A TOPOLOGICAL H-COBORDISM THEOREM FOR $n \geq 5$

BY

E. H. CONNELL[1]

An H-cobordism is a compact manifold M with boundary components N and $\bar{N}$ which are deformation retracts of M. If $M = M^n$ is a simply connected differentiable manifold and $n \geq 6$, then M is diffeomorphic to $N \times I$ [11]. If M is a combinatorial manifold and $n \geq 5$, then $M - \bar{N}$ is piecewise-linearly homeomorphic to $N \times [0, 1)$ (p. 251 of [14]). In this paper it will be shown that if M is a topological n-manifold and $n \geq 5$, then $M - \bar{N}$ is homeomorphic to $N \times [0, 1)$. This is done by a type of topological engulfing (see Lemma 1).

A stronger form of Lemma 1 has independently (and previously) been obtained by M. H. A. Newman [1]. A corollary to these procedures is that if Y is a closed topological manifold which is a homotopy sphere, and $n \geq 5$, then Y is homeomorphic to S^n. The reader is assumed familiar with the proof of the combinatorial engulfing lemma [2], [5], [8].

Notation. Suppose M is a metric space with the distance between x and $y \in M$ denoted by $d(x, y)$. If $Y \subset M$ is any subset of M, $d(x, Y)$ will denote the distance from x to Y, $d(Y)$ will denote the diameter of Y, and for any $\varepsilon > 0$, $V(Y, M, \varepsilon)$ will denote the set $\{z \in N : d(z, Y) < \varepsilon\}$. If K is a finite complex, the statement that $f : K \to R^n$ is piecewise-linear (p.w.l.) means $\exists$ a subdivision K_1 of K such that any simplex σ of K_1 is mapped linearly into R^n by f. If M is a topological manifold, the interior and boundary of M are denoted by Int M and ∂M respectively. D^n denotes the closed n-cell in R^n,

$$D^n = \{(x_1, x_2, \cdots x_n) : -1 \leq x_i \leq 1, i = 1, 2, \cdots n\}.$$

Hypothesis I. $M = M^n$ is a compact, connected topological n-manifold ($n \geq 5$) with boundary consisting of two components, $\partial M = N \cup \bar{N}$; $\pi_i(M, N) = \pi_i(M, \bar{N}) = 0$ for $i = 1, 2, \cdots, n - 3$;

$$g : N \times [0, 1] \to M - \bar{N} \quad \text{and} \quad \bar{g} : \bar{N} \times [0, 1] \to M - N$$

are topological embeddings with $g(x, 0) = x$ for all $x \in N$ and $\bar{g}(y, 0) = y$ for all $y \in \bar{N}$. (Note: If M is any topological manifold with boundary components N and $\bar{N}$, then it follows from [13] that the embeddings g and $\bar{g}$ exist.)

LEMMA 1. *Suppose Hypothesis I. Suppose $K \subset R^n$ is a finite m-complex (a rectilinear complex in R^n), $m \leq n - 3$, $h : R^n \to$ Int M is a topological embedding, and ε is a number with $0 < \varepsilon < 1$. Then $\exists$ a homeomorphism*

Received April 25, 1966.

[1] The author has been supported by the Sloan Foundation and by a National Science Foundation grant.

$H : M \to M$ satisfying:
(1) $H(x) = x$ *for* $x \in \bar{N} \cup g(N \times [0, 1 - \varepsilon])$
(2) $H(g[N \times [0, 1)]) \supset h(K)$.

Proof. The proof is given for $m \leq n - 4$. The case $m = n - 3$ contains an extra difficulty that makes the proof less transparent. This difficulty may be handled in a way completely analogous to the combinatorial case (see note at end of Case 1).

The proof is by induction on $m = \dim K$. Suppose $m \leq n - 4$ and the lemma is true when $\dim K \leq m - 1$. The proof below actually shows without any induction on m that the lemma is true when $2(1 + \dim K) < n$. This is because no singularities are encountered in these dimensions.

Let each of $h_1, h_2, \cdots h_k : R^n \to \operatorname{Int} M$ be a topological embedding with

$$[\textstyle\bigcup_{1 \leq i \leq k} h_i(R^n)] \cup g(N \times [0, 1 - \varepsilon]) \cup \bar{g}(\bar{N} \times [0, 1 - \varepsilon]) = M.$$

Let $\delta > 0$ such that $V\{h(K), M, 2\delta\} \subset h(R^n)$.

Let K_1 be a subdivision of K with Q_1 and Q subcomplexes of K_1 satisfying

$$\dim(Q_1 \cap Q) \leq m - 1, \qquad K_1 = Q_1 \cup Q, \qquad h(Q_1) \subset g[N \times [0, 1)],$$
$$h(Q) \subset \operatorname{Int} M - g(N \times (0, 1 - \varepsilon]),$$

and thus

$$h(Q_1 \cap Q) \subset g[N \times (1 - \varepsilon, 1)].$$

Let $f : Q \times I \to \operatorname{Int} M - g(N \times (0, 1 - \varepsilon])$ be a continuous function satisfying:

(a) $f(x, 1) = h(x)$ for $x \in Q$.
(b) $f(x, t) = h(x) \in V\{h(K), M, \delta\} \cap g[N \times (1 - \varepsilon, 1)]$ for $x \in Q_1 \cap Q$ and $t \in [0, 1]$.
(c) $f(x, 0) \in g[N \times (1 - \varepsilon, 1)]$ for $x \in Q$.

Such an f exists because

$$\pi_i\{\operatorname{Int} M - g(N \times (0, 1 - \varepsilon]), g[N \times (1 - \varepsilon, 1)]\} = 0$$

for $i = 1, 2, \cdots m$. Let K_2 be a subdivision of K_1 with L_1 and L the induced subdivision of Q_1 and Q. Let $\sigma_1^i, \sigma_2^i, \cdots \sigma_{r(i)}^i$ be the closed i-simplexes of $(L, L_1 \cap L)$ for $i = 0, 1, \cdots, m$. Finally, let $0 = t_0 < t_1 < \cdots < t_v = 1$ be a partition of $[0, 1]$. If the subdivision K_2 and the partition $t_0 < t_1 \cdots < t_v$ are fine enough, then $f : L \times I \to M$ will satisfy Property P below.

DEFINITION. A continuous function $f : L \times I \to M$ has Property P provided
(1) $f(L \times I) \subset \operatorname{Int} M - g[N \times (0, 1 - \varepsilon]]$
(2) $f(x, 1) = h(x)$ for $x \in L$
(3) $f(L_1 \cap L \times [0, 1]) \subset V(h(K), M, \delta)$

(4) $f(\sigma_j^i \times [t_{a-1}, t_a]) \subset h_b(R^n)$ for some $b = b(i, j, a)$ when $0 \leq i \leq m$, $1 \leq j \leq r(i)$, and $1 \leq a \leq k$.

(5) $d[f(\sigma_j^i \times [t_{a-1}, t_a])] < \delta$ for $0 \leq i \leq m$, $1 \leq j \leq r(i)$ and $1 \leq a \leq k$.

Now suppose that the subdivision K_2 and the partition $t_0 < t_1 < \cdots < t_v$ are given so that f satisfies Property P. Note also that, in addition to (3), f satisfies

$$f(L_1 \cap L \times [0, 1]) \subset g[N \times (1 - \varepsilon, 1)].$$

For the remainder of this proof, the simplexes σ_j^i, the partition $t_0 < t_1 < \cdots < t_v$, and the function $b = b(i, j, a)$ are fixed. The statement that some $\alpha : L \times I \to M$ satisfies Property P means with respect to this fixed data. Notice that if α satisfies Property P and $\beta : L \times I \to M$ has $\beta(x, 1) = h(x)$ and β is a close enough approximation to α, then β will also have Property P.

Definition. For $0 \leq i \leq m$, $1 \leq j \leq r(i)$, $1 \leq a \leq v$,

$$X(i, j, a) \subset (L \times I)$$

is defined by

$$\begin{aligned} X(i, j, a) = {} & L \times 0 \cup [L \cap L_1] \times [0, 1] \\ & \cup L \times [0, t_{a-1}] \cup \{\sigma_t^s \times [t_{a-1}, t_a] : s < i, 1 \leq t \leq r(s)\} \\ & \cup \{\sigma_t^i \times [t_{a-1}, t_a] : 1 \leq t \leq j\}. \end{aligned}$$

Inductive Hypothesis $(i, j, a) = IH(i, j, a)$. There exists a continuous function

$$\alpha_{(i,j,a)} : L \times I \to M$$

which satisfies Property P and a homeomorphism

$$H_{(i,j,a)} : M \to M$$

ssatisfying

(1) $H_{(i,j,a)}(x) = x$ for $x \in \bar{N} \cup g(N \times [0, 1 - \varepsilon])$.

(2) $H_{(i,j,a)}(g[N \times [0, 1)]) \supset h(L_1) \cup \alpha_{(i,j,a)}[X(i, j, a)]$.

The purpose of the proof is to show that $IH(m, r(m), v)$ is true.

Fact 1. $IH(0, 1, 1)$ is true.

Fact 2. $IH(i, j - 1, a) \Rightarrow IH(i, j, a)$ for $0 \leq i \leq m$, $2 \leq j \leq r(i)$, $1 \leq a \leq v$.

Fact 3. $IH(i, r(i), a) \Rightarrow IH(i + 1, 1, a)$ for $0 \leq i < m$, $1 \leq a \leq v$.

Fact 4. $IH(m, r(m), a) \Rightarrow IH(0, 1, a + 1)$ for $1 \leq a < v$.

The proof of Fact 2 is presented in detail. The proofs of Facts 1, 3, and 4 require only trivial modifications and are not included.

Suppose $0 \leq i \leq m$, $2 \leq j \leq r(i)$, $1 \leq a \leq v$, and $IH(i, j - 1, a)$ is true.

For simplicity of notation, let

$$H = H_{(i,j-1,a)} : M \to M \quad \text{and} \quad \alpha = \alpha_{(i,j-1,a)} : L \times I \to M.$$

Then α has Property P and

(1) $H(x) = x$ for $x \in \bar{N} \cup g(N \times [0, 1 - \varepsilon])$
(2) $H(g[N \times [0, 1)]) \supset h(L_1) \cup \alpha[X(i, j - 1, a)]$

Proof of Fact 2, *Case* 1. *Suppose*

$$\alpha(\sigma_j^i \times [t_{a-1}, t_a]) \cap \{Y = h(L_1) \cup \alpha(L_1 \cap L \times [0, 1] \cup L \times 1\} = \emptyset.$$

Let U_1, U_2, U_3 be open subsets of Int M with $\alpha(\sigma_j^i \times [t_{a-1}, t_a]) \subset U_1$, $\mathrm{Cl}(U_1) \subset U_2$, $\mathrm{Cl}(U_2) \subset U_3$, $\mathrm{Cl}(U_3) \subset h_{b(i,j,a)}(R^n)$, and $U_3 \cap Y = \emptyset$. Let $Z \subset L \times I$ be a finite subcomplex of some subdivision of $L \times I$ with $\alpha^{-1}(U_2) \subset Z \subset \alpha^{-1}(U_3)$. Now by a general position approximation argument, $\exists$ a continuous

$$\alpha_{(i,j,a)} = \beta : L \times I \to M$$

which satisfies Property P and

(1) $\beta(\sigma_j^i \times [t_{a-1}, t_a]) \subset U_1$.
(2) $\beta^{-1}(U_1) \subset \alpha^{-1}(U_2) \subset Z$.
(3) $\beta | \alpha^{-1}(M - U_3) = \alpha | \alpha^{-1}(M - U_3)$.
(4) $h_{b(i,j,a)}^{-1} \beta \mid Z : Z \to R^n$ is p.w.l. and in general position. In particular, if

$$S = \mathrm{Cl}\ \{x \in \sigma_j^i \times [t_{a-1}, t_a] : \exists y \in Z \text{ with } x \neq y,\ \beta(x) = \beta(y)\},$$

then $\dim S \leq 2(m + 1) - n \leq (n - 4) + m + 2 - n = m - 2$.

In addition, it is assumed that β approximates α close enough that

$$H(g[N \times [0, 1)] \supset h(L_1) \cup \beta[X(i, j - 1, a)]$$

(see (2) above).

Let $\pi : \sigma_j^i \times [t_{a-1}, t_a] \to \sigma_j^i$ be the projection. Since β has Property P,

$$\beta(\pi(S) \times [t_{a-1}, t_a]) \subset h_{b(i,j,a)}(R^n).$$

Since $h_{b(i,j,a)}^{-1} \beta(\pi(S) \times [t_{a-1}, t_a])$ is a rectilinear complex in R_n of dimension $\leq m - 1$, the inductive hypothesis on m may be applied. (Note that if $2(m + 1) < n$, then no induction on m is necessary.)

Let $0 < \Delta < \varepsilon$ such that

$$H(g[N \times [0, 1 - \Delta)]) \supset h(L_1) \cup \beta[X(i, j - 1, a)].$$

Then $\exists$ a homeomorphism $G_1 : M \to M$ satisfying

(a) $G_1(x) = x$ for

$$x \in \bar{N} \cup H(g(N \times [0, 1 - \Delta])) \supset \bar{N} \cup g(N \times [0, 1 - \varepsilon]).$$

(b) $G_1(H(g[N \times [0, 1)])) \supset \beta(\pi(S) \times [t_{a-1}, t_a])$.

Now since $\sigma_j^i \times [t_{a-1}, t_a]$ collapses to

$$[X(i, j - 1, a) \cap (\sigma_j^i \times [t_{a-1}, t_a])] \cup (\pi(S) \times [t_{a-1}, t_a]) = (\sigma_j^i \times t_{a-1}) \cup (\pi(S) \cup \partial\sigma_j^i) \times [t_{a-1}, t_a],$$

$\exists$ a homeomorphism $G_2 : M \to M$ satisfying

(A) $G_2(x) = x$ for $x \in (M - U_1) \cup g(N \times [0, 1 - \varepsilon])$
(B) $G_2 G_1 H(g[N \times [0, 1)] \supset h(L_1) \cup \beta[X(i, j, a)]$.

(See p. 486 of [2].)

The homeomorphism $H_{(i,j,a)}$ is given by

$$H_{(i,j,a)} = G_2 G_1 H = G_2 G_1 H_{(i,j-1,a)}.$$

$H_{(i,j,a)}$ and $\alpha_{(i,j,a)} = \beta$ satisfy $IH(i, j, a)$. (Note: The changes necessary for the case $m = n - 3$ are almost identical to the changes necessary in the combinatorial case. The inductive hypothesis $IH(i, j - 1, a)$ would require covering only the m-skeleton of $\alpha_{(i,j-1,a)}[X(i, j - 1, a)]$, i.e., the $(m + 1)$-cells need not be contained in $H_{(i,j-1,a)}(g[N \times [0, 1)])$. The singular set S would be defined by intersections of $\alpha(\sigma_j^i \times [t_{a-1}, t_a])$ with $\alpha(Z^m)$, where Z^m is the m-skeleton of Z.)

Proof of Fact 2, *Case* 2. Suppose

$$\alpha(\sigma_j^i \times [t_{a-1}, t_a]) \cap \{Y = h(L_1) \cup \alpha(L_1 \cap L \times [0, 1] \cup L \times 1)\} \neq \emptyset.$$

This case is similar to Case 1 except $h(R^n)$ is used instead of $h_{b(i,j,a)}(R^n)$. Note that Case 2 always holds when $a = v$.

Since

$$h(L_1) \cup \alpha(L_1 \cap L \times [0, 1] \cup L \times 1) \subset V(h(K), M, \delta)$$

and

$$d[\alpha(\sigma_j^i \times [t_{a-1}, t_a])] < \delta,$$

it follows that

$$\alpha(\sigma_j^i \times [t_{a-1}, t_a]) \subset V(h(K), M, 2\delta) \subset h(R^n).$$

Let U_1, U_2, U_3 be open subsets of Int M with

$$h(K) \cup \alpha(\sigma_j^i \times [t_{a-1}, t_a] \cup L_1 \cap L \times [0, 1]) \subset U_1,$$

$$\mathrm{Cl}\,(U_1) \subset U_2, \qquad \mathrm{Cl}\,(U_2) \subset U_3, \qquad \mathrm{Cl}\,(U_3) \subset h(R^n).$$

Let $Z \subset L \times I$ be a finite subcomplex of some subdivision of $L \times I$ with

$$\alpha^{-1}(U_2) \subset Z \subset \alpha^{-1}(U_3).$$

Now by a relative general position approximation argument, $\exists$ a continuous

$$\alpha_{(i,j,a)} = \beta : L \times I \to M$$

which satisfies Property P and

(1) $\beta(\sigma_j^i \times [t_{a-1}, t_a] \cup L \cap L_1 \times [0, 1]) \subset U_1$

(2) $\beta^{-1}(U_1) \subset \alpha^{-1}(U_2) \subset Z$
(3) $\beta \mid \alpha^{-1}(M - U_3) = \alpha \mid \alpha^{-1}(M - U_3)$
(4) $h^{-1}\beta \mid Z : Z \to R^n$ is p.w.l. and in general position relative to L_1.

In particular, if $S = \mathrm{Cl}\,\{x \in \sigma_j^i \times [t_{a-1}, t_a] : (\exists y \in Z, y \neq x, \beta(x) = \beta(y))$ or $(\exists w \in L_1 - L$ with $\beta(x) = h(w)\}$ then

$$\dim S \leq 2(m + 1) - n \leq n - 4 + m + 2 - n = m - 2.$$

The remainder of the proof is now a repeat from Case 1. Since

$$\dim (\pi(S) \times [t_{a-1}, t_a]) < m,$$

it may be engulfed without uncovering

$$h(L_1) \cup \beta(L_1 \cap L \times [0, 1] \cup X(i, j - 1, a)).$$

Then using the collapsing technique, engulf all of $\beta(\sigma_j^i \times [t_{a-1}, t_a])$. This completes Lemma 1.

LEMMA 2. *Suppose Hypothesis I, b is a number with $0 < b < 1$,*

$$g(N \times [0, 1]) \subset M - \bar{g}(\bar{N} \times [0, 1 - b]),$$

$$\bar{g}(\bar{N} \times [0, 1]) \subset M - g(N \times [0, 1 - b]),$$

and $h : R^n \to \mathrm{Int}\, M$ is a topological embedding. Then for any number a with $0 < a < b$, $\exists$ homeomorphisms $f : M \to M$ and $\bar{f} : M \to M$ with

$$f \mid g(N \times [0, 1 - a]) \cup \bar{g}(\bar{N} \times [0, 1 - b]) = \mathrm{Id}.$$

$$\bar{f} \mid \bar{g}(\bar{N} \times [0, 1 - a]) \cup g(N \times [0, 1 - b]) = \mathrm{Id}.$$

and

$$fg[N \times [0, 1)] \cup \bar{f}\bar{g}[\bar{N} \times [0, 1)] \supset h(D^n).$$

Proof. Let T be a rectilinear triangulation of R^n which has D^n as a subcomplex. Let X be the subcomplex of T composed of all closed simplexes $\sigma \subset D^n$ with $h(\sigma) \cap$

$$\{M - [g(N \times [0, 1 - a/2]) \cup \bar{g}(\bar{N} \times [0, 1 - a/2])]\} \neq \emptyset$$

and let Y be the closed star of X in T (in all of R^n). Suppose that the triangulation T is fine enough that

$$h(Y) \subset \{M - [g(N \times [0, 1 - 3a/4]) \cup \bar{g}(\bar{N} \times [0, 1 - 3a/4])]\}.$$

Let $\Delta > 0 \ni$

$$V\{h(X), M, 3\Delta\} \subset h(Y) \quad \text{and} \quad V\{g(N \times [0, 1 - a/2]), M, \Delta\} \subset g(N \times [0, 1 - a/4]).$$

Let T_1 be a subdivision of $T \ni$ for any simplex σ_1 of T_1, $d(h(\sigma_1)) < \Delta$. Let X_1 and Y_1 be the sets X and Y under the triangulation T_1. Let K be the $(n - 3)$-skeleton of Y_1 and $\bar{K}$ be the maximal complex of the first derived of Y_1 which does not intersect K. Then $\dim \bar{K} = 2 \leq n - 3$. Now apply

Lemma 1 to the H-cobordism $M - \bar{g}[\bar{N} \times [0, 1 - b)]$ and obtain a homeomorphism

$$f_1 : M - \bar{g}[\bar{N} \times [0, 1 - b)] \to M - \bar{g}[\bar{N} \times [0, 1 - b)]$$

such that $f_1(x) = x$ for $x \in \bar{g}(\bar{N}, 1 - b) \cup g(N \times [0, 1 - a/4])$ and

$$f_1(g[N \times [0, 1)] \supset h(K).$$

Extend f_1 to a homeomorphism $f_1 : M \to M$ satisfying

(1) $f_1(x) = x$ for $x \in \bar{g}(\bar{N}, [0, 1 - b]) \cup g(N \times [0, 1 - a/4])$
(2) $f_1(g[N \times [0, 1)]) \supset h(K)$.

In the same manner, apply Lemma 1 to the H-cobordism $M - g[N \times [0, 1 - b)]$ and obtain a homeomorphism $\bar{f} : M \to M$ satisfying

(1) $\bar{f}(x) = x$ for $x \in g(N \times [0, 1 - b]) \cup \bar{g}(\bar{N} \times [0, 1 - a/4])$.
(2) $\bar{f}(\bar{g}[\bar{N} \times [0, 1)]) \supset h(\bar{K})$.

Statement A. $\exists$ a homeomorphism $f_2 : M \to M \ni$

(i)
$$f_2(x) = x$$
for $x \in M - h(Y_1) \supset g(N \times [0, 1 - 3a/4]) \cup \bar{g}(\bar{N} \times [0, 1 - 3a/4])$

(ii) $f_2 f_1(g[N \times [0, 1)]) \cup \bar{f}(\bar{g}[\bar{N} \times [0, 1)]) \supset h(X_1)$

(iii) $d(f_2(x), x) < \Delta$ for any $x \in M$.

Statement B. The proof of Lemma 2 is completed by setting $f = f_2 f_1$.

Proof of Statement B *assuming Statement* A. It must be shown that if $p \in D^n$,

$$h(p) \in f_2 f_1(g[N \times [0, 1)]) \cup \bar{f}(\bar{g}[\bar{N} \times [0, 1)]).$$

If $p \in X_1$, then this follows from Statement A (ii). Now suppose $p \in D^n - X_1$. Then it follows from the definition of X that

$$h(p) \in g(N \times [0, 1 - a/2]) \cup \bar{g}(\bar{N} \times [0, 1 - a/2]).$$

Case 1. $h(p) \in \bar{g}(\bar{N} \times [0, 1 - a/2])$. Since $\bar{f} \mid \bar{g}(\bar{N} \times [0, 1 - a/2]) = \text{Id.}$, it follows that

$$h(p) \in \bar{f}(\bar{g}[\bar{N} \times [0, 1)])$$

and this case is immediate.

Case 2. $h(p) \in g(N \times [0, 1 - a/2])$. The sequence of facts

(a) $f_1 \mid g(N \times [0, 1 - a/4]) = \text{Id.}$
(b) $V\{g(N \times [0, 1 - a/2]), M, \Delta\} \subset g(N \times [0, 1 - a/4])$
(c) $d(f_2(x), x) < \Delta$ for $x \in M$.

imply that $h(p) \in f_1 f_2(g[N \times [0, 1)])$. This completes the proof of Statement B.

Sketch of Proof of Statement A. The ideas here are taken from p. 499–500 of [5]. Each point $y \in Y_1$ can be described in terms of "barycentric coordinates", $\lambda(y) \in K$, $\bar{\lambda}(y) \in \bar{K}$, and $t(y) \in [0, 1]$, such that

$$y = t(y)\lambda(y) + [1 - t(y)]\bar{\lambda}(y).$$

Using these coordinates it is possible to define a homeomorphism $U : Y_1 \to Y_1 \ni$ each interval $[\lambda(y), \bar{\lambda}(y)]$ is mapped onto itself and

$$Uh^{-1}\{f_1(g[N \times [0, 1)]) \cap h(Y_1)\} \cup h^{-1}\{\bar{f}(\bar{g}[\bar{N} \times [0, 1)]) \cap h(Y_1)\} = Y_1 .$$

Define a homeomorphism $W : Y_1 \to Y_1$ by

$$W(y) = \frac{1}{\Delta} d[V\{h(X), M, \Delta\}, h(\lambda(y)]y + \left(1 - \frac{1}{\Delta} d[V\{h(X), M, \Delta\}, h(\lambda(y)]\right) U(y)$$

when

$$y \in Y_1 - (K \cup \bar{K}) \quad \text{and} \quad \Delta \geq d[V\{h(X), M, \Delta\}, h(\lambda(y))],$$

$$W(y) = y \quad \text{otherwise.}$$

Define a homeomorphism $f_2 : M \to M$ by

$$f_2(x) = hWh^{-1}(x) \quad \text{for } x \in h(Y_1)$$

$$f_2(x) = x \qquad \text{for } x \in M - h(Y_1).$$

The facts

$$W([\lambda(y), \bar{\lambda}(y)]) = [\lambda(y), \bar{\lambda}(y)] \quad \text{for } y \in Y_1 - (K \cup \bar{K}),$$

$$d[h(\sigma)] < \Delta \qquad \text{for each simplex } \sigma \text{ of } Y_1 ,$$

and

$$V\{h(X), M, 3\Delta\} \subset h(Y_1)$$

imply

$$W \mid X = U \mid X, \qquad W \mid \partial Y_1 = \mathrm{Id},$$

$$f_2 \mid M - h(Y_1) = \mathrm{Id}, \qquad d(f_2(x), x) < \Delta$$

and

$$f_2 f_1(g[N \times [0, 1)]) \cup \bar{f}(\bar{g}[\bar{N} \times [0, 1)]) \supset h(X_1)$$

This completes the proof of Statement A and Lemma 2.

THEOREM 1. *Suppose Hypothesis I, and that*

$$g(N \times [0, 1]) \cap \bar{g}(\bar{N} \times [0, 1]) = \emptyset.$$

Then if b is a number, $0 < b < 1$, $\exists$ homeomorphisms $f : M \to M$ and $\bar{f} : M \to M \ni$

$$f \mid g(N \times [0, 1 - b]) \cup \bar{g}(\bar{N} \times [0, 1 - b]) = \mathrm{Id}$$

$$\bar{f} \mid g(N \times [0, 1 - b]) \cup \bar{g}(\bar{N} \times [0, 1 - b]) = \mathrm{Id}$$

and

$$f(g[N \times [0, 1)]) \cup \bar{f}(\bar{g}[\bar{N} \times [0, 1)]) = M.$$

Also $\exists$ *a homeomorphism* $H : g[N \times [0, 1)] \to M - \bar{N}$.

Proof. Let each of $h_1, h_2, \cdots h_k : R^n \to \operatorname{Int} M$ be a topological embedding with

$$\bigcup_{1 \leq i \leq k} h_i(D^n) \cup g(N \times [0, 1 - b]) \cup \bar{g}(\bar{N} \times [0, 1 - b]) = M.$$

Inductive Hypothesis $(i) = IH(i)$ $i = 1, 2, \cdots k$. $\exists$ homeomorphisms f_i and $\bar{f}_i : M \to M \ni$

$$\text{each of } f_i \text{ and } \bar{f}_i \mid g(N \times [0, 1 - b]) \cup \bar{g}(\bar{N} \times [0, 1 - b]) = \mathrm{Id}$$

and

$$f_i(g[N \times [0, 1)]) \cup \bar{f}_i(\bar{g}[\bar{N} \times [0, 1)]) \supset \bigcup_{1 \leq t \leq i} h_t(D^n).$$

The proof involves showing $IH(k)$ is true and setting $f = f_k$ and $\bar{f} = \bar{f}_k$. $IH(1)$ follows immediately from Lemma 2. Suppose $IH(i)$ is true for some i, $1 \leq i < k$, and show $IH(i + 1)$ is true. The collar neighborhoods of Lemma 2 will be

$$f_i g(N \times [0, 1]) \subset M - \bar{g}(\bar{N} \times [0, 1 - b]) = M - \bar{f}_i \bar{g}(\bar{N} \times [0, 1 - b])$$

and

$$\bar{f}_i \bar{g}(\bar{N} \times [0, 1]) \subset M - g(N \times [0, 1 - b]) = M - f_i g(N \times [0, 1 - b]).$$

Now $\exists$ a number a, $0 < a < b$ with

$$f_i g[N \times [0, 1 - a)] \cup \bar{f}_i \bar{g}[\bar{N} \times [0, 1 - a)] \supset \bigcup_{1 \leq t \leq i} h_t(D^n)$$

By Lemma 2, $\exists$ homeomorphisms α and $\bar{\alpha} : M \to M$ with

$$\alpha \mid f_i\, g(N \times [0, 1 - a]) \cup \bar{f}_i\, \bar{g}(\bar{N} \times [0, 1 - b]) = \mathrm{Id}$$

$$\bar{\alpha} \mid \bar{f}_i \bar{g}(\bar{N} \times [0, 1 - a]) \cup f_i g(N \times [0, 1 - b]) = \mathrm{Id}$$

and

$$\alpha f_i g[N \times [0, 1)] \cup \bar{\alpha} \bar{f}_i \bar{g}[\bar{N} \times [0, 1)] \supset h_{i+1}(D^n).$$

The induction is completed by setting

$$f_{i+1} = \alpha f_i : M \to M \quad \text{and} \quad \bar{f}_{i+1} = \bar{\alpha} \bar{f}_i : M \to M.$$

This completes the proof of the first part of Theorem 1. (The f and $\bar{f}$ constructed here are actually isotopic to the identity.)

Note that $\bar{f}^{-1} f : M \to M$ satisfies

$$\bar{f}^{-1} f \mid g(N \times [0, 1 - b]) = \mathrm{Id}$$

and

$$\bar{f}^{-1} f(g[N \times [0, 1)]) \cup \bar{g}[\bar{N} \times [0, 1)] = M.$$

Thus the existence of the homeomorphism $H : g[N \times [0, 1)] \to M - \bar{N}$

follows in a standard way from a countable number of applications of the first part of the theorem.

COROLLARY 1. *If Y is a compact topological n-manifold ($n \geq 5$) without boundary, which has the homotopy type of S^n, then Y is homeomorphi to S^n.*

Sketch of proof. Let B^n and B_1^n be disjoint topological n-cells in Y and $p \in B_1^n$. Then $Y - B_1^n$ is homeomorphic to $Y - p$. It follows from Theorem 1 and the fact that

$$Y - (\operatorname{Int} B^n \cup \operatorname{Int} B_1^n)$$

is a topological H-cobordism that $Y - B_1^n$ is homeomorphic to R^n. Thus $Y - p$ is homeomorphic to R^n and Y is homeomorphic to S^n.

REFERENCES

1. M. H. A. NEWMAN, *The engulfing theorem for topological manifolds*, Ann. of Math., vol. 84 (1966), pp. 555–571.
2. J. STALLINGS, *The piecewise-linear structure of euclidean space*, Proc. Cambridge Philos. Soc., vol. 58 (1962), pp. 481–488.
3. ———, *Groups with infinite products*, Bull. Amer. Math. Soc., vol. 68 (1962), pp. 388–389.
4. ———, *Polyhedral homotopy-spheres*, Bull. Amer. Math. Soc., vol. 66 (1960), pp. 485–488.
5. ———, *On topologically unknotted spheres*, Ann. of Math., vol. 77 (1963), pp. 490–503.
6. E. C. ZEEMAN, *The Poincaré conjecture for $n \geq 5$*, Topology of 3-manifolds by M. K. Fort, Jr., Englewood Cliffs, Prentice-Hall, 1962, pp. 198–204.
7. ———, *The generalized Poincaré conjecture*, Bull. Amer. Math. Soc., vol. 67 (1961), p. 270.
8. ———, *Seminar on combinatorial topology*, Institute Des Hautes Etudes Scientifiques, 1963, revised 1965.
9. E. C. ZEEMAN AND D. R. McMILLAN, *On contractible open manifolds*, Proc. Cambridge Philos. Soc., vol. 58 (1962), pp. 221–224.
10. B. MAZUR, *Relative neighborhoods and the theorems of Smale*, Ann. of Math., vol. 77 (1963), pp. 232–249.
11. S. SMALE, *Generalized Poincaré's conjecture in dimensions greater than* 4, Ann. of Math., vol. 74 (1961), pp. 391–406.

11a. ———, *Differentiable and combinatorial structures on manifolds*, Ann. of Math., vol. 74 (1961), pp. 498–502.

12. ———, *On the structure of manifolds*, Amer. J. Math., vol. 84 (1962), pp. 387–399.
13. M. BROWN, *Locally flat imbeddings of topological manifolds*, Ann. of Math., vol. 75 (1962), pp. 331–341.
14. S. CAIRNS, *Differential and combinatorial topology*, Princeton, Princeton University Press, 1965.

RICE UNIVERSITY
HOUSTON, TEXAS