TENSOR PRODUCTS AND COMPACT GROUPS

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1. Introduction

In [3], there was defined a tensor product of H^*-algebras A and B that are modules over another H^*-algebra C. In this paper the tensor product is redefined in a slightly different fashion, its structure is discussed, and the special case in which the algebras are group algebras of compact groups is investigated in detail.

2. Tensor products

Proposition 1 and Theorem 1 of this section have analogues in §2 of [3]. The proofs are in each case similar to and, in fact, somewhat simpler than those in [3], and so they are omitted. Throughout the section A, B, and C denote H^*-algebras, A is a right C-module and B is a left C-module.

DEFINITION. $F(A, B)$ is the free algebra over $\mathbf{C}$ generated by $A \times B$, i.e. $F(A, B)$ is the collection of all (finite) formal sums of the form $\sum_{i=1}^{n} \lambda_i(a_i, b_i)$, $\lambda_i \in \mathbf{C}$, $a_i \in A$, and $b_i \in B$, with the usual operations. $F(A, B)$ is also a pseudo-inner product space if we define

$$((a_1, b_1), (a_2, b_2)) = (a_1, a_2)\ (b_1, b_2)$$

and extend by linearity.

Denote by I_1' the ideal in $F(A, B)$ spanned by the set of all elements of the following forms:

(1) $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$,
(2) $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$,
(3) $\lambda(a, b) - (\lambda a, b)$, and
(4) $\lambda(a, b) - (a, \lambda b)$.

Denote by I_2' the ideal in $F(A, B)$ spanned by the set of all elements of the form

(5) $(ac, b) - (a, cb)$, $c \in C$.

Then set $I' = I_1' + I_2'$, the ideal spanned by all elements of the forms (1) through (5).

PROPOSITION 1. $I_1' = \{X \in F(A, B) : (X, X) = 0\}$.

$F(A, B)$ is a pseudo-normed space, with $\| X \|^2 = (X, X)$. Denote by $\mathfrak{F}(A, B)$ its pseudo-normed completion, i.e. all Cauchy sequences from

Received October 17, 1966.

$F(A, B)$. All the operations on $F(A, B)$ extend to $\mathfrak{F}(A, B)$ and $\mathfrak{F}(A, B)$ is a complete pseudo-normed algebra over $\mathbf{C}$. Let I_1, I_2, and I denote the closures in $\mathfrak{F}(A, B)$ of I_1', I_2', and I', respectively. By Proposition 1, I_1 is the closure of (0), and so $I = I_2$ is the closed ideal of $\mathfrak{F}(A, B)$ spanned by all elements of the form (5).

DEFINITION. $A \otimes_C B$, the tensor product of A and B over C, is the quotient algebra $\mathfrak{F}(A, B)/I$. We denote the element $(a, b) + I$ by $a \otimes b$.

THEOREM 1. *$A \otimes_C B$ is isometric and isomorphic with a closed ideal E in $A \otimes B$; E is the orthogonal complement of the ideal D spanned by all elements of the form $ac \otimes b - a \otimes cb$, $a \in A$, $b \in B$, $c \in C$.*

COROLLARY 1. *$A \otimes_C B$ is an H^*-algebra, its minimal closed ideals can be identified with those minimal closed ideals of $A \otimes B$ that are orthogonal to D.*

COROLLARY 2. *If A and B are strongly semi-simple, then $A \otimes_C B$ is strongly semi-simple.*

It should be pointed out that $A \otimes_C B$, as defined here, is not necessarily isomorphic with the algebra defined in [3]. Suppose, for example, that A, B, and C are closed ideals in an H^*-algebra $\mathfrak{A}$. If M denotes the direct sum of all the one-dimensional minimal ideals in $A \cap B \cap C$, then computations similar to those following Proposition 3 of [3] show that $A \otimes_C B$ is isomorphic with

$$M \oplus ((A \cap C^{\perp}) \otimes (B \cap C^{\perp})).$$

In particular, if M is the direct sum of all the one-dimensional ideals in A, then $A \otimes_A A \cong M$.

We show next that $A \otimes_C B$ can be characterized in terms of certain universal mapping properties (the development here parallels that in §12 of [1]).

DEFINITION. If H is an H^*-algebra, a mapping $\varphi : A \times B \to H$ is called *balanced* if and only if it satisfies

(1) φ is bilinear,
(2) $\varphi(\lambda_1 a_1, b_1) = \varphi(a_1, \lambda_1 b_1) = \lambda_1 \varphi(a_1, b_1)$,
(3) $\varphi(a_1 a_2, b_1 b_2) = \varphi(a_1, b_1)\varphi(a_2, b_2)$,
(4) $\varphi(a_1 c, b_1) = \varphi(a_1, cb_1)$, and
(5) $\| \sum_{i=1}^{n} \lambda_i \varphi(a_i, b_i) \|^2 \leq \sum_{i,j} \lambda_i \bar{\lambda}_j (a_i, a_j)(b_i, b_j)$

for all λ_i, a_i, b_i, and c.

PROPOSITION 2. *The map $t : A \times B \to A \otimes_C B$, defined by $t(a, b) = a \otimes b$, is a balanced map, and linear combinations of elements in the range of t are dense in $A \otimes_C B$.*

Proof. Conditions (1)–(4) for balanced maps obviously hold for t. As for

(5), we have

$$\begin{aligned}\|\textstyle\sum \lambda_i t(a_i, b_i)\|^2 &= \|\textstyle\sum \lambda_i(a_i \otimes b_i)\|^2\\ &= \inf\{\|\textstyle\sum \lambda_i(a_i, b_i) + X\|^2 : X \in I\}\\ &\le \|\textstyle\sum \lambda_i(a_i, b_i)\|^2\\ &= \textstyle\sum \lambda_i \bar{\lambda}_j (a_i, a_j)(b_i, b_j).\end{aligned}$$

The second statement is obvious since the elements $\sum \lambda_i t(a_i, b_i)$ comprise the image of $F(A, B)$ under the quotient map.

THEOREM 2. *If $\varphi : A \times B \to H$ is a balanced map, and t is the map defined in Proposition 2, then there is a unique continuous homomorphism*

$$\varphi^* : A \otimes_C B \to H$$

such that $\varphi = \varphi^* t$. Conversely, if T is an H^*-algebra and $t_1 : A \times B \to T$ is a balanced map with the properties that every balanced map $\varphi : A \times B \to H$ "factors through" T via t_1 (as above), and that linear combinations of range elements of t_1 are dense in T, then T is isomorphic and isometric with $A \otimes_C B$.

Proof. Extend φ to a mapping φ' on all of $F(A, B)$ by defining

$$\varphi'(\textstyle\sum \lambda_i(a_i, b_i)) = \textstyle\sum \lambda_i \varphi(a_i, b_i).$$

Since φ is balanced, φ' is easily seen to be an algebra homomorphism on $F(A, B)$. By the definition of the pseudonorm on $F(A, B)$, condition (5) for balanced maps simply says that φ' is bounded, with bound at most one. Conditions (1) through (4) insure that $\varphi' \mid I' = 0$. Since φ' is continuous it extends uniquely to a homomorphism on $\mathfrak{F}(A, B)$ to H that vanishes on I and has the same bound. As a result, φ' gives rise to a continuous homomorphism

$$\varphi^* : \mathfrak{F}(A, B)/I = A \otimes_C B \to H,$$

defined by $\varphi^*(X + I) = \varphi'(X)$, again with the same bound (see [5, p. 16]). Observe that

$$\varphi^* t(a, b) = \varphi^*((a, b) + I) = \varphi'(a, b) = \varphi(a, b),$$

and also that the uniqueness of φ^* follows from the fact that linear combinations of the elements in the range of t are dense in $A \otimes_C B$.

As for the converse, we have homomorphisms

$$t^* : T \to A \otimes_C B \quad \text{and} \quad t_1^* : A \otimes_C B \to T,$$

each with bound at most one, such that $t = t^* t_1$ and $t_1 = t_1^* t$. Thus $t_1 = t_1^* t^* t_1$ and $t = t^* t_1^* t$, and so $t_1^* t^*$ and $t^* t_1^*$ are both identity maps when restricted to the linear spans of the ranges of t_1 and t, respectively. Since these are dense, t^* is an isometric isomorphism on T onto $A \otimes_C B$ (isometry is immediate since $\|t^* X\| \le \|X\| = \|t_1^* t^* X\| \le \|t^* X\|$, all $X \in T$).

3. Group algebras of compact groups

Suppose G, H, and K are compact groups. Let us denote elements of the roup algebra $L^2(G)$ by $g, g_1, g_2, \cdots$, elements of $L^2(H)$ by $h, h_1, \cdots$, and elements of $L^2(K)$ by $k, k_1, \cdots$. Denote by gh that function on $G \times H$ whose value at (x, y) is $g(x)h(y)$.

Suppose $\theta : K \to G$ and $\varphi : K \to H$ are continuous homomorphisms. For example, G and H might be subgroups of some common group, K a closed subgroup of $G \cap H$, and θ and φ inclusion maps. As another example, K might be a closed subgroup of $G \times H$ and θ and φ the restrictions to K of projection maps into G and H. Module actions of $L^2(K)$ on $L^2(G)$ and $L^2(H)$ can be defined as follows:

$$(g*k)(x) = \int_K g(x\theta z^{-1})k(z)\, dz,$$

and

$$(k*h)(y) = \int_K k(z)h((\varphi z^{-1})y)\, dz$$

for all $x \in G$, $y \in H$.

As was observed in [3], the map $gh \to g \otimes h$ extends to an isometric isomorphism on $L^2(G \times H)$ onto $L^2(G) \otimes L^2(H)$. Thus if we set $A = L^2(G)$, $B = L^2(H)$, and $C = L^2(K)$ we have, by Theorem 1, that $A \otimes_C B$ is isomorphic and isometric with the ideal J of $L^2(G \times H)$ that is the orthogonal complement of the ideal generated by all functions of the form $(g*k)h - g(k*h)$. If $F \in J$, then

$$((g*k)h - g(k*h), F) = 0$$

for all g, h, and k. In other words

$$\int k(z) \iint (g(x\theta z^{-1})h(y) - g(x)h((\varphi z^{-1})y))\overline{F(x,y)}\, dy\, dx\, dz = 0$$

for all k, and so

$$\iint g(x\theta z^{-1})h(y)\overline{F(x,y)}\, dy\, dx = \iint g(x)h((\varphi z^{-1})y)\overline{F(x,y)}\, dy\, dx.$$

Changing variables, we have

$$\iint g(x)h(y)\overline{F(x\theta z, y)}\, dydx = \iint g(x)h(y)\overline{F(x, (\varphi z)y)}\, dy\, dx,$$

or $(gh, F^{(\theta z, e)}) = (gh, F_{(e, \varphi z)})$, where, in general, $f^u(v) = f(vu)$ and $f_u(v) = f(uv)$. Each equality holds for all g and h, and almost every $z \in K$. Since linear combinations of the functions gh are dense in $L^2(G \times H)$, it follows that $F^{(\theta z, e)} = F_{(e, \varphi z)}$, i.e. that $F(x\theta z, y) = F(x, (\varphi z)y)$ for almost every pair $(x, y) \in G \times H$ and almost all $z \in K$. The next theorem asserts that this property characterizes J when θ and φ are central.

THEOREM 3. *If θK and φK are subgroups of the centers of G and H, respectively, then $A \otimes_C B$ is isometric and isomorphic with the ideal in $L^2(G \times H)$ consisting of all functions F such that $F^{(\theta z, e)} = F_{(e, \varphi z)}$ for almost all $z \in K$.*

Proof. Since θ and φ are central, it is easily seen that $(g*k)_x = g_x*k$ and $(g*k)^x = g^x*k$. Thus

$$((g*k)h - g(k*h))_{(x,y)} = (g_x*k)h_y - g_x(k*h_y),$$

$$((g*k)h - g(k*h))^{(x,y)} = (g^x*k)h^y - g^x(k*h^y),$$

and the closed linear subspace L of $L^2(G \times H)$ spanned by all

$$(g*k)h - g(k*h)$$

is translation invariant. It follows that L is an ideal (see [5, p. 125]), and hence that $L = J^{\perp}$. Thus in order to show that $F \in J$ it suffices to show that $(F, (g*k)h) = (F, g(k*h))$ for all g, h, and k.

Suppose then that $F \in L^2(G \times H)$ and that $F^{(\theta z, e)} = F_{(e, \varphi z)}$ for almost all $z \in K$. Then

$$\begin{aligned}((g*k)h, F) &= \iiint g(x\theta z^{-1})k(z)h(y)\overline{F(x, y,)}\, dz\, dx\, dy \\ &= \iiint g(x)k(z)h(y)\overline{F(x\theta z, y)}\, dx\, dy\, dz \\ &= \iiint g(x)k(z)h(y)\overline{F(x, y(\varphi z)y)}\, dy\, dx\, dz \\ &= \iiint g(x)k(z)h((\varphi z^{-1})y)\overline{F(x, y)}\, dy\, dx\, dz \\ &= (g(k*h), F) \qquad \text{for all } g, h, \text{ and } k.\end{aligned}$$

Thus $F \in J$ and the theorem is proved.

For G and H compact, the next theorem is a generalization of Theorem 4.1 in [2].

THEOREM 4. *If θ and φ are central, then $A \otimes_C B$ is isomorphic and isometric with $L^2((G \times H)/Q)$, where Q is a closed normal subgroup of $G \times H$.*

Proof. Define Q to be the set of all pairs $(\theta z, \varphi z^{-1})$, $z \in K$. Since θ and φ are continuous and central, it is immediate that Q is a closed normal subgroup of $G \times H$. If $F \in J$ then F is (essentially) constant on the cosets of Q, for if

$$(x, y) = (u\theta z, v\varphi z^{-1}) = (x\theta z, (\varphi z^{-1})y),$$

then

$$F(x, y) = F(u\theta z, (\varphi z^{-1})v) = F(u, \varphi z(\varphi z^{-1})v) = F(u, v).$$

Suppose, conversely, that $F \in L^2(G \times H)$ is constant on the cosets of Q. Then $F(x\theta z, y) = F(x\theta z\theta^{-1}z, (\varphi z)y) = F(x, (\varphi z)y)$, and $F \in J$.

Let us denote by m_1 and m_2 the normalized Haar measures on $G \times H$ and $(G \times H)/Q$, respectively. The discussion of "quotient measures" in §33 of [5] shows that the map $F \to F^{\#}$, where $F^{\#}((x, y)Q) = F(x, y)$, is a 1-1 linear map from the collection of continuous functions in J onto the set of all continuous functions on $(G \times H)/Q$. Furthermore,

$$\int F(x, y)\, dm_1(x, y) = \int F^{\#}((x, y)Q)\, dm_2((x, y)Q)$$

for all continuous $F \in J$ (that the measures are correctly normalized becomes apparent upon integration of a constant function). It follows immediately, since the norms and algebra products are defined in terms of integrals, that the map $F \to F^{\#}$ extends to an isometric algebra isomorphism on J onto $L^2((G \times H)/Q)$. Since J and $A \otimes_C B$ were identified in Theorem 3, the proof is completed.

As an example of the sort of situation to which Theorem 4 might apply, suppose that $\mathcal{G}$ is a finite-dimensional compact connected group. It is shown in [6, p. 479] that we may assume $\mathcal{G} = (G \times H)/K$, where G is a simply connected, compact, semi-simple Lie group, H is a finite-dimensional, compact, connected Abelian group, and K is a finite normal subgroup of $G \times H$. By the Pontrjagin Duality Theorem, H can be described algebraically as follows: it is the dual of a (discrete) torsion-free Abelian group of finite rank (see [4, pp. 385–386]). For $(x, y) \in K$ define $\theta(x, y) = x$ and $\varphi(x, y) = y^{-1}$. Since K is finite and $G \times H$ is connected, K is in the center of $G \times H$, and so θ and φ are central homomorphisms. By Theorem 4, $L^2((G \times H)/Q))$. is isometric and isomorphic with $A \otimes_C B$, with $A = L^2(G)$, $B = L^2(H)$, and $C = L^2(K)$. But

$$Q = \{(\theta(x, y), \varphi(x, y)^{-1}) : (x, y) \in K\} = \{(x, (y^{-1})^{-1}) : (x, y) \in K\} = K$$

in this case, and so we have $L^2(\mathcal{G})$ isomorphic and isometric with $A \otimes_C B$. As a result, all irreducible representations S of $\mathcal{G}$ over $\mathbf{C}$ may be obtained (to within equivalence) in the following manner. Choose an irreducible representation T of G and a character α of H with the property that $T(x) = \alpha(y)I$ for each of the finitely many pairs $(x, y) \in K$. Then set $S((x, y)K) = \alpha(y)T(x)$ for each $(x, y)K \in \mathcal{G}$.

It is not known whether the requirement that θ and φ be central is essential in Theorem 4. If the requirement were to be dropped then Q would have to be redefined, probably as the closed normal subgroup generated by the set of all pairs $(\theta z, \varphi z^{-1})$, $z \in K$. With that definition of Q the conclusion of theorem 4 can be shown to hold in one special case where θ and φ may be highly noncentral. Suppose, in fact, that $G = H = K$, and $\theta = \varphi$ is the identity map. As observed in §2 above, $A \otimes_C B$ is then isometric and isomorphic with the direct sum M of all one-dimensional minimal ideals in $L^2(G)$. This in turn can be identified with $L^2(G/G')$, where G' is the *closure* of the commutator subgroup of G. Finally, $(G \times G)/Q$ is topologically isomorphic with G/G', and so $A \otimes_C B \cong L^2((G \times G)/Q)$.

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