STABLE PARALLELIZABILITY OF FIBER BUNDLES OVER SPHERES[1]

BY

RICHARD R. PATTERSON

1. Introduction

A manifold E is called stably parallelizable if the Whitney sum of its tangent bundle $\tau(E)$ and a trivial bundle is trivial. The object of this paper is to determine necessary and sufficient conditions that certain manifolds have this property. The manifolds we consider are those which appear as fiber bundles over spheres.

The notation we use for a bundle ξ is (E, π, S^n, Y, G), where E and Y are compact differentiable manifolds, π the projection to S^n, $n \geq 2$, Y the fiber and G the structural group. If $Y = V_{q,k}$, the real Stiefel manifold of orthonormal k-frames in R^q, we get the following result which appears as Corollary 4.

THEOREM. *Let $(E, \pi, S^n, V_{q,k}, O(q))$ be a bundle characterized by*

$$\delta : S^{n-1} \to O(q), \qquad n \geq 2.$$

Then E is stably parallelizable if and only if $k\{\delta\}$ is contained in the image of $(\sigma \circ \delta)^* : [V_{q,k}, \mathbf{O}] \to [S^{n-1}, \mathbf{O}]$.

Here $\mathbf{O}$ is the stable orthogonal group and $\{\delta\}$ the homotopy class of

$$\delta : S^{n-1} \to O(q) \to \mathbf{O}.$$

The map $\sigma : O(q) \to V_{q,k}$ is the natural projection. The case $k = 1$ was proved by Kosinski [5] and Sutherland [9].

Since $\tau(E)$ is the sum of the bundle $\tau_F(\xi)$ along the fibers of ξ, and the bundle $\pi^*\tau(S^n)$ orthogonal to the fibers of ξ which is always stably trivial, we see we must devote our attention to $\tau_F(\xi)$. Now $\tau_F(\xi)$ is an example of a wider class of bundles over E; those which can be obtained by *"lifting a bundle from the fiber."* In this case the bundle which is lifted is $\tau(Y)$. The techniques of this paper apply to all such lifted bundles. Some of the notation used here is taken directly from [7] without further comment.

The author wishes to thank A. A. Kosinski, whose paper [5] gave the initial ideas for this attack on the problem and who was most helpful during the research.

2. The lifted bundle

Suppose we are given a bundle

$$\xi = (E, \pi, S^n, Y, G)$$

Received December 10, 1966.

[1] This research was supported by the National Science Foundation.

characterized by $\delta : S^{n-1} \to G$ and a vector bundle

$$\eta = (H, \pi_1, Y, R^m, O(m))$$

defined over the fiber of ξ [8]. We shall assume that G acts on η; i.e., there is also given an action of G on the left on H which covers the action on Y and is orthogonal on fibers. We know that E is constructed as $D_+^n \times Y \cup_{\gamma_1} D_-^n \times Y$, where

$$\gamma_1 : S^{n-1} \times Y \to S^{n-1} \times Y$$

by $\gamma_1(x, y) = (x, \delta(x)y)$. We can define a bundle

$$\tilde{\eta} = (L, \pi_2, E, R^m, O(m))$$

over E by $L = D_+^n \times H \cup_{\gamma_1} D_-^n \times H$, where $\gamma_2 : S^{n-1} \times H \to S^{n-1} \times H$ by $\gamma_2(x, h) = (x, \delta(x)h)$ and π_2 is induced by π_1. We shall say $\tilde{\eta}$ is obtained by *lifting* η *to* E. (There is another more general way to construct $\tilde{\eta}$ which does not require the base space of ξ to be a sphere. See [2, p. 478].)

In the special case in which η is a trivial bundle we can be much more explicit in the construction of $\tilde{\eta}$. We choose a framing F of η. For $y \in Y$, $F(y)$ is an ordered orthonormal basis for the fiber of η over y. Other framings are, up to homotopy, in one to one correspondence with $[Y, O(m)]$ by letting $\beta : Y \to O(m)$ correspond to the framing $F_\beta(y) = F(y)\beta(y)$. Since G acts on η an element $g \in G$ takes the frame $F(y)$ at y to some frame at gy which we can compare to the frame $F(gy)$ already defined there. By this means we define functions

$$M : G \times Y \to O(m) \quad \text{and} \quad M_\beta : G \times Y \to O(m)$$

by $gF(y) = F(gy)M(g, y)$ and $gF_\beta(y) = F_\beta(gy)M_\beta(g, y)$. Elementary calculations show

(2.1) $M_\beta(e, y)$ is the identity of $O(m)$, where e is the identity of G,

$$M_\beta(g, y) = \beta(gy)^{-1}M(g, y)\beta(y). \tag{2.2}$$

The lifted bundle is constructed from two bundles

$$\eta_\pm = (D_\pm^n \times H, \mathrm{id} \times \pi_1, D_\pm^n \times Y, R^m, O(m))$$

induced by the projections of $D_\pm^n \times Y$ to Y. The framings F and F_β of η induce framings $F_\pm$ and $F_{\pm\beta}$ of $\eta_\pm$. We can let G act on $D_\pm^n \times H$ and $D_\pm^n \times Y$ by $g(x, h) = (x, gh)$ and $g(x, y) = (x, gy)$. Then the relations of before are still true:

$$gF_\pm(x, y) = F_\pm(x, gy)M(g, y) \quad \text{and} \quad gF_{\pm\beta}(x, y) = F_{\pm\beta}(x, gy)M_\beta(g, y).$$

When we identify $S_+^{n-1} \times H$ to $S_-^{n-1} \times H$ by means of γ_2 to construct $\tilde{\eta}$, we identify $F_+(x, y)$ to $\gamma_2 F_+(x, y)$. But on the fiber over (x, y) for $x \in S_+^{n-1}$ and any y, mapping by γ_2 is the same as operating by the element $\delta(x)$ of

$G : \gamma_2(x, h) = (x, \delta(x)h) = \delta(x)(x, h)$. So

$$\gamma_2 F_+(x, y) = \delta(x)F_+(x, y) = F_-(x, \delta(x)y)M(\delta(x), y).$$

Thus the bundle $\tilde{\eta}$ is formed by taking trivial bundles over $D^n_\pm \times Y$ and the framings $F_\pm$ and identifying for all $(x, y) \in S^{n-1}_+ \times Y$

$$F_+(x, y) \equiv F_-(x, \delta(x)y)M(\delta(x), y). \tag{2.3}$$

Had we used F_β we could express this as

$$F_{+\beta}(x, y) \equiv F_{-\beta}(x, \delta(x)y)M_\beta(\delta(x), y).$$

Note that the function $M \circ \delta \times \mathrm{id} : S^{n-1} \times Y \to O(m)$ shows exactly how the lifted bundle is determined by ξ (the function δ) and the way G acts on η (the function M).

3. The main theorem

Our technique will be to take a lifted bundle $\tilde{\eta} \in [E, BO(m)]$ and try to pull it back twice in the Wang sequence, the relevant section of which is

$$[SY, BO(m)] \xrightarrow{S\gamma^* - Sp^*} [S^nY, BO(m)] \oplus [S^n, BO(m)]$$

$$\xrightarrow{\varphi^*} [E, BO(m)] \xrightarrow{j^*} [Y, BO(m)].$$

As $j^*\tilde{\eta} = \eta$ we see the necessity for assuming η to be a trivial bundle.

We let $\theta : [X, O(m)] \to [SX, BO(m)]$ be the natural transformation of functors which makes correspond the characteristic map of an m-plane bundle over SX and its classifying map. If $\theta[\beta] = [f]$ then the bundle over SX classified by $f : SX \to BO(m)$ can be constructed using β as a clutching function.

THEOREM 1. *The set* $\mathcal{S} = \{\theta[M_\beta \circ \delta \times \mathrm{id}]\}$, *where* $\beta \in [Y, O(m)]$, *is a coset of the image of* $S\gamma^* - Sp^*$ *in* $[S^nY, BO(m)] \oplus [S^n, BO(m)]$ *which maps to* $\tilde{\eta}$ *under* φ^*.

COROLLARY 2. *The bundle* $\tilde{\eta}$ *is trivial if and only if* $[M \circ \delta \times \mathrm{id}]$ *is in the image of*

$$\gamma^* - p^* : [Y, O(m)] \to [S^{n-1}Y, O(m)] \oplus [S^{n-1}, O(m)].$$

Proof of the corollary. This follows using the commutativity of the diagram:

$$\begin{array}{ccc} [SY, BO(m)] & \xrightarrow{S\gamma^* - Sp^*} & [S^nY, BO(m)] \oplus [S^n, BO(m)] \\ \uparrow \theta & & \uparrow \theta \\ [Y, O(m)] & \xrightarrow{\gamma^* - p^*} & [S^{n-1}Y, O(m)] \oplus [S^{n-1}, O(m)]. \end{array}$$

Proof of the theorem. First we observe by (2.1) that $\mathcal{S}$ does lie in the indicated subgroup of $[S(S^{n-1} \times Y), BO(m)]$. That $\mathcal{S}$ is a coset follows from formula (2.2).

To show $\varphi^*\theta[M \circ \delta \times \mathrm{id}]$ classifies $\tilde{\eta}$, we use the following definition of φ as the composition of three maps, $\varphi = w \circ v \circ u$. Let $\bar{S}X$ and $\bar{C}X$ denote the unreduced suspension and cone on X, respectively. Then let

$$u : E \to E/(Y, Y_+)$$

be the map which shrinks the fiber Y over the south pole to a point and the fiber Y_+ over the north pole to another point;

$$v : E/(Y, Y_+) \to \bar{S}(S^{n-1} \times Y)$$

is the identity on the top cone $\bar{C}_+(S^{n-1} \times Y)$ but "untwists" the bottom cone $\bar{C}_-(S^{n-1} \times Y)$, mapping $S^{n-1} \times Y \times I$ to itself by $(x, y, t) \to (x, \delta(x)^{-1}y, t)$;

$$w : \bar{S}(S^{n-1} \times Y) \to S(S^{n-1} \times Y)$$

shrinks the line to the basepoint.

Remark. In [7] we used a different definition of φ because we had the total space of the bundle defined as $D^n \times Y$ with $S^{n-1} \times Y$ identified with Y by means of γ. Call this space E' for the moment and let φ' be the map φ of [7]. It is straightforward but tedious to validate our definition of φ by defining a homeomorphism $f : E' \to E$ and proving the following diagram homotopy commutative:

$$\begin{tikzcd}
& E' \arrow[dl, "\varphi'"'] \arrow[dd, "f"] & \\
S(S^{n-1}\times Y) & & Y \arrow[ul, "j"'] \arrow[dl, "j"] \\
& E \arrow[ul, "\varphi"] &
\end{tikzcd} .$$

The proof is skipped.

Now $\theta[M \circ \delta \times \mathrm{id}]$ classifies the bundle over $S(S^{n-1} \times Y)$ formed by identifying

$$F'_+(x, y) \equiv F'_-(x, y)M(\delta(x), y),$$

where $F'_\pm$ frames the trivial bundle over $C_\pm(S^{n-1} \times Y)$. The pull-back to $E/(Y, Y_+)$ is formed by identifying

$$F''_+(x, y) \equiv F''_-(x, \delta(x)y)M(\delta(x), y),$$

where $F''_\pm$ are the framings of the trivial bundles over $\bar{C}_\pm(S^{n-1} \times Y)$ induced by $w \circ v$ from $F'_\pm$. Finally we get framings $F'''_\pm$ of the trivial bundles over $D^n_\pm \times Y$ induced by u from $F''_\pm$ and form the induced bundle over E by identifying

$$F'''_+(x, y) \equiv F'''_-(x, \delta(x)y)M(\delta(x), y).$$

As all framings of the trivial bundle over $D_{\pm}^n \times Y$ are given by $F_{\pm\beta}$ for some β, an equivalent bundle is gotten by identifying

$$F_{+\beta}(x, y) \equiv F_{-\beta}(x, \delta(x)y)M(\delta(x), y),$$

or by definition of F_β and homotopy commutativity,

$$F_+(x, y) \equiv F_-(x, \delta(x)y)M(\delta(x), y)\beta(\delta(x)y)\beta(y)^{-1}.$$

But $\beta(\delta(x)y)\beta(y)^{-1}$ represents $\varphi^* \circ \theta \circ (\gamma^* - p^*)[\beta]$, so is homotopic to the identity by exactness of the Wang sequence. Thus by (2.3) we have constructed $\tilde{\eta}$. This completes the proof.

4. Applications

We now assume Y and E to be compact, differentiable manifolds, Y of dimension r and stably parallelizable, and that G acts as a group of diffeomorphisms of Y. Theorem 1 will allow us to study the stable tangent bundle to E whenever we can compute the function $M : G \times Y \to O(m)$. The concept needed is that of an equivariant framed embedding, which we now define.

There exists an embedding ψ of Y into some R^m which is equivariant with respect to some representation $\rho : G \to O(m)$; i.e. is such that $\psi(gy) = \rho(g)\psi(y)$ [6]. Furthermore as Y is stably parallelizable, by choosing larger m if necessary we can be sure that the normal bundle ν_ψ has a framing [4, Lemma 3.3]. The elements g of G have differentials g_* which act on the tangent space $T(Y)$. Similarly $\rho(g)_*$ acts on $T(R^m)$. Thus we define actions of G on $\tau(Y)$ and ν_ψ. We shall say ψ is an *equivariant framed embedding* if there exists a framing of ν_ψ such that the associated function M is the constant map to the identity of the group.

To state the theorem we need one more map $\sigma : G \to Y$. If $y_0 \in Y$ is the basepoint, $\sigma(g) = gy_0$.

THEOREM 3. *If there exists an equivariant framed embedding of Y into R^m relative to some representation $\rho : G \to O(m)$, then the stable tangent bundle $\{\tau(E)\} = 0$ if and only if $\{\rho \circ \delta\}$ is contained in the image of*

$$(\sigma \circ \delta)^* : [Y, \mathbf{O}] \to [S^{n-1}, \mathbf{O}].$$

Proof. The equivariance of the framed embedding ψ means we can lift $\tau(Y) \oplus \nu_\psi$ to $\{\tau(E)\}$. The framed embedding itself allows us to compute the required function M; that is, if F_0 is the standard framing of $\tau(R^m)$, F is the induced framing of $\tau(Y) \oplus \nu_\psi$, and M is defined relative to F, then $M(g, y) = \rho(g)$. For let $N(Y)$ be the total space of ν_ψ and

$$\psi' : T(Y) \oplus N(Y) \to T(R^m)$$

be the bundle map. The definition of M then says that for all g and y,

$$\psi' g_* F(y) = \psi' F(gy)M(g, y).$$

By the equivariance of ψ and the definition of F this is

$$\rho(g)_* F_0(\psi(y)) = F_0(\psi(gy))M(g, y).$$

By the linearity of $\rho(g)$ and equivariance again,

$$F_0(\rho(g)\psi(y))\rho(g) = F_0(\rho(g)\psi(y))M(g, y),$$

so $M(g, y) = \rho(g)$.

Let $i_1 : S^{n-1} \to S^{n-1} \times Y$ be the inclusion $i_1(x) = (x, y_0)$. As $[M \circ \delta \times \mathrm{id}]$ is independent of Y, it can be considered to lie in $[S^{n-1}, O(m)]$ where it is equal to $[\rho \circ \delta]$ and where it is hit by $i_1^* \circ (\gamma^* - p^*) = (\sigma \circ \delta)^*$ if and only if $\{\tau(E)\} = 0$ by Corollary 2. Making the groups stable, we complete the proof.

Corollary 4. *Let $(E, \pi, S^n, V_{q,k}, O(q))$ be a bundle characterized by $\delta : S^{n-1} \to O(q)$. Then $\{\tau(E)\} = 0$ if and only if $k\{\delta\}$ is contained in the image of $(\sigma \circ \delta)^* : [V_{q,k}, \mathbf{O}] \to [S^{n-1}, \mathbf{O}]$.*

Proof. An embedding ψ of $V_{q,k}$ in R^{qk} puts the columns of the matrix of an element x of $V_{q,k}$ over each other into one long column vector of R^{qk}. This embedding is equivariant with respect to the diagonal representation $\Delta_k : O(q) \to O(qk)$. Hsiang and Szczarba in [3, p. 700] display a set of vectors $s_{ij}(x)$, $1 \leq i \leq j \leq k$, which frame the normal space to $\psi(V_{q,k})$ at $\psi(x)$. It is easy to see by direct computation that $\Delta_k(g)_* s_{ij}(x) = s_{ij}(gx)$, hence the framing is equivariant. Finally, $\Delta_k \circ \delta$ is homotopic to $k\delta$. This completes the proof.

Corollary 5. *Let $(E, \pi, S^n, V_{q,k}, O(q))$ be a bundle characterized by δ. Then $\{\tau(E)\} = 0$ if*

(i) $n \equiv 3, 5, 6, 7 \pmod 8$,
(ii) $n \equiv 1, 2 \pmod 8$ *and k is even,*
(iii) $n \equiv 0, 4 \pmod 8$ *and $q \leq n/2$.*

Also in the following cases $\{\tau(E)\} = 0$ if and only if $k\{\delta\} = 0$:

(iv) $n = 4s,\ s > 4,\ q > n,\ q - k \geq 2s + 1$,
(v) $n = 8s + 1,\ s > 2,\ q > n,\ q - k \geq 4s + 2$,
(vi) $n = 8s + 2,\ s > 2,\ q > n,\ q - k \geq 4s + 2$.

Proof. (i). Here $\{\delta\} = 0$ as $\pi_{n-1}(\mathbf{O}) = 0$.

(ii). Here $k\{\delta\} = 0$ as $\pi_{n-1}(\mathbf{O}) = Z_2$ and k is even.

(iii). Here $\{\delta\} = 0$ as is seen by looking at the top Pontrjagin class of the bundle classified by δ.

(iv), (v) and (vi). These follow directly from the following theorem of Barratt and Mahowald [1]:

Theorem. *A non-trivial stable real vector bundle over S^{4s} is the sum of a $(2s + 1)$-plane bundle and a trivial bundle if $s > 4$; over S^{8s+1} is the sum of a*

$(4s + 2)$-plane bundle and a trivial bundle if $s > 2$; over S^{8s+2} is the sum of a $(4s + 2)$-plane bundle and a trivial bundle if $s > 2$.

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The University of California
Berkeley, California
Northwestern University
Evanston, Illinois