APPLICATIONS OF A FUNCTION TOPOLOGY ON RINGS WITH UNIT

BY

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Let R be a ring with 1 (we assume this throughout) and let M be a left R-module. Then $M^* = \mathrm{Hom}_R(M, R)$, the dual of M, becomes a right R-module with a module operation defined by $(fr)(x) = f(x)r$ for each $f \in M^*$, $r \in R$, and $x \in M$. Let T be a submodule of M^* and $S = \{\ker t \mid t \in T\}$. The topology on M whose subbase for the neighborhood system of 0 is the set S is called the T-topology on M. It is easy to see that the T-topology is the weakest (coarsest) topology on M such that every element of T is a continuous homomorphism from M into R with the discrete topology (see Chase [1]). In [7], L. E. T. Wu proved that a necessary and sufficient condition that R be left self-injective is that T is precisely the set of all continuous homomorphisms from M with the T-topology into R with the discrete topology for any left R-module M and any submodule T of M^* such that the T-topology is Hausdorff.

Our aim here is to study R when M is restricted to be the left regular R-module, ${}_RR$. In this case, M^* is the right regular R-module, R_R and the submodules of M^* are right ideals in R. Since every element of M^* is represented by right multiplication by an element of R, S is just the set of left annihilators of the elements of T. (If X is a non-empty subset of R, we define X^l, the *left-annihilator* of X, to be $\{r \in R \mid rx = 0 \text{ for all } x \in X\}$. If $X = \{a\}$, we write $X^l = (a)^l$. The right annihilator, X^r, is defined analogously.) Thus, if I is a right ideal in R, we define the T_I topology on R to be the topology whose neighborhood system at 0 has as subbasis $\{(a)^l \mid a \in I\}$.

If I is a right ideal in R, we define C_I to be the set of all $a \in R$ such that the mapping $f_a : (R, T_I) \to (R, \text{discrete})$ defined by $f_a(x) = xa$ is continuous for all $x \in R$. We note here that $x \in C_I$ if and only if $(x)^l \supseteq \bigcap_{i=1}^n (y_i)^l$ for some positive integer n and for some $y_1, y_2, \cdots, y_n$ in I. Hence, $C_I \subseteq (I^l)^r$ and if I is finitely generated, then $C_I = (I^l)^r$. In Section 1 we consider a generalization of Wu's result [7] in determining necessary and sufficient conditions for $I = C_I$. We also consider other implications of $I = C_I$. In Section 2 we consider implications of the T_I topology being compact and/or Hausdorff.

1. Implications of $I = C_I$

We first prove the generalization of Wu's result. Note that we obtain a weaker (but similar) condition than left-self injectivity.

THEOREM 1.1. *The following three conditions are equivalent:*

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(i) *If J is a finitely generated right deal in R, $J = (J^l)^r$.*

(ii) *If F is a finitely generated free left R-module and A is a cyclic submodule of F then any R-homomorphism from A into R may be extended to an R-homomorphism of F into R.*

(iii) *$I = C_I$ for all right ideals I in R.*

Proof. (i) implies (ii). Let $f : A \to R$ be an R-homomorphism where $A = Rm$ for some $m \in F$. Then $m = (a_1, \cdots, a_n)$ for some $a_1, \cdots, a_n$ in R. Suppose $f(m) = b$. Let $L = \bigcap_{i=1}^n (a_i)^l$. Then $L \subseteq (b)^l$ and $L = (\sum_{i=1}^n a_i R)^l$. Let $I = \sum_{i=1}^n a_i R$. Then I is a finitely generated right ideal in R so that $I = (I^l)^r$ and so

$$b \in ((b)^l)^r \subseteq L^r = (I^l)^r = I.$$

Consequently, $b = a_1 r_1 + \cdots + a_n r_n$ for some $r_1, \cdots, r_n$ in R. Define $\bar{f} : F \to R$ by

$$\bar{f}(x_1, \cdots, x_n) = x_1 r_1 + \cdots + x_n r_n$$

for any $x_1, \cdots, x_n$ in R. Then $\bar{f}$ is an R-homomorphism from F into R and $\bar{f}(a) = f(a)$ for any $a \in A$.

(ii) implies (iii). Suppose I is a right ideal in R such that there is an a in C_I such that a is not in I. Then there exists a finite number of elements $x_1, \cdots, x_n$, for some positive integer n, in I such that $\bigcap_{i=1}^n (x_i)^l \subseteq (a)^l$. Let $F = R \oplus \cdots \oplus R$ (n copies). Define a mapping f by $f(rx_1, \cdots, rx_n) = ra$ for all $r \in R$. Then f is an R-homomorphism from the cyclic submodule $R(x_1, \cdots, x_n)$ of F into R. If f were extended to an R-homomorphism $\bar{f} : F \to R$, then

$$\begin{aligned} ra = \bar{f}(rx_1, \cdots, rx_n) &= \bar{f}(rx_1, 0, \cdots, 0) \\ &\quad + \bar{f}(0, rx_2, 0, \cdots, 0) + \cdots + \bar{f}(0, \cdots, 0, rx_n). \end{aligned}$$

In particular, if $r = 1$ we have

$$\begin{aligned} a &= \bar{f}(x_1, 0, \cdots, 0) + \cdots + \bar{f}(0, \cdots, 0, x_n) \\ &= x_1 \bar{f}(1, 0, \cdots, 0) + \cdots + x_n \bar{f}(0, \cdots, 0, 1), \end{aligned}$$

so that $a \in I$ since $x_i \in I$, $i = 1, \cdots, n$ and I is a right ideal in R. This contradiction confirms that I cannot be properly contained in C_I. (Note: $I \subseteq C_I$ always.)

(iii) implies (i). Let $x_1, \cdots, x_n \in R$ be such that $J = x_1 R + \cdots + x_n R$. Then $J^l = \bigcap_{i=1}^n (x_i)^l$ and since $J = C_J$ we have $(J^l)^r \subseteq J$. Since $J \subseteq (J^l)^r$ always, we have $J = (J^l)^r$.

That this theorem is a generalization of Wu's result can be seen from the following theorem and the fact that a regular ring is not necessarily left self-injective (Utumi [5, Example 3]).

THEOREM 1.2. *If R is a regular ring with 1 and I is a right ideal of R, then $I = C_I$.*

Proof. Let $J = x_1 R + \cdots + x_n R$ be a finitely generated right ideal in R. By [6, Lemma 15], $J = eR$ for some idempotent e in R. Then

$$J^l = (eR)^l = R(1 - e) \quad \text{and } (J^l)^r = (R(1 - e))^r \subseteq eR$$

and so $(J^l)^r = J$. By 1.1 we have $I = C_I$ for all right ideals I in R.

In general, (R, T_I) will not be a topological ring (of course, the additive group, R^+, of R will be a topological group with the T_I topology). We end this section with a necessary and sufficient condition for (R, T_I) to be a topological ring.

THEOREM 1.3. *If $I = C_I$ for a right ideal I in R, then (R, T_I) is a topological ring if and only if I is a two-sided ideal in R.*

Proof. Assume (R, T_I) is a topological ring. Let u be in I and r be in R. Since R is a topological ring, right multiplication by r is a continuous map from (R, T_I) to (R, T_I). Since u is in I, right multiplication by u is a continuous map from (R, T_I) to $(R$, discrete$)$. The composition of these maps, which is right multiplication by ru, is then a continuous map from (R, T_I) to $(R$, discrete$)$. Thus ru is in $C_I = I$, and I is a left (hence two-sided) ideal of R.

Conversely, let I be a two-sided ideal in R. Then, if $a \in R$, $i \in I$, $(ai)^l$ is open in T_I. We will show $(x, a) \to xa$ is continuous in T_I. Let

$$xa + \bigcap_{i=1}^n (x_i)^l$$

be an open set containing xa. Then $x + \bigcap_{i=1}^n (ax_i)^l$ is an open set containing x and $a + \bigcap_{i=1}^n (x_i)^l$ is an open set containing a and

$$\begin{aligned}
&[x + \textstyle\bigcap_{i=1}^n (ax_i)^l][a + \bigcap_{i=1}^n (x_i)^l] \\
&\qquad = xa + x[\textstyle\bigcap_{i=1}^n (x_i)^l] + [\bigcap_{i=1}^n (ax_i)^l]a + [\bigcap_{i=1}^n (ax_i)^l][\bigcap_{i=1}^n (x_i)^l] \\
&\qquad \subseteq xa + \textstyle\bigcap_{i=1}^n (x_i)^l
\end{aligned}$$

since $\bigcap_{i=1}^n (x_i)^l$ is a left ideal in R and $[\bigcap_{i=1}^n (ax_i)^l]a \subseteq \bigcap_{i=1}^n (x_i)^l$.

Thus, $(x, a) \to xa$ is continuous and so (R, T_I) is a topological ring.

2. Implications of (R, T_I) being compact and/or Hausdorff

We now consider the implications of (R, T_I) being compact Hausdorff. It is easy to see that (i) (R, T_I) is Hausdorff if and only if $I^l = (0)$ and that (ii) if (R, T_I) is compact then $R/(x)^l$ finite for all x in I. We shall use these facts subsequently.

It is known that a compact Hausdorff topological group cannot be countably infinite [2, 4.26, p. 31]. We first give two sufficient conditions that it be finite.

THEOREM 2.1. *Let (R^+, T) be a compact topological group (R^+ is the additive group of R) such that there is an $x \neq 0$ in R with $(x)^l$ open in T. If R is a prime ring, then it is finite.*

Proof. Since $R/(x)^l$ is compact and discrete and $Rx \cong R/(x)^l$, Rx is a finite left ideal in R. Hence, R is a primitive ring with a finite minimal left ideal. By the Wedderburn Theorem [4, p. 445], R is finite.

COROLLARY 2.2. *If* (R, T_I) *is compact for some right ideal* $I \neq 0$ *and* R *is prime, then* R *is finite.*

2.2 is not true if we replace the hypothesis that R be a prime ring with the hypothesis that R be a semi-prime ring: Let R be the complete direct product of the rings $Z/(p)$, p a positive prime, $R = \Pi Z/(p)$. Addition and multiplication are component-wise. Let I be the direct sum $\Sigma Z/(p)$. (The subset of $\Pi Z/(p)$ where all but finitely many of the components are zero.) The T_I topology is the product topology (Tichonoff) on $\Pi Z/(p)$ where each $Z/(p)$ has the discrete topology. Thus (R, T_I) is compact Hausdorff and R is an infinite, semi-prime ring.

Denote by C^I the set of all a in R such that for each x in R and each open set W containing xa there are open sets U, V containing x, a such that $UV \subseteq W$.

LEMMA 2.3. $I \subseteq C^I = \overline{C^I}$

Proof. Let $t \in I$, $x \in R$, and $xt + \bigcap_{i=1}^n (y_i)^l$ an open set containing xt $(y_i \in I)$. Let

$$U = x + (t)^l, \qquad V = t + \bigcap_{i=1}^n (y_i)^l.$$

Then

$$UV = xt + x[\bigcap_{i=1}^n (y_i)^l] + (t)^l \cdot t + (t)^l[\bigcap_{i=1}^n (y_i)^l] \subseteq xt + \bigcap_{i=1}^n (y_i)^l$$

since $(t)^l t = 0$ and $\bigcap_{i=1}^n (y_i)^l$ is a left ideal in R. Thus $I \subseteq C^I$.

Let $t \in \overline{C^I}$. If $x \in I$, $[t + (x)^l] \cap C^I \neq \emptyset$. Let $f \in C^I$ such that $f = t + s$ for some $s \in (x)^l$. Then $fx = tx$. Since $f \in C^I$ there is an open set U containing 0 such that $Uf \subseteq (x)^l$. Consequently, $Ut \subseteq (x)^l$ so that

$$U[t + (x)^l] = Ut + U(x)^l \subseteq (x)^l.$$

Thus, $t \in C^I$.

THEOREM 2.4. *Let* (R, T_I) *be compact Hausdorff for some right ideal* I *in* R. *If* I *is a prime ring,* R *is finite.*

Proof. Let $x \in I$, $x \neq 0$. (Since the space is Hausdorff, $I \neq (0)$.) For each $t \in \bar{I}$ there are open sets U_t, V_t containing 0, t, respectively, such that $U_t V_t \subseteq (x)^l$ by 2.3. Since $\bar{I}$ is compact in the subspace topology, finitely many of the V_t cover $\bar{I}$. Let $U_1, \cdots, U_n$ be the corresponding U_t's and let $U = \bigcap_{i=1}^n U_i$. Then $U\bar{I} \subseteq (x)^l$ and so $UIx = (0)$. If there were a $y \neq 0$ in $U \cap I$, then $yIx = 0, x, y \in I, x \neq 0 \neq y$, contradicting the hypothesis that I be a prime ring. Thus $U \cap I = (0)$ and so I is discrete in the subspace topology. By [2, Th. 5.10, p. 35], I is closed and consequently must be compact and is thus finite. Let $I = \{x_1, \cdots, x_n\}$. Then $(0) = I^l = \bigcap_{i=1}^n (x_i)^l$ so that (0) is an open set. Hence, T_I is discrete.

The next theorem is an analogue of a theorem of Kaplansky [3, Cor. 1, p. 162]. We first prove the following lemma.

LEMMA 2.5. *If (R, T_I) is Hausdorff and $a \in R$, $(a)^r$ is closed in T_I.*

Proof. Suppose there is a $t \in \overline{(a)^r}$, $t \notin (a)^r$. Then $at \neq 0$. Let $at + \bigcap_{i=1}^n (y_i)^l$ be an open set containing at and not containing 0. Then

$$a[t + \bigcap_{i=1}^n (y_i)^l] \subseteq at + \bigcap_{i=1}^n (y_i)^l.$$

However,

$$[t + \bigcap_{i=1}^n (y_i)^l] \cap (a)^r \neq 0.$$

So there is a b in $t + \bigcap_{i=1}^n (y_i)^l$ such that $b \in (a)^r$. Consequently,

$$0 = ab \in at + \bigcap_{i=1}^n (y_i)^l,$$

a contradiction. Thus, $(a)^r$ is closed.

THEOREM 2.6. *Let (R, T_I) be compact Hausdorff for some right ideal I. If R has no proper closed two-sided ideals, then R is a finite prime ring.*

Proof. Suppose $x \in R$, $x \neq 0$. Then $xR \neq 0$ and so xR is a non-zero right ideal of R. Since $(xR)^r$ is the intersection of all sets of the form $(t)^r$, $t \in xR$, by 2.5 we have that $(xR)^r$ is closed in T_I. By hypothesis, $(xR)^r = (0)$. Consequently, $xRt \neq (0)$ if $t \neq 0$. Thus R is a prime ring so by 2.1, R is finite.

Finally, we give two theorems which are concerned with the structure of rings which are compact Hausdorff in a T_I topology. The first involves semi-prime rings. In the second we remove this restriction.

THEOREM 2.7. *Let R be a semi-prime ring and (R, T_I) compact Hausdorff for some maximal right ideal I in R. Then, either R/I is a division ring, or R is isomorphic to a subdirect sum of finite simple rings.*

Proof. Since R is a semi-prime ring, $\bigcap_{\alpha \in A} P_\alpha = (0)$, where $\{P_\alpha\}_{\alpha \in A}$ is the set of all prime ideals in R. If $I = P_\alpha$ for some $\alpha \in A$, R/I is a division ring. If R/I is not a division ring, $I \not\subseteq P_\alpha$ for any $\alpha \in A$. Let $Q(T_I)$ be the quotient space topology induced on R/P_α by T_I (see [2, p. 36]). If $(x)^l + P_\alpha = R$ for all $x \in I$ then $1 = t + p$ for some $t \in (x)^l$ and $p \in P_\alpha$. Hence, $x = tx + px = px$ and $I \subseteq P_\alpha$. Thus $Q(T_I)$ is not indiscrete. Since $(R/P_\alpha, Q(T_I))$ is compact and there is x in R not I such that

$$(x + P_\alpha)^l \supseteq ((x)^l + P_\alpha)/P_\alpha,$$

by Theorem 2.1, R/P_α is a finite prime ring. Let $h_\alpha : R \to R/P_\alpha$ be the canonical mapping and define f from R onto $\mathfrak{S}_{\alpha \in A} R/P_\alpha$, the subdirect sum of the R/P_α, by

$$f(r) = (h_1(r), \cdots, h_\alpha(r), \cdots).$$

Then f is an isomorphism.

THEOREM 2.8. *If (R, T_I) is compact Hausdorff for some right ideal I in R, then there is a mapping f from R onto a subdirect sum of rings, $\mathfrak{S}_{\alpha \in A} S_\alpha$, such* *that*

(i) *f is a ring isomorphism*

(ii) *f is a homeomorphism from (R, T_I) onto ($\mathfrak{S}_{\alpha \in A} S_\alpha$, Product Topology)*

(iii) *Characteristic $(S_\alpha) \neq 0$.*

Proof. If Characteristic $(R) \neq 0$, the theorem follows. Suppose Characteristic $(R) = 0$. Then for each $x_\alpha \in I$, $x_\alpha \neq 0$, $R/(x_\alpha)^l$ is finite and has more than one coset. Let n_α be the number of cosets in $R/(x_\alpha)^l$. Since $R/(x_\alpha)^l \cong Rx_\alpha$, Rx_α has n_α elements and so $n_\alpha x_\alpha = 0$. Thus, $n_\alpha \cdot 1$ is in $(x_\alpha)^l$. Note that $n_\alpha \cdot 1 \neq 0$ since Ch $(R) = 0$. Define a mapping

$$g : (R, T_I) \to (R, T_I)$$

by $g(r) = n_\alpha r$. Then g is an R-homomorphism from ${}_RR$ into ${}_RR$ and, furthermore, g is continuous. Thus, $g(R) = n_\alpha R$ is compact and, hence, is a closed subgroup of R in T_I. In the quotient space topology $R/n_\alpha R$ is Hausdorff [2, Th. 5.26, p. 40]. Let $S_\alpha = R/n_\alpha R$ and let $h_\alpha : R \to S_\alpha$ be the canonical mapping. Define f by

$$f(r) = (h_1(r), \cdots, h_\alpha(r), \cdots).$$

Then f satisfies (i) and (ii) by [2, Th. 5.29, p. 42] and Ch $(S_\alpha) \leq n_\alpha$ so that Ch $(S_\alpha) \neq 0$.

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References

1. Stephen U. Chase, *Function topologies on Abelian groups*, Illinois J. Math., vol. 7 (1963), pp. 593–608.
2. Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis*, Berlin, Springer-Verlag, 1963.
3. Irving Kaplansky, *Topological rings*, Amer. J. Math., vol. 69 (1947), pp. 153–183.
4. Serge Lang, *Algebra*, Reading, Mass., Addison-Wesley, 1965.
5. Yuzo Utumi, *On continuous regular rings and semisimple self injective rings*, Canad. J. Math., vol. 12 (1960), pp. 597–605.
6. John Von Neumann, *On regular rings*, Proc. Nat. Acad. Sci. U.S.A., vol. 22 (1936), pp. 707–713.
7. L. E. T. Wu, *A characterization of self-injective rings*, Illinois J. Math., vol. 9 (1965), pp. 61–65.

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