LOCALLY CONNECTED SPACES AND THEIR COMPACTIFICATIONS[1]

BY

J. DE GROOT AND R. H. McDOWELL

Introduction

This paper arose from the following problem: to find conditions under which a locally connected, rim-compact Hausdorff space H has a locally connected Hausdorff compactification. This proves to be the case (Theorem 4.2) if and only if at most finitely many of the components of H are compact. In trying to find a simple proof, the authors realized that a systematic and simple approach to the theory of locally connected spaces can be obtained by stressing—even more than Wilder [13]—the use of quasicomponents. We use, among other notions (e.g. "paddedness") the property of being quasilocally connected at a point [13, p. 40], and observe that the useful Proposition 1.7 holds for quasicomponents but not for components. The quasicomponent approach leads naturally to the result that in a connected, compact Hausdorff space, the property "components coincide with quasicomponents on every open subset" is equivalent to local connectedness (Theorem 3.3). (Recall that components and quasicomponents coincide on every *closed* subset of *any* compact Hausdorff space.)

To ask for a reasonable necessary and sufficient condition that a locally connected completely regular Hausdorff space have a locally connected Hausdorff compactification seems hopeless. However, it is known [6] that *every* compactification of such a space is locally connected if and only if the space is pseudo-compact. We readily obtain a proof of this theorem, and add some corollaries.

Example 5.3 seems to be of interest. Here we exhibit a subspace S of Euclidean three-space which is the union of a countable number of pairwise disjoint closed intervals, each nowhere dense in S, but S is nevertheless connected and locally connected.

The authors wish to thank the referee for his many helpful suggestions and corrections, which improved the paper substantially.

1. Components and quasicomponents

1.1. DEFINITIONS. Let p be a point in the space X. If the subset S of X is both open and closed in X, we say S is *clopen* in X.

The *component* of p in X is the maximal connected subset of X containing p.

Received April 23, 1964; received in revised form August 11, 1966.

[1] This paper was begun while the second author was supported by the Charles F. Kettering Foundation, and completed while both authors were working under contracts from the National Science Foundation.

The *quasicomponent* of p in X is the intersection of all the clopen subsets of X containing p.

A *neighborhood* of p is a set containing p in its interior.

1.2. PROPOSITION. *(Quasi)components are closed and pairwise disjoint, and every quasicomponent is a union of components.*

1.3. PROPOSITION. *Every open quasicomponent is a component.*

Proof. If Q is an open quasicomponent, then Q is clopen (1.2). Now if Q were the disjoint union of two proper relatively clopen sets, these sets would be clopen in the whole space, and Q would not be a quasicomponent.

1.4. PROPOSITION. *Components and quasicomponents coincide in every compact Hausdorff space.* (This is well known. See, e.g., [8, §42, II, 2].)

1.5. PROPOSITION. *If $X \subset Y$, and C_X is a (quasi)component in X, there is a (quasi)component C_Y in Y such that $C_X \subset C_Y$.*

Proof. A component C_X in X is still connected in Y, and is contained in the largest connected set in Y containing it. As to quasicomponents, if H is clopen in Y, then $H \cap X$ is clopen in X. The result follows at once.

1.6. PROPOSITION. *If Q is a quasicomponent in the space X, and K a compact set in X disjoint from Q, then there exists a clopen set S containing Q and missing K; moreover, Q is a quasicomponent of $X\backslash K$.*

Proof. For each $k \in K$, there is a clopen set S_k containing Q and missing k. The open cover $\{X\backslash S_k\}_{k \in K}$ of K has a finite subcover. The intersection of the corresponding S_k is the desired S. Since S is clopen, and misses K, the quasicomponents of S are also quasicomponents of X, hence Q is also a quasicomponent of $X\backslash K$.

1.7. PROPOSITION. *If U is an open subset of the connected space X such that $\bar{U}\backslash U$ is compact and non-empty, then every quasicomponent Q of $\bar{U}$ meets $\bar{U}\backslash U$.*

Proof. Suppose the contrary. Then, by Proposition 1.6, there is a set S, clopen in $\bar{U}$, which contains Q and misses $\bar{U}\backslash U$, so $S \subset U$. Hence S is a proper clopen subset of the connected space X; a contradiction.

In this lemma, "quasicomponent" cannot be replaced by "component", as is well known (see Example 5.1).

2. Local connectedness

2.1. DEFINITIONS. A space X is *locally connected at* p if every neighborhood of p contains an open connected neighborhood of p.

X is *weakly locally connected* (connected "im kleinen") *at* p if every neighborhood of p contains a connected neighborhood of p; equivalently, for every neighborhood U of p, the component of p in U is a neighborhood of p. In regular spaces, this is equivalent to: every neighborhood of p contains a closed connected neighborhood of p.

X is *quasilocally connected at* p if for every neighborhood U of p, the quasicomponent of p in U is a neighborhood of p [13, p. 40].

X is *padded at* p if for every neighborhood U of p, there exist open sets W and V such that $p \in W \subset \bar{W} \subset V \subset U$, and $V \backslash \bar{W}$ has onlp finitely many components ([5]; see also [12, p. 19]).

The space X is said to have any of the properties defined here if it has that property at each of its points.

2.2. Proposition [13, pp. 40–41]. *If X is locally connected at p, then X is weakly locally connected at p.*

If X is weakly locally connected at p, then X is quasilocally connected at p.

Neither of these implications can be reversed.

See Example 5.4 for a space which is quasilocally connected at a point, but is not weakly locally connected there. For a space that is weakly locally connected at a point, but not locally connected, see, for example, [7, p. 113].

2.3. Proposition. *If the connected space X is padded at p, then X is locally connected at p.* (The converse is false—see Example 5.1.)

Proof. Let U be a neighborhood of p. Choose open neighborhoods W and V of p such that $\bar{W} \subset V \subset U$, and $V \backslash \bar{W}$ has only finitely many distinct components $C_1, \cdots, C_n$. For each i, $1 \leq i \leq n$, there is a quasicomponent Q_i of V such that $C_i \subset Q_i$ (the Q_i need not be distinct). We assert that each $v \in V$ is in some Q_i. If not, for each $i \leq n$ there is a set V_i, clopen in V, containing v and missing Q_i. But then $\bigcap_{i=1}^{n} V_i$ is open in V and closed in $\bar{W}$, and is therefore clopen in X, which is impossible.

Since V has only finitely many quasicomponents, each of them is open, and is therefore a component (1.3). Then the component of p in V is an open connected neighborhood of p lying in U.

2.4. Proposition. *In a locally connected space, components and quasicomponents coincide in every open subset.*

Proof. If X is locally connected and U is open in X, then any component in U is a neighborhood of each of its points, and is therefore open. Hence every quasicomponent in U, being a union of components (1.2), is open. But then every quasicomponent is a component (1.3).

2.5. Theorem. *The following conditions on a space X are equivalent:*

(i) *X is quasilocally connected*

(ii) *X is weakly locally connected*

(iii) *X is locally connected*

(iv) *(quasi)components in every open subset of X are open.*

Proof. First, observe that the two assertions in (iv) are equivalent by (1.2) and (1.3). Then, since a set is open if and only if it is a neighborhood of each of its points, (iv) is equivalent to (i), (ii) and (iii).

2.6. COROLLARY. *A locally connected space is the topological union of its (quasi)components, i.e., each (quasi)component is clopen.*

It follows that *in all problems of an internal nature concerning a locally connected space, one may assume that the space is connected.*

2.7. PROPOSITION. *A space X fails to be weakly locally connected at the point p if and only if for some neighborhood U of p, there is a collection of distinct components $\{C_\alpha\}$ in U such that every neighborhood of p meets infinitely many of the C_α.*

2.8. PROPOSITION. *If X is a dense subspace of Y, and X is locally connected at $p \in X$, then Y is locally connected at p.*

The proofs of 2.7 and 2.8 are straightforward.

3. Compact and rim-compact spaces

3.1. DEFINITION. *X is rim-compact* (*semicompact*, or *locally peripherally compact*) *at p* if every neighborhood of p contains an open neighborhood of p with compact boundary. (Clearly, every locally compact space is rim-compact. Every rim-compact Hausdorff space is completely regular; see [9].)

X has *dimension* 0 *at* p if every neighborhood of p contains an open neighborhood of p with empty boundary.

A compact, connected Hausdorff space is called a *continuum.*

3.2. PROPOSITION [3]. *If X is a rim-compact Hausdorff space, then X has a compactification X^* such that $X^*\backslash X$, as a subspace of X^*, has dimension* 0.

In continua, or in rim-compact connected Hausdorff spaces, the converses of Propositions 2.3 and 2.4 are valid. This gives a new characterization of local connectedness in these spaces.

3.3. THEOREM. *If X is a continuum, or more generally, a rim-compact connected Haudorff space, the following are equivalent:*

(i) *X is locally connected*

(ii) *X is padded*

(iii) *Components and quasicomponents coincide on every open subset of X.*

Proof. In view of Propositions 2.3 and 2.4, we need only prove that (i) implies (ii) and (iii) implies (i).

To prove (i) implies (ii) we proceed as follows. Let p be a point in the open set U, and let W be an open neighborhood of p such that $\bar{W} \subset U$, and $\bar{W}\backslash W$ is compact. Since X is locally connected, each point $x \in \bar{W}\backslash W$ is contained in an open connected set U_x such that $\bar{U}_x \subset U\backslash\{p\}$. It follows that $\bar{W}\backslash W$ can be covered by a finite collection of connected open sets whose union lies in

U. Consequently, $\bar{W}\backslash W$ can be covered by a finite collection of disjoint connected open sets whose union lies in U; denote these by $U_1, \cdots, U_n$, and let O be their union. Set $V = W \cup O$, and let $G = V\backslash\bar{O}$. Then $p \in G \subset \bar{G} \subset V \subset U$, and $O \subset V \subset \bar{G} \subset \bar{O}$, so that $V\backslash\bar{G}$ is a finite union of connected sets; i.e., X is padded at p.

The proof that (iii) implies (i) is more complicated. We shall assume that X is not locally connected, and construct an open set U^* on which components and quasicomponents do not agree. There is an open set U in X such that some quasicomponent Q of U is not open (Theorem 2.5); i.e., there is a $q \in Q$ such that every neighborhood of q contains points outside Q. Let U_1 and U_2 be open sets such that

$$q \in U_1 \subset \bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset U,$$

and such that $\bar{U}_1\backslash U_1$ and $\bar{U}_2/U_2$ are compact. The quasicomponent of q in $\bar{U}_1$ meets $\bar{U}_1\backslash U_1$, and the quasicomponent of q in U_2 meets $\bar{U}_2\backslash U_2$ (Proposition 1.7). Hence Q meets these two boundaries by Proposition 1.5. Clearly, any quasicomponent of U which meets U_1 meets these same boundaries.

The set $H = Q \cap (\bar{U}_1\backslash U_1)$ is compact. We set $U^* = U\backslash H$. The component C_q of q in U^* is wholly contained in U_1, since $\bar{U}_1\backslash U_1$ separates U_1 from $X\backslash\bar{U}_1$, and $C_q \subset Q$. We shall show that the quasicomponent Q_q of q in U^* meets $\bar{U}_2\backslash U_2$, so that $C_q \neq Q_q$.

Let S be any clopen set in U^* containing Q_q. By the choice of q, S contains a point $p \in U_1\backslash Q$. The quasicomponent P of p in U meets $\bar{U}_2\backslash U_2$. P is disjoint from H, since $P \neq Q$ and $H \subset Q$ (the quasicomponents P and Q are disjoint). Hence (Proposition 1.5) P is also a quasicomponent of p in $U^* = U\backslash H$. Since $p \in S, P \subset S$, for S is clopen in U^*. Hence S meets the compact set $\bar{U}_2\backslash U_2$. It follows from Proposition 1.6 that Q_q meets $\bar{U}_2\backslash U_2$.

According to Theorem 2.5, (i) above is equivalent to "X is quasilocally connected" and to "X is weakly locally connected" in any space. Recall also that components and quasicomponents always coincide on every *closed* subset of a compact Hausdorff space (Proposition 1.4).

3.4. THEOREM [13, p. 104]. *Let X be a continuum, and let F be the set of all points of X at which X fails to be locally connected. Then either F is empty, or F contains a continuum consisting of more than one point.*

Actually we shall prove that F contains a continuum contained in the topological limit superior of a collection of continua. Under the stronger hypothesis that X is a metrizable continuum, it has been shown (see [8, p. 176]) that the set of points at which X fails to be weakly locally connected contains a continuum which is the topological limit of a *sequence* of continua.

Wilder (loc. cit.) proved 3.4 for locally compact connected spaces. The proof below (which of course yields the same result) is considerably shorter.

Proof. If F is non-empty, there is a point p in F at which X fails to be weakly locally connected (Theorem 2.5). By Proposition 2.7, there is an open neighborhood V of p, and a collection $\{C_\alpha\}$ of components of $\bar{V} = H$, none of which contain p, such that every neighborhood of p meets infinitely many of the C_α. H is not connected, and therefore (since X is connected) H is not clopen; i.e., the boundary B of H is nonempty. Since components and quasicomponents coincide on the compact set H (Proposition 1.4), each C_α meets B (Proposition 1.7). Consider the compact set

$$S = \overline{\bigcup_\alpha C_\alpha}, \qquad \text{so } p \in S,$$

and let C_p be the (quasi)component of p in $S \cup B$. Now every clopen neighborhood of p in $S \cup B$ contains a connected C_j, hence every such clopen set meets the compact set B. Then, by Proposition 1.6, C_p meets B. So C_p is a continuum containing more than one point, which, in turn, contains a subcontinuum C containing p properly and lying in the interior of H (for example, the component of p in $\bar{U} \cap S$, where U is an open neighborhood of p whose closure lies in the interior of H). Moreover, by the definitions of S and C_p, every neighborhood of any point in C meets infinitely many of the C_α, so by Proposition 2.7, X is not weakly locally connected, and a fortiori not locally connected, at any point of C.

4. Locally connected compactifications

Here we investigate conditions under which a locally connected space has locally connected compactifications. *Unless otherwise specified, "space", in this section, means "completely regular Hausdorff space".* Our first theorem is an immediate consequence of Theorems 3.4 and 2.8.

4.1. THEOREM. *Let X be a connected, locally connected space, and let X^* be a compactification of X. If $X^*\backslash X$ contains no continuum consisting of more than one point, then X^* is locally connected. In particular, X^* is locally connected whenever $X^*\backslash X$ is totally disconnected, or of dimension* 0, *and the one-point compactification of a locally compact, connected, locally connected Hausdorff space is locally connected.*

We may ask for conditions that a locally connected but not necessarily connected space X have a locally connected compactification. Clearly, if a space has more than finitely many compact components, any compactification must contain a point p such that every neighborhood of p meets infinitely many of these components, hence no compactification of such a space can be locally connected. Therefore, for a space to have any locally connected compactification, it must have at most finitely many compact components. For rim-compact spaces, this condition is also sufficient.

4.2. THEOREM. *A locally connected, rim-compact Hausdorff space X has a locally connected compactification if and only if at most finitely many of the components of X are compact.*

If this condition is satisfied, such a space has, in fact, a compactification by a *set of dimension* 0. *If in addition the space is locally compact, its one-point compactification is already locally connected.*

Rinow [10] treats so-called "perfect", locally connected extensions in the locally compact case. His results can easily be generalized to the rim-compact case by stressing the use of Wilder's Theorem 3.4.

Proof. We have already seen that the condition is necessary.

To prove sufficiency, recall that since X is locally connected, it is the topological union of its (clopen) components (2.6). First, suppose X is locally compact, and let $X \cup \{p\}$ be its one-point compactification.

Let $\{C_\alpha\}$ be the collection of non-compact components of X. Then for each α,

$$C_\alpha \cup \{p\} \subset X \cup \{p\}$$

is the one-point compactification of C_α. It suffices to show that $\bigcup_\alpha C_\alpha \cup \{p\}$ is locally connected at p. Let U be a neighborhood of p in this subspace. Evidently $C_\alpha \subset U$ for all but finitely many values of α. Call these $\alpha_1, \cdots, \alpha_n$, and, for each i, let $U_{\alpha_i} \subset U$ be an open connected neighborhood of p in $C_\alpha \cup \{p\}$ (4.1). Then $\bigcup_{i=1}^n U_{\alpha_i} \cup \bigcup_{\alpha \neq \alpha_i} C_\alpha$ is a connected open neighborhood of p contained in U.

If X is rim-compact, each non-compact component C_α of X is connected, locally connected and rim-compact. For each α, compactify C_α to C_α^* by a set of dimension 0 (3.2). By Theorem 4.1, each C_α^* is locally connected. Now remove one point from $C_\alpha^* \backslash C_\alpha$ for each α. The resulting spaces are locally compact but not compact, hence so is their union. Let X^* be the one-point compactification of this union, together with the union of the finitely many compact components of X. Then X^* is a locally connected compactification of X by a set of dimension 0.

4.3. *Example.* It is not true that every compactification by a set of dimension 0 of a rim-compact space (even a locally compact space) with finitely many compact components is locally connected. Let X be the topological union of infinitely many open intervals. The one-point compactification of the topological union of infinitely many closed intervals is a non-locally connected compactification of X by a set of dimension 0.

4.4. *Remark.* In Example 5.2, we construct a connected, locally connected subset of the plane, rim-compact at every point but two, which has *no* locally connected compactification.

It is natural to ask for a necessary and sufficient condition on a locally connected space X that every compactification of X be locally connected. This problem has been solved by Henriksen and Isbell [6]; the required condition is that X be pseudocompact (every real continuous function on X is bounded). We give here a short proof of this and related results (4.6, 4.12).

Recall that every completely regular Hausdorff space X has a compactification βX (the Stone-Čech compactification) with the following property: every continuous function from X into a compact space can be extended continuously to βX. A discussion of βX, and of further properties of pseudocompact spaces, may be found in [4].

4.5. LEMMA. *Let X be a locally connected pseudocompact space, and let W and V be open sets in X, with $\bar{W} \subset V$. Then only finitely many components in V intersect W.*

Proof. Suppose an infinite collection $\{C_n\}$ of components of V intersect W. For each n, choose one component D_n of $C_n \cap W$. Then D_n is open for each n. Every component in V is open, hence each point in $\bar{W}$ has a neighborhood meeting at most one D_n, i.e., the collection $\{D_n\}$ is discrete. For each n, choose $x_n \in D_n$, and let f_n be a continuous function from X into $[0, n]$ such that $f_n(x_n) = n$, $f_n(X \backslash D_n) = 0$. Define f on X by $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Then f is a continuous real unbounded function on X, and X is not pseudocompact.

4.6. COROLLARY. *A locally connected pseudocompact space X has only finitely many components.*

Proof. Take $W = V = X$ in Lemma 4.5.

4.7. THEOREM. *If X is locally connected and pseudocompact, and X is dense in the regular space Y, then Y is locally connected.*

Proof. Let p be a point in $Y \backslash X$, and V an open neighborhood of p. Choose W open so that $p \in W \subset \bar{W} \subset V$. By Lemma 4.5, only finitely many components of $V \cap X$ meet $W \cap X$; denote these by $C_1, \cdots, C_n$. Then p is in the closure in Y of some of these components, say $C_1, \cdots, C_k$, $1 \leq k \leq n$. Now p is in the interior of $\bar{W}$; it follows that $\bar{C}_1 \cup \cdots \cup \bar{C}_k$ is a connected set in $\bar{W}$ with p in its interior. So Y is weakly locally connected at every point (recall 2.8) and is therefore locally connected.

4.8. COROLLARY. *If X is locally connected and pseudocompact, then every compactification of X is locally connected. In particular, βX is locally connected.*

4.9. THEOREM. *If βX is locally connected, then X is pseudocompact.*

Proof. Let Y be the graph in the plane of the function $\sin 1/x$ $(0 < x \leq 1)$ together with the segment S joining the points $(0, -1)$ and $(0, 1)$. Then Y is compact, and $Y \backslash S$ is homeomorphic to the non-negative reals. If X is not pseudocompact, there is a continuous $f : X \rightarrow Y \backslash S$ such that $\overline{f(X)} \backslash f(X) \supset S$. Now f has a continuous extension $f^* : \beta X \rightarrow Y$; since f^* is a closed map, and $f^*(\beta X)$ is not locally connected, βX is not locally connected [2, I, §11, 6].

4.10. THEOREM (Banaschewski [1]). *If βX is locally connected then X is locally connected.*

The following theorem sums up the results in 4.8, 4.9 and 4.10. (Compare [11, Lemma 4], for normal spaces.)

4.11. THEOREM (Henriksen-Isbell). *The following properties of a completely regular Hausdorff space X are equivalent:*
(i) *βX is locally connected*
(ii) *Every space in which X is dense is locally connected*
(iii) *X is locally connected and pseudocompact.*

4.12. THEOREM. *A continuous image of a locally connected pseudocompact space is locally connected.*

Proof. Let f be a continuous mapping from the locally connected pseudocompact space X onto the completely regular space Y. Extend f to a continuous mapping $\bar{f}$ from βX onto βY. βX is locally connected (4.8) and $\bar{f}$ is a closed map, so βY is locally connected [2, I, §11, 6] and therefore Y is locally connected (4.10).

5. Examples

5.1. *Example.* We construct a space X as follows. Consider a closed rectangle, and a sequence of points $\{x_n\}$ on one side of the rectangle converging to a point p on that side. From each x_n, and from p, extend a closed unit interval away from the rectangle, perpendicular to the side. From the set so obtained, remove all the points from the boundary of the rectangle except the points $\{x_n\}$ and p, and remove the interior of the interval extending from p. The resulting subset of the plane is our space X. This space has the following two properties.

(1) X is connected, and locally connected at p, but not padded at p.
(2) Let U be the subset of X consisting of all points of X not in the original closed rectangle, so U consists of a countable family of half open intervals together with a point q. Then U is open in X, and $\bar{U}\backslash U$ is non-empty and compact. The quasicomponent of q in $\bar{U}$ meets $\bar{U}\backslash U$, but the component of q in $\bar{U}$ does not.

5.2. *Example.* Let X be the subset of the plane consisting of the segment joining $(0, 1)$ to $(1, 1)$, the segment joining $(0, -1)$ to $(1, -1)$ and the segments $\{(1 - 1/n, y) : -1 < y < 1\}, n = 1, 2, \cdots$. Then X is connected and locally connected, and is rim-compact except at $(1, 1)$ and $(1, -1)$, but X has no locally connected compactification. In fact, X is dense in no locally connected space. To see this, consider the subset K of X consisting of all points in X not lying between the graphs of $y = x$ and $y = -x$; i.e.,

$$K = \{(x, y) \in X : | y | \geq x\}.$$

Then K is compact, and its complement in X is an infinite discrete collection of open intervals. Now if X is dense in Y, and $p \in Y\backslash X$, then $U = Y\backslash K$ is a neighborhood of p whose closure has infinitely many compact components. The same is clearly true of any smaller neighborhood of p, so Y fails to be locally connected at p.

5.3. *Example.* We construct here a space which helps delimit the notion of local connectedness. Our space S is a *connected, locally connected subspace of three dimensional Euclidean space which is a countable union of disjoint line segments, each nowhere dense in* S.

The space S is constructed as follows. Let Q be the rational numbers in the closed unit interval, and let D be a countable dense set of irrationals in that interval. Let

$$S_1 = \{(x, y, z) : 0 \leq x \leq 1, y \in Q, z \in Q\},$$

$$S_2 = \{(x, y, z) : x \in Q, y \in D, 0 \leq z \leq 1\},$$

$$S_3 = \{(x, y, z) : x \in D, 0 \leq y \leq 1, z \in D\}.$$

Evidently each S_i is a countable union of closed intervals, and the S_i are pairwise disjoint. Let $S = S_1 \cup S_2 \cup S_3$. We shall show that S is connected (similar reasoning shows that S is locally connected, since each point has a base of "cubes", each very similar to S itself).

Let F, then, be a non-empty clopen subset of S. We shall show that $F = S$. Clearly, if F contains any point on one of the line segments comprising S, then F contains the entire segment. Now suppose F contains a point p, say in S_1, and let L_p be the segment in S containing p. Choose a sequence of segments in S_2, all having the same first coordinate, such that each of them contains a point of F, and such that their distances from L_p tend to zero (this can be done, since F is an open neighborhood of L_p). Since F is closed, F then contains all the segments in S_1 whose second coordinate is the same as that of p. Now every segment in S_3 contains a limit point of this subset of S_1; it follows that $S_3 \subset F$. But S_3 is dense in S, so $S = F$.

Our space S fails, of course, to be compact. Indeed, by a theorem of Sierpinski [8, §42, III, 6], *no compact connected Hausdorff space is the union of a countable collection of non-empty disjoint closed subsets.* "Compact" is essential here; it cannot be replaced by "locally compact, metrizable and topologically complete" (see, for example, [8, §42, III, 6a]). However, we can say this. *No separable metrizable topologically complete space M which is connected and rim-compact is the union of a countable collection of disjoint compact subsets.* The proof is not difficult; M can be compactified by a countable set to a compact metrizable space $\bar{M}$ [14]. Since $\bar{M}$ is connected, the result is an immediate consequence of Sierpinski's theorem. Even here, it is not possible to omit rim-compactness (see Example 5.4) or topological completeness.

If we form a space S' by identifying the end points of each interval in S, then S' is a *homogeneous* connected, locally connected space which is a countable union of its (compact) constituents [8, §42, VII], each of which is homeomorphic to a circle.

5.4. *Example.* Consider the sequence of points in the plane

$$p_n = (1 - 1/n, 1 - 1/n),$$

$n = 1, 2, \cdots$. For each n, let S_n be the square with p_n and p_{n+1} at opposite ends of a diagonal, and let R_n be the rectangle two of whose sides are sides of S_n and S_{n+1} whose fourth vertex is $(1 - 1/(n + 2), 1 - 1/n)$. For each S, let S'_n and R'_n be the reflections of S_n and R_n in the point $(\frac{1}{4}, 0)$.

The space X consists of the point $(1, 1) = p$, and the point p' (the reflection of p in $(\frac{1}{4}, 0)$), together with a countable collection of closed intervals chosen as follows. In each S_k (each R_k) choose a sequence of segments with endpoints on opposite sides of S_k (R_k) whose distance from the line common to S_k and R_k (R_k and S_{k+1}) becomes arbitrarily small. Similarly, in each $R_{k'}$ (each $S_{k'}$) choose a sequence of segments approaching the common boundary of $R_{k'}$ and $S_{k'}$ ($S_{k'}$ and $R_{(k-1)'}$, where $R_{0'} = S_1$).

There is a base at p consisting of the intersections with X of the interiors of squares in the plane whose sides are parallel to the sides of the S_k. If such a basic set V fails to contain p' in its closure, then it is easy to see that the components in V are p and the line segments (or parts of them) lying in V, so X is not weakly locally connected at p. On the other hand, the quasicomponent of p in V contains all the S_k and R_k from some k on (namely, the smallest k for which V meets S_{k-1}), since any clopen subset of V containing p has this property. Hence X is quasilocally connected at p. Notice that X is connected and topologically complete.

REFERENCES

1. B. BANASCHEWSKI, *Local connectedness of extension spaces*, Canad. J. Math., vol. 8 (1956), pp. 395–398.
2. N. BOURBAKI, *Topologie générale*, Éléments de Mathématique, Livre III, Chapters I and II, Paris, Hermann, 1951.
3. H. FREUDENTHAL, *Kompaktisierungen und Bikompaktisierungen*, Proc. Kon. Ned. Akad. Series A, vol. 54 (1951), pp. 184–192.
4. L. GILLMAN AND M. JERISON, *Rings of continuous functions*, Princeton, Van Nostrand, 1960.
5. J. DE GROOT, *Local connectedness and quasiorder*, Indag. Math., vol. 10 (1948), pp. 313–318.
6. M. HENRIKSEN AND J. ISBELL, *Local connectedness in the Stone-Čech compactification*, Illinois J. Math., vol. 1 (1957), pp. 574–582.
7. J. HOCKING AND G. YOUNG, *Topology*, Reading, Massachusetts, Addison-Wesley, 1961.
8. C. KURATOWSKI, *Topologie*, vol. II, Warszawa, 1961.
9. K. MORITA, *On bicompactification of semibicompact spaces*, Sci. Rep. Tokyo Bunrika Daigaku Section A4, no. 92 (1952), pp. 200–207.
10. W. RINOW, *Perfekte lokal zusammenhängende Kompaktifizierungen und Primendentheorie*, Math. Zeitschr., vol. 84 (1964), pp. 294–304.

11. A. D. WALLACE, *Extensional invariance*, Trans. Amer. Math. Soc., vol. 70 (1951), pp. 97–102.
12. G. T. WHYBURN, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942.
13. R. L. WILDER, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 42, 1949.
14. L. ZIPPIN, *On semi-compact spaces*, Amer. J. Math., vol. 57 (1935), pp. 327–341.

UNIVERSITY OF AMSTERDAM
AMSTERDAM, THE NETHERLANDS
WASHINGTON UNIVERSITY
ST. LOUIS, MISSOURI