-1}H^*(E) = \{x \in H^*(E) \mid \exists y \in H^*(B \times B_F \times E_F)$$

with $(p_F - \varphi)^*(y) = \delta(x)\}$.

Proof. See [12, Theorem 1.3.3]. □

Note that if $B = *$ we then obtain the Eilenberg-Moore spectral sequence of the fibration $F \to E_F \to B_F$ and

COROLLARY 5.2. $F^{-1}H^*(F) = \operatorname{im}\{s^* : H^*(B_F) \to H^*(F)\}$ *where* s^* *is the cohomology suspension.* □

PROPOSITION 5.3. *There exist elements* $x_1, \cdots y_1, \cdots \in Q$ sub-ker φ^* *such that*

$$\mathcal{G}H^*(E) \cong H^*(B)/\ker p^* \otimes T[\{sx_1, \cdots\} \cup \{sy_1, \cdots\}].$$

Proof. This follows from Theorem 4.8 and Corollary 4.9. □

For simplicity in the sequel we will write $R = H^*(B)/\ker p^*$.

Choose elements $u_1, \cdots, v_1, \cdots \in F^{-1}H^*(E)$ to represent $sx_1, \cdots sy_1, \cdots$ respectively.

PROPOSITION 5.4. $i^*(u_i) = \sigma^*(x_i)$, $i^*(v_i) = \sigma^*(y_i)$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H^*(B \times B_F \times E_F) \arrow[r, "(\bar{p}_F - \varphi)^*"] \arrow[d] & H^*(B \times E_F, E) \arrow[d] & H^*(E) \arrow[l, "\delta"'] \arrow[d, "i^*"] \\
H^*(B \times E_F) \arrow[r, "\bar{p}_F^*"] & H^*(E_F, F) & H^*(F) \arrow[l, "\delta"']
\end{tikzcd}$$

where the vertical maps are induced by inclusion. Now it is clear that

$$(\bar{p}_F - \bar{\varphi})^*(1 \times x_i \times 1) = \delta(u_i), \qquad (\bar{p}_F - \bar{\varphi})^*(1 \times y_i \times 1) = \delta(v_i)$$

since the bottom line determines the cohomology suspension in $F \to E_F \to B_F$ commutativity of the diagram yields the result. □

Let $M = \operatorname{im}\{s^* : Q \text{ sub-ker } \varphi^* \to H^*(F)\}$ and let $S \subset H^*(F)$ be the sub-$\mathcal{A}(p)$-algebra generated by M.

PROPOSITION 5.5. *The sequence of Hopf algebras*

$$R \xrightarrow{p^*} H^*(E) \xrightarrow{i^*} S$$

has the following properties:

(i) p^* *is one to one*

(ii) i^* *is onto hence* $S = \operatorname{im} i^*$

(iii) $\ker i^* = \bar{R} \cdot H^*(E)$.

Proof. Both (i) and (ii) are trivial so we turn to (iii). To prove (iii), filter $H^*(E)$ by $\{F^{-n}H^*(E)\}$ as in Theorem 4.8.

Filter $H^*(F)$ by the filtration determined by the Eilenberg-Moore spectral sequence of $F \to E_F \to B_F$. Then this determines a filtration on $A \subset H^*(F)$. It is not too difficult to see that $i^* : H^*(E) \to A \subset H^*(F)$ is a filtration-preserving map.

Passing to associated gradeds we have

$$\mathcal{G}A = T[\{sx_1, \cdots\} \cup \{sy_1, \cdots\}]$$

$$\mathcal{G}H^*(E) = R \otimes T[\{sx_1, \cdots\} \cup \{sy_1, \cdots\}]$$

and

$$\mathcal{G}i^* : R \otimes T \to T \quad \text{by} \quad r \otimes t \to 0, \quad \deg r > 0$$
$$\to rt, \quad \deg r = 0$$

Since $H^*(E)$ is a free R-module it follows that

$$\ker \{i^* : H^*(E) \to S\} = \{rt\} \mid r \in R \text{ and } \deg r > 0\}. \ \square$$

DEFINITION. A sequence of commutative algebras

$$\Lambda_{i-1} \xrightarrow{f_{i-1}} \Lambda_i \xrightarrow{f_i} \Lambda_{i+1}$$

is called co-exact at Λ_i if

$$\ker f_i = \overline{(\operatorname{im} f_{i-1})} = \overline{f_{i-1}(\Lambda_{i-1})}\,\Lambda_i$$

COROLLARY 5.6. *The sequence of Hopf algebras*

$$Z_p \to R \xrightarrow{p^*} H^*(E) \xrightarrow{i^*} S \to Z_p$$

is co-exact; the maps p^* and i^* being maps of $\mathfrak{A}(p)$-algebras. □

PROPOSITION 5.7. (i) *As an algebra, $S \cong S[L]$ where*

$$L = Q \operatorname{im} \{\sigma^* : Q \text{ sub-ker } \varphi^* \to H^*(F)\}.$$

(ii) *As an algebra over $\mathfrak{A}(p)$, $S \cong U(M)$ where*

$$M = \operatorname{im} \{\sigma^* : Q \text{ sub-ker } \varphi^* \to H^*(F)\}.$$

Proof. Let us begin by proving (ii). By Cartan's results we have $H^*(F) = U(X)$ where $X = \operatorname{im}\{\sigma^* : QH^*(B_F) \to H^*(F)\}$.

Since Q sub-ker $\varphi^* \subset QH^*(B_F)$, it follows that $M \subset X$. Hence it follows that $S = U(M)$.

To prove (i) note that by (ii) $S = U(M) \subset U(X)$ is a sub-Hopf algebra of $U(X)$. Since $U(X)$ is a free commutative algebra on QX by the results of Cartan, Borel's structure theorem for Hopf algebras over Z_p yields (i). (If $p = 2$ we merely substitute the results of Serre [9] for those of Cartan.) □

If we define $\lambda : Q$ sub-ker $\varphi^{*-} \to Q$ sub-ker φ^* by $\lambda(x) = P^t x$, $2t + 1 = \deg x$, it is not too difficult to see that

$$(\dim L^j) = \dim (Q \text{ sub-ker } \varphi^*)^j - \dim (\lambda Q \text{ sub-ker } \varphi^{*-})^j - \dim (\beta\lambda Q \text{ sub-ker } \varphi^{*-})^j.$$

THEOREM 5.7. *As an algebra,*

$$H^*(E) \cong R \otimes S.$$

Proof. Consider the co-exact sequence

$$Z_p \to R \xrightarrow{p^*} H^*(E) \xrightarrow{i^*} S \to Z_p.$$

By Proposition 5.6, $S \cong S[L]$. Therefore we can construct a map

$$\mu : A \to H^*(E)$$

such that $i^* \cdot \mu = 1 : S \to S$.

Therefore the result follows. □

Following MacLane we say that

$$Z_p \to R \to H^*(E) \to A \to Z_p$$

is "cleft" as a sequence of algebras.

Warning. $S = U(M)$ but M need not be a free unstable module. It is therefore not true in general that the sequence

$$Z_p \to R \xrightarrow{p^*} H^*(E) \xrightarrow{i^*} A \to Z_p$$

is a cleft $\mathfrak{A}(p)$ extension. In fact simple examples show that this extension is not always cleft over $\mathfrak{A}(p)$.

THEOREM 5.8. *The sequence*

$$Z_p \to R \xrightarrow{p^*} H^*(E) \xrightarrow{i^*} U(M) \to Z_p$$

is a co-exact sequence of $\mathfrak{A}(p)$-algebras that splits as a sequence of algebras. □

The study of the above extension when $p = 2$ "reduces" to the extension problem of [6]. To make this precise we will show that

THEOREM 5.9. *As an algebra over $\mathfrak{A}(p)$*

$$H^*(E) \cong U_R(F^{-1}).$$

Proof. We remind the reader that U_R is defined in [6, §3]. It was also shown in [12, I.4] that $F^{-1}H^*(E)$ is an $\mathfrak{A}(p)$-submodule of $H^*(E)$.

To prove this result we proceed as follows. In the notation of Proposition 5.3 choose elements $u_1, \cdots, v_1, \cdots \in F^{-1}H^*(E)$ to represent $sx_1, \cdots, sy_1, \cdots \in \mathcal{G}H^*(E)$.

Since $F^{-1} \subset H^*(E)$ is a map of $\mathfrak{A}(p)$-modules the universal properties of the algebra $U_R(F^{-1})$ assure that the inclusion induces a map

$$\zeta : U_R(F^{-1}) \to H^*(E)$$

of $\mathfrak{A}(p)$-algebras.

We contend that ζ is an isomorphism. For convenience in what follows we will denote by $\bar{u}_i, \bar{v}_j$ the elements corresponding to u_i, v_j when thought of as elements of $U_R(F^{-1})$.

Now $1, \bar{u}_1, \cdots, \bar{v}_1, \cdots$ are an R-basis for F^{-1}. This follows from the fact that they correspond to $1, sx_1, \cdots, sy_1, \cdots$ in $\mathcal{G}H^*(E)$ which are an R-basis for $E_\infty^{0,*} \oplus E_\infty^{-1,*}$. Therefore we have

(i) the monomials $1, u_{i_1}^{r_1} \cdots u_{i_n}^{r_n} v_{j_1}^{\varepsilon_1} \cdots v_{j_m}^{\varepsilon_m}$ where $0 < r_t < p$ and $\varepsilon_s = 0$ 1 are an R-basis for $H^*(E)$, for they correspond to the monomials $sx_{i_1}^{r_1} \cdots sx_{i_n}^{r_n} sy_{j_1}^{\varepsilon_1} \cdots sy_{j_m}^{\varepsilon_m}$ that are an R-basis for $\mathcal{G}H^*(E)$;

(ii) the monomials $1, \bar{u}_{i_1}^{r_1} \cdots \bar{u}_{i_n}^{r_n} \bar{v}_{j_1}^{\varepsilon_1} \cdots \bar{v}_{j_m}^{\varepsilon_m}$ where $0 < r_t < p$ and $\varepsilon_s = 0, 1$ are an R-generating set for $U_R(F^{-1})$. This follows directly from the definition of $U_R(F^{-1})$.

Now since $\zeta : U_R(F^{-1}) \to H^*(E)$ is a map of algebras we have

$$\zeta(\bar{u}_{i_1}^{r_1} \cdots \bar{u}_{i_n}^{r_n} \bar{v}_{j_1}^{\varepsilon_1} \cdots \bar{v}_{j_m}^{\varepsilon_m}) = u_{i_1}^{r_1} \cdots u_{i_n}^{r_n} v_j^{\varepsilon_1} \cdots v_{j_m}^{\varepsilon_m}$$

and hence ζ takes an R-generating set in a one-one fashion to an R-basis. Therefore ζ is an isomorphism. □

Thus we see that a knowledge of the $\mathcal{A}(p)$-module structure of $F^{-1}H^*(E)$ will determine the $\mathcal{A}(p)$-algebra structure of $H^*(E)$. Determining this $\mathcal{A}(p)$-structure seems to be a hard problem.

Acknowledgment. Theorem 5.7 was first proved in the case $p = 2$ by Kristensen in [5] using his theory of cohomology operations in the Serre spectral sequence of a fibre space. Massey and Peterson [7] have obtained a neat proof based on [6]. Theorem 5.9 for $p = 2$, also follows from the results of [6] and we assume that this will appear in [7].

Appendix. Rational coefficients

The results of §1 can be used to yield some simple results on multiplicative fibre maps when the ground field is the rational numbers Q. In this section all cohomology is taken with Q as coefficients.

THEOREM. *If*

$$\xi : F \to E \xrightarrow{p} B$$

is a multiplicative fibre map and $H^*(B)$ *is co-commutative then*

$$H^*(F) \cong \mathrm{Tor}_{H^*(B)}(Q, H^*(E))$$

as an algebra.

Proof. Consider the Eilenberg-Moore spectral sequence of ξ. Using the results of §1 we see that

$$E_2 = \mathrm{Tor}_{H^*(B)}(Q, H^*(E)) \cong H^*(E) /\!\!/ p^* \otimes \mathrm{Tor}_{\text{sub-ker } p^*}(Q, Q).$$

Now sub-ker $p^* \subset H^*(B)$ is a sub-Hopf algebra. Therefore by Borel's structure theorem for Hopf algebras over Q we see that sub-ker $p^* = S[V]$. It follows that

$$\mathrm{Tor}_{\text{sub-ker } p^*}(Q, Q) = S[sV]$$

and

$$E_2 = H^*(E) /\!\!/ p^* \otimes S[sV].$$

Since $sV \subset E_2^{-1,*}$ we see that $E_2^{0,*}$ and $E_2^{-1,*}$ generate E_2 as an algebra. Hence $E_2 = E_\infty$.

Since E_2 is a free commutative algebra and $H^*(E)$ is commutative with $\mathcal{G}H^*(E) = E_2$ it follows that $H^*(E) \cong E_2$ as an algebra. □

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