QUADRATIC MAPS AND STABLE HOMOTOPY GROUPS OF SPHERES

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The original proof [4] of the Bott periodicity theorem used Morse theory. Recent work on this theorem has, however, been algebraic in nature [1], [2], [3], [7]. The new proofs of the Bott periodicity theorem center around showing that a stable homotopy class can be represented by a specially simple sort of polynomial map. In [1] Atiyah and Bott ask whether it might be possible to use this approach on other homotopy problems. Is there, for example, some specially simple class of polynomial maps which carries the stable homotopy of spheres? As a possible first step towards selecting such a class we shall indicate that probably one wants to examine the properties of quadratic maps. In detail, we shall show that:

1. The stable J-homomorphism can be interpreted as an algebraic operation which converts a linear map into a quadratic map.
2. Any element of a k-stem can be represented by a quadratic map $q : R^n \to R^l$ such that $q(S^{n-1}) \subset R^l - \{0\}$.

In view of these results it is not surprising that many classical examples of non-trivial maps from S^n to S^k are quadratic. The Hopf map $S^3 \to S^2$ is, for instance, given by

$$(x_1, x_2, x_3, x_4) \to (2x_1 x_3 - 2x_2 x_4, 2x_1 x_4 + 2x_2 x_3, -x_1^2 - x_2^2 + x_3^2 + x_4^2).$$

We remark that the results given here are mainly suggestive. No actual computations of k-stems are done. But perhaps eventually the k-stem may be envisaged as a group of equivalence classes of quadratic forms.

Our main technical lemma is 2.9. The proof of this lemma describes a procedure for lowering the degree of a polynomial map. This procedure resembles the linearization procedure of [1].

The general idea of this paper is due to M. F. Atiyah. It is a pleasure to thank Professor Atiyah for several most enjoyable conversations. Thanks go also to N. Steenrod for his interest and comments.

R. Wood has also, independently, proved that any element in a k-stem can be represented by a quadratic map.

1. The J-Homomorphism

1.1. *Notation.* R = the real numbers.

R^n = the space of all n-tuples a, $a = (a_1, \cdots, a_n) a_i \in R$, $\| a \| = (a_1^2 + \cdots + a_n^2)^{1/2}$.

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$S^{n-1} = \{a \in R^n \mid \| a \| = 1\}$.

$\pi_k = \lim_{n\to\infty} \pi_{k+n}(S^n)$.

$L(n, m)$ = the vector space of all $n \times m$ matrices of real numbers.

$O(n)$ = the group of all $n \times n$ orthogonal matrices.

$\pi_k(O) = \lim_{n\to\infty} \pi_k(O(n))$.

C_k = the kth Clifford algebra over R. This is an algebra with generators $e_1, e_2, \cdots, e_k$ subject to the relations $e_i^2 = -1$, $e_i e_j + e_j e_i = 0$ if $i \neq j$. For Clifford algebras see [2]. As a vector space over R, C_k has dimension 2^k.

R^{k+1} shall be identified with the subspace of C_k spanned by $1, e_1, \cdots, e_k$. The identification is given by

$$(a_1, a_2, \cdots, a_{k+1}) \leftrightarrow a_1 \cdot 1 + \sum_{i=1}^{k} a_{i+1} \cdot e_i .$$

$R^n \times R^l$ shall be identified with R^{n+l}. The identification is given by

$$(a, b) \leftrightarrow (a_1, \cdots, a_n, b_1, \cdots, b_l)$$

where $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_l)$.

All rings are associative and have a unit element 1. 1 acts as the identity on all modules. If Λ is a ring then $\Lambda[X_1, \cdots, X_n]$ denotes the ring of all polynomials in the indeterminates $X_1, \cdots, X_n$ with coefficients in Λ. If Λ_1 and Λ_2 are rings with $\Lambda_1 \subset \Lambda_2$ and W is a Λ_2-module, then $W \mid \Lambda_1$ denotes the Λ_1-module obtained by considering W to be a Λ_1-module.

1.1. LEMMA. *Let Λ_1 be a ring such that every Λ_1-module is projective. Let Λ_2 be a ring with $\Lambda_1 \subset \Lambda_2$ and suppose that as a Λ_1-module Λ_2 is finitely generated. Then if V is any finitely generated Λ_1-module there exists a finitely generated Λ_1-module U and a finitely generated Λ_2-module W such that $W \mid \Lambda_1$ is isomorphic to $V \oplus U$.*

Proof. Let $W = \Lambda_2 \otimes_{\Lambda_1} V$.

1.2. DEFINITION OF H_k. Identifying $e_i \in C_k$ with $e_i \in C_{k+1}$, $i = 1, 2, \cdots, k$ gives an inclusion $C_k \subset C_{k+1}$. If V_1 and V_2 are two finitely generated C_k-modules set $V_1 \sim V_2$ if and only if there exist finitely generated C_{k+1}-modules W_1 and W_2 such that

$$V_1 \oplus (W_1 \mid C_k) \cong V_2 \oplus (W_2 \mid C_k).$$

H_k is the set of equivalence classes. The operation of forming the direct sum of two modules makes H_k into an abelian semi-group. The existence of an additive universe follows from the preceding lemma, so H_k is an abelian group.

1.3. LEMMA. *If V is a finitely generated C_k-module then it is possible to choose a norm for V such that $\| av \| = \| a \| \cdot \| v \|$ for all $a \in R^{k+1}$, $v \in V$.*

Proof. Let Pin (k) be as in [2]. Pin (k) is a compact group. Let $\langle\ ,\ \rangle$

be any inner product for V and set

$$\langle v_1, v_2\rangle' = \int_{\mathrm{Pin}(k)} \langle xv_1, xv_2\rangle \, dx,$$

where the integral is taken with respect to the normalized Haar measure on Pin (k). Then for any $x \in$ Pin (k), $\langle xv_1, xv_2\rangle' = \langle v_1, v_2\rangle'$. It may now be easily checked that in fact $\langle xv_1, xv_2\rangle' = \langle v_1, v_2\rangle'$ for every $x \in S^k$.

1.4. Definition of a homomorphism $\eta : H_k \to \pi_k(O)$. Given a finitely generated C_k-module V choose a norm for V such that $\| av \| = \| a \| \cdot \| v \|$ for all $a \in R^{k+1}$, $v \in V$. Let $v_1, \cdots, v_n$ be an orthonormal R-basis for V. For each $a \in R^{k+1}$ let $T(a)$ be the matrix of $v \to av$ with respect to the basis $v_1, \cdots, v_n$. Then $T : R^{k+1} \to L(n, n)$ is a linear map of R-vector spaces. Since $T(S^k) \subset O(n)$, T determines an element of $\pi_k(O)$. It is straightforward to verify that:

(i) This element of $\pi_k(O)$ depends only on the isomorphism class of V.
(ii) If $\eta(V)$ denotes the element of $\pi_k(O)$ determined by V, then

$$\eta(V_1 \oplus V_2) = \eta(V_1) + \eta(V_2).$$

(iii) If W is any finitely generated C_{k+1}-module, then $\eta(W \mid C_k) = 0$.

So η defines a group homomorphism $H_k \to \pi_k(O)$.

1.5. Theorem (Atiyah, Bott, Shapiro [2]). *For all $k \geqq 0$, $\eta : H_k \to \pi_k(O)$ is an isomorphism.*

Proof. The theorem is verified in three steps:
(i) Periodicity homomorphisms $\pi_k(O) \to \pi_{k+8}(O)$, $H_k \to H_{k+8}$ are defined and proved to be isomorphisms.
(ii) For $k = 0, 1, \cdots, 7$, $\eta : H_k \to \pi_k(O)$ is proved to be an isomorphism.
(iii) The diagram

$$\begin{array}{ccc} H_k & \xrightarrow{\eta} & \pi_k(O) \\ \downarrow & & \downarrow \\ H_{k+8} & \xrightarrow[\eta]{} & \pi_{k+8}(O) \end{array}$$

is proved to be commutative where the vertical arrows are the periodicity isomorphisms.

A corollary of the theorem is:

1.6. Corollary. *Any element of $\pi_k(0)$ can be represented by a linear map T of R-vector spaces $T : R^{k+1} \to L(n, n)$ such that $T(S^k) \subset O(n)$.*

1.7. *The Hopf construction* [6]. Let $f : S^k \times S^{n-1} \to S^{n-1}$ be a continuous map. Extend f to a continuous map

$$\tilde{f} : R^{k+1} \times R^n \to R^n$$

by setting

$$\tilde{f}(t_1 a, t_2 b) = t_1 t_2 f(a, b)$$

for $0 \leqq t_1, t_2 \in R$ and $\| a \| = \| b \| = 1$. Let

$$J(f) : R^{k+1} \times R^n \to R^n \times R^1$$

be

$$J(f)(a, b) = (2\tilde{f}(a, b), \| b \|^2 - \| a \|^2).$$

$J(f)$ maps S^{k+n} into S^n. $J(f)$ is the map obtained by applying the *Hopf construction* to f.

1.8. *The J-homomorphism* [6]. Let $w \in \pi_k(O)$. Choose a map

$$\varphi : S^k \to O(n)$$

representing w. Let $f : S^k \times S^{n-1} \to S^{n-1}$ be $f(a, b) = \varphi(a)(b)$. Then $J(f)$ determines an element of π_k and this operation defines a homomorphism $J : \pi_k(O) \to \pi_k$.

1.9. Definition. A map $f : R^n \to R^l$ is a *polynomial map* if for some l-tuple $(P_1, \cdots, P_l)$, $P_i \in R[X_1, \cdots, X_n]$, $f(a) = (P_1(a), \cdots, P_l(a))$ for all $a \in R^n$. (If f is polynomial then the l-tuple $(P_1, \cdots, P_l)$ is uniquely de-determined by f). The *degree* of a polynomial f is the maximum of the degrees of the P_i. A polynomial map f is *homogeneous* if each P_i is a homogeneous polynomial and degree P_i = degree P_j for all i, j. A polynomial map f is *quadratic* if degree $f \leqq 2$.

1.10. Corollary. *An element of π_k in the image of the J-homomorphism can be represented by a homogeneous quadratic map $q : R^{n+k+1} \to R^{n+1}$ such that $q(S^{n+k}) \subset S^n$.*

Proof. Let $w \in \pi_k(O)$. Choose a linear map T of R-vector spaces

$$T : R^{k+1} \to L(n, n)$$

such that $T(S^k) \subset O(n)$ and T represents w. Then $J(T)$ gives the required q.

Thus the (stable) J-homomorphism may be viewed as an algebraic operation converting linear maps into homogeneous quadratic maps.

2. Quadratic maps

2.1. Definition. A continuous map $f : R^n \to R^l$ is *admissible* if $f(0) = 0$ and $f(a) \neq 0$ for all $a \in S^{n-1}$. If $f : R^n \to R^l$ is admissible then the *i-th suspension* of f, denoted $f^{(i)}$, is the map

$$f^{(i)} : R^n \times R^i \to R^l \times R^i$$

by $f^{(i)}(a, b) = (f(a), b)$.

2.2. *Notation.* $M_{k,n}$ = the space of all admissible maps $f : R^n \to R^{n-k}$, topologized by the compact-open topology. $f \to f^{(1)}$ gives an inclusion $M_{k,n} \subset M_{k,n+1}$.

$M_k = \lim_{n\to\infty} M_{k,n}$. M_k is topologized by the direct limit topology.

$Q_{k,n}$ = the subspace of $M_{k,n}$ consisting of all admissible quadratic maps.

$Q_k = \lim_{n\to\infty} Q_{k,n}$.

$[Y, Z]$ = the set of homotopy classes of continuous maps from the topological space Y to the topological space Z.

2.3. DEFINITION. If Z_1 and Z_2 are topological spaces and $f : Z_1 \to Z_2$ is a continuous map, then f is a *weak homotopy equivalence* if for every compact Hausdorff space Y, f indices a bijection of sets $[Y, Z_1] \to [Y, Z_2]$.

2.4. THEOREM. *For all k, the inclusion $Q_k \subset M_k$ is a weak homotopy equivalence.*

Before the proof, two definitions and three lemmas.

2.5. DEFINITION. Let Y be a topological space. A continuous map

$$f : Y \times R^n \to R^l$$

is *admissible* if $f(y, 0) = 0$ for all $y \in Y$, and $f(y, a) \neq 0$ whenever $a \in S^{n-1}$. The set of all admissible maps from $Y \times R^n$ into R^l can be identified, in the standard fashion [5], with the set of all continuous maps from Y into $M_{n-l,n}$. Two admissible maps $f, g : Y \times R^n \to R^l$ are *ad-homotopic* if as maps of Y into $M_{n-l,n}$ they are homotopic.

2.6. DEFINITION. Let Y be a topological space and let $R(Y)$ denote the ring of all continuous real-valued functions on Y. A map

$$f : Y \times R^n \to R^l$$

is *polynomial* if there exists an l-tuple $(P_1, \cdots, P_l)$,

$$P_i \in R(Y)[X_1, \cdots, X_n]$$

such that

$$f(y, a) = (P_1(y, a), \cdots, P_l(y, a))$$

for all $(y, a) \in Y \times R^n$. (If f is polynomial then $(P_1, \cdots, P_l)$ is uniquely determined by f.) The *degree* of a polynomial f is the maximum of the degrees of $P_1, \cdots, P_l$. If degree $f \leqq 2$, then f is *quadratic*. Two admissible polynomial maps

$$f, g : Y \times R^n \to R^l$$

are *polynomially ad-homotopic* if there exists an admissible polynomial map

$$h : (Y \times [0, 1]) \times R^n \to R^l$$

such that $h(y, 0, a) = f(y, a)$ and $h(y, 1, a) = g(y, a)$ for all $(y, a) \in Y \times R^n$.

Polynomial ad-homotopy is an equivalence relation on the set of all admissible polynomial maps from $Y \times R^n$ to R^l.

2.7. LEMMA. *Let Y be a compact Hausdorff space, and let*

$$f, g : Y \times R^n \to R^l$$

be two admissible polynomial maps. Then f and g are ad-homotopic if and only if they are polynomially ad-homotopic.

Proof. Let $\rho : Y \times [0, 1] \times R^n \to R^l$ be an admissible map such that $\rho(y, 0, a) = f(y, a)$ and $\rho(y, 1, a) = g(y, a)$. By the Stone-Weierstrass approximation theorem there exists an admissible polynomial map

$$h : Y \times [0, 1] \times R^n \to R^l$$

such that h approximates ρ on $Y \times [0, 1] \times S^{n-1}$. Let

$$h_i : Y \times R^n \to R^l$$

be $h_i(y, a) = h(y, i, a)$, $i = 0, 1$. Then h_0 is polynomially ad-homotopic to f. A polynomial ad-homotopy is given by $(1 - t)h_0 + tf$, $0 \leqq t \leqq 1$. Similarly h_1 and g are polynomially ad-homotopic. Since h gives a polynomial ad-homotopy from h_0 to h_1 it now follows that f is polynomially ad-homotopic to g and the lemma is proved.

2.7. *Notation.* $\alpha = (\alpha_1, \cdots, \alpha_n) =$ an n-tuple of non-negative integers.
$| \alpha | = \alpha_1 + \cdots + \alpha_n$.
$X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$.
If Λ is a ring and $\sum_\alpha c_\alpha X^\alpha = P \in \Lambda[X_1, \cdots, X_n]$, then

$$P_i = \sum_{\{\alpha \mid \alpha_i > 0\}} c_\alpha \frac{\alpha_i}{|\alpha|} \frac{X^\alpha}{X_i}.$$

2.8. LEMMA. *If Λ is a ring and $P \in \Lambda[X_1, \cdots, X_n]$ then*

$$P = P(0) + \textstyle\sum_{i=1}^n X_i P_i$$

2.9. LEMMA. *Let Y be a compact Hausdorff space and let*

$$f_1 : Y \times R^n \to R^l$$

be an admissible polynomial map with $2 <$ degree f_1. *Then there is an admissible polynomial map*

$$f_2 : Y \times R^{n+nl} \to R^{l+nl}$$

such that $2 \leqq$ degree $f_2 <$ degree f_1 *and f_2 is polynomially ad-homotopic to $f_1^{(nl)}$.*

Proof. Let $(P_1, \cdots, P_l)$, $P_i \in R(Y)[X_1, \cdots, X_n]$ be the l-tuple determined by f_1. Let A be the $n \times l$ matrix with entries in $R(Y)[X_1, \cdots, X_n]$ given by $A_{ij} = P_{ji}$. Consider each $a \in R^n$ to be a $1 \times n$ matrix and form the matrix product $a \cdot A(y, a)$. Then by Lemma 2.8 $f_1(y, a) = a \cdot A(y, a)$ for all $(y, a) \in Y \times R^n$. For $k = 1, 2, \cdots, n$ let A_k be the $n \times l$ matrix whose (i, j) entry is A_{ijk}. Then $A = A(0) + \sum_{k=1}^{n} X_k A_k$.

Let B be the $n \times nl$ matrix

$$B = \begin{array}{|c|c|c|c|} \hline A_1 & A_2 & \cdots & A_n \\ \hline \end{array}$$

Choose a positive real number λ such that whenever $(y, a) \in Y \times R^n$ has $f_1(y, a) = 0$ and $\| a \| < 1$ then $\| a \|^2 + \lambda^{-2} \cdot \| a \cdot B(y, a) \|^2 < 1$. Let I denote the $l \times l$ identity matrix and let C be the $nl \times l$ matrix

$$C = \begin{array}{|c|} \hline -X_1 \lambda I \\ \hline -X_2 \lambda I \\ \hline \vdots \\ \hline -X_n \lambda I \\ \hline \end{array}$$

Set

$$D = \begin{array}{|c|cccc|} \hline A(0) & & B & & \\ \hline & \lambda I & & & \\ C & & \lambda I & & \\ & & & \ddots & \\ & & & & \lambda I \\ \hline \end{array} = \begin{array}{|c|cccc|} \hline A(0) & A & & & A_n \\ \hline -\lambda X_1 I & \lambda I & & & \\ & & & & \\ & & & & \\ -\lambda X_n I & & & & \lambda I \\ \hline \end{array}$$

Let $f_2 : T \times R^{n+nl} \to R^{l+nl}$ be $f_2(y, a) = a \cdot D(y, a)$. Then f_2 is a polynomial map and degree $f_2 <$ degree f_1. To see that f_2 is admissible let E_t, $t \in R$, be the $l + nl \times l + nl$ matrix

$$E_t = \begin{array}{|c|cccc|} \hline I & & & 0 & \\ \hline tX_1 I & I & & & \\ tX_2 I & & I & & \\ \vdots & & & \ddots & \\ tX_n I & & & & I \\ \hline \end{array}$$

and let $f_3 : Y \times R^{n+nl} \to R^{l+nl}$ be $f_3(y, a) = a \cdot D(y, a) \cdot E_1(a)$. For all $a \in R^n$, $t \in R$, $E_t(a)$ is a non-singular matrix so $f_3^{-1}(0) = f_2^{-1}(0)$

$$D\cdot E_1 = \left[\begin{array}{c|cccc} A & & & B & \\ \hline & \lambda I & & & \\ 0 & & \lambda I & & \\ & & & \ddots & \\ & & & & \lambda I \end{array}\right]$$

So $f_3(y, a) = 0$ if and only if $f_1(y, a_1, \cdots, a_n) = 0$ and

$$(a_1, \cdots, a_n)\cdot B(y, a_1, \cdots, a_n) + (\lambda a_{n+1}, \cdots, \lambda a_{n+nl}) = 0.$$

By the choice of λ this implies that f_3 is admissible, and therefore f_2 is admissible.

$$Y \times [0, 1] \times R^{n+nl} \to R^{l+nl}$$

by $(y, t, a) \to a\cdot D(y, a)\cdot E_t(a)$ is an ad-homotopy from f_2 to f_3.

$$(1 - t)f_3 + tf_1^{(nl)}$$

is an ad-homotopy from f_3 to $f_1^{(nl)}$. This proves the lemma.

Proof of 2.4. Let Y be a compact Hausdorff space and let $f : Y \to M_k$ be a continuous map. For some integers n, l with $n - l = k, f$ maps Y into $M_{n-l,n}$. Let $f_0 : Y \times R^n \to R^l$ be the admissible map so obtained from f. By the Stone-Weierstrass approximation theorem there exists an admissible polynomial map $f_1 : Y \times R^n \to R^l$ such that f_1 approximates f_0 on $Y \times S^{n-1}$. f_0 and f_1 are ad-homotopic. An ad-homotopy is given by $(1 - t)f_0 + tf_1$, $0 \leqq t \leqq 1$. Suppose degree $f_1 > 2$. Then by 2.9 there is an admissible map $f_2 : Y \times R^{n+nl} \to R^{l+nl}$ such that $2 \leqq$ degree $f_2 <$ degree f_1 and f_2 is polynomially ad-homotopic to f_1^{nl}. Thus the map of Y into M_k determined by f_2 is homotopic to f and $2 \leqq$ degree $f_2 <$ degree f_1. Hence by repeated applications of 2.9 a quadratic map is obtained so $[Y, Q_k] \to [Y, M_k]$ is surjective.

Now let $g, g' : Y \to Q_k$ be two continuous maps which are homotopic as maps into M_k. Then by 2.7 there exist integers n, l with $n - l = k$ and an admissible polynomial map $h_1 : Y \times [0, 1] \times R^n \to R^l$ such that g is given by

$$h_{1,0} = h_1 \mid Y \times 0 \times R^n$$

and g' is given by

$$h_{1,1} = h_1 \mid Y \times 1 \times R^n.$$

Suppose that degree $h_1 > 2$. Let $(\tilde{P}_1, \cdots, \tilde{P}_l)$ be the l-tuple determined by h_1.

$$\tilde{P}_i \in R(Y \times [0, 1])[X_1, \cdots, X_n].$$

Let $\tilde{A}$ be the $n \times l$ matrix with entries in $R(Y \times [0, 1])[X_1, \cdots, X_n]$ given by $\tilde{A}_{ij} = \tilde{P}_{ji}$. For $k = 1, 2, \cdots, n$ let $\tilde{A}_k$ be the $n \times l$ matrix whose (i, j)

entry is A_{ijk}. Let $\tilde{B}$ be the $n \times nl$ matrix

$$B = \begin{array}{|c|c|c|c|} \hline \tilde{A}_1 & \tilde{A}_2 & \cdots & \tilde{A}_n \\ \hline \end{array}$$

Choose a positive real number $\tilde{\lambda}$ such that whenever

$$(y, t, a) \in Y \times [0, 1] \times R^n$$

has $h_1(y, t, a) = 0$ and $\| a \| < 1$, then

$$\| a \|^2 + \tilde{\lambda}^{-2} \| a \cdot B(y, t, a) \| < 1.$$

Let I denote $l \times l$ identity matrix and let

$$\tilde{D} = \begin{array}{|c|cccc|} \hline \tilde{A}(0) & & \tilde{B} & & \\ \hline -\tilde{\lambda} X_1 I & \tilde{\lambda} I & & & \\ -\tilde{\lambda} X_2 I & & \tilde{\lambda} I & & \\ \vdots & & & \ddots & \\ -\tilde{\lambda} X_n I & & & & \tilde{\lambda} I \\ \hline \end{array}$$

Let $h_2 : Y \times [0, 1] \times R^{n+nl} \to R^{l+nl}$ be $h_2(y, t, a) = a \cdot \tilde{D}(y, t, a)$. Then as above h_2 is an admissible polynomial map and degree $h_2 <$ degree h_1. Let

$$h_{2,0} = h_2 \mid Y \times 0 \times R^n.$$

Then as in the proof of 2.9 $h_{2,0}$ is ad-homotopic to $h_{1,0}^{(nl)}$. Moreover the ad-homotopy obtained as in the proof of 2.9 is given by an admissible quadratic map

$$q : Y \times [0, 1] \times R^{n+nl} \to R^{l+nl}.$$

Similarly for $h_{2,1} = h_2 \mid Y \times 1 \times R^n$ and $h_{1,1}$. Hence a polynomial ad-homotopy

$$h_2' : Y \times [0, 1] \times R^{n+nl} \to R^{l+nl}$$

from $h_{1,0}^{(nl)}$ to $h_{1,1}^{(nl)}$ is obtained and $2 \leqq$ degree $h_2' <$ degree h_1. By iteration it follows that there exists an integer i and an admissible quadratic map

$$\tilde{q} : Y \times [0, 1] \times R^{n+i} \to R^{l+i}$$

such that $\tilde{q}(y, 0, a) = h_{1,0}^{(i)}(y, a)$ and $\tilde{q}(y, 1, a) = h_{1,1}^{(i)}(y, a)$. Thus

$$[Y, Q_k] \to [Y, M_k]$$

is injective and 2.4 is proved.

2.10. Definition. An admissible map $f : R^n \to R^{n-k}$ *represents* $w \in \pi_k$ if the stable homotopy class of $S^{n-1} \to S^{n-k-1}$ by $a \to f(a)/\| f(a) \|$ is w.

2.11. COROLLARY. (i) *For all k every $w \in \pi_k$ can be represented by an admissible quadratic map $q : R^n \to R^{n-k}$.*

(ii) *Two admissible quadratic maps*

$$q : R^n \to R^{n-k} \quad \textit{and} \quad q' : R^m \to R^{m-k}$$

represent the same $w \in \pi_k$ if and only if there exists a positive integer i and an admissible quadratic map

$$h : [0, 1] \times R^{n+i} \to R^{n-k+i}$$

such that $h \mid 0 \times R^{n+i} = q^{(i)}$ and $h \mid 1 \times R^{n+i} = q'^{(n-m+i)}$.

Proof. In 2.4 take Y to be a point.

3. Problems

Problem 1. Can every $w \in \pi_k$ be represented by a quadratic map

$$q : R^n \to R^{n-k}$$

such that $q^{-1}(0) = 0$?

Problem 2. Can every $w \in \pi_k$ be represented by a quadratic map

$$q : R^n \to R^{n-k}$$

such that $q(0) = 0$ and $q(S^{n-1}) \subset S^{n-k-1}$?

It can be proved that

(i) For any $w \in \pi_k$, $w + w$ can be represented by a quadratic map

$$q : R^n \to R^{n-k}$$

with $q^{-1}(0) = 0$.

(ii) If

$$q : R^n \to R^{n-k}$$

is a quadratic map with $q(0) = 0$ and $q(S^{n-1}) \subset S^{n-k-1}$ then $q^{-1}(0) = 0$.

REFERENCES

1. M. F. ATIYAH AND R. BOTT, *On the periodicity theorem for complex vector bundles*, Acta Math., vol. 112 (1964), pp. 229–247.
2. M. F. ATIYAH, R. BOTT AND A. SHAPIRO, *Clifford modules*, Topology, vol. 3, Suppl. 1 (1964), pp. 3–38.
3. H. BASS, *An algebraic analogue of Bott's complex periodicity theorem*, mimeographed notes, Columbia University.
4. ———, *The stable homotopy of the classical groups*, Ann. of Math., vol. 70 (1959), pp. 313–337.
5. R. H. FOX, *On topologies for function spaces*, Bull. Amer. Math. Soc., vol. 51 (1945), pp. 429–432.
6. G. W. WHITEHEAD, *A generalization of the Hopf invariant*, Ann. of Math., vol. 51 (1950), pp. 192–237.
7. R. WOOD, *Banach algebras and Bott periodicity*, Topology, vol. 4 (1966), pp. 371–389.

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