A LINEAR EXTENSION THEOREM

BY

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1. Introduction

Let T be a topological space, S a closed subset of T, and $C(S)$ and $C(T)$ the Banach spaces of bounded, continuous complex (or real) functions on S and T, respectively. Let $E \subset C(S)$ and $H \subset C(T)$ be closed subspaces. A continuous linear map $u : E \to H$ is called a *linear extension* if $u(f)$ is an extension of f for every $f \in E$. The purpose of this paper is to study the existence of linear extensions of norm one.

If $H = C(T)$, our problem was completely settled by Borsuk [3] for separable metric T, and subsequently by Dugundji [6, Theorem 5] for all metric T.[3]

THEOREM 1.1 (Borsuk-Dugundji). *If T is metrizable, there exists a linear extension $u : C(S) \to C(T)$ of norm one.*

If H is a proper subspace of $C(T)$, the situation becomes more complicated, and Example 9.2 shows that no linear extension $u : C(S) \to H$ need exist even when every $f \in C(S)$ can be extended to some $f' \in H$. We therefore introduce the following concept:

DEFINITION 1.2. The pair (E, H) has the *bounded extension property* if, given any $\varepsilon > 0$, every $f \in E$ has a bounded family of extensions

$$\{f_{\varepsilon,W} : W \supset S, W \text{ open in } T\} \subset H$$

such that $|f_{\varepsilon,W}(x)| \leq \epsilon$ whenever $x \in T - W$.

Note that the pair $(C(S), C(T))$ has this property whenever T is normal. The following result was proved by the second author in [13] and [14].[4]

THEOREM 1.3. *If T is compact metric, and if $(C(S), H)$ has the bounded extension property, then there exists a linear extension $u : C(S) \to H$ of norm one.*

Perhaps the most interesting application of Theorem 1.3 was to the case where T is the unit circle in the complex plane, $H \subset C(T)$ is the disc algebra (i.e. H consists of boundary values of continuous functions on the unit disc

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[3] Strictly speaking, Borsuk and Dugundji stated the theorem for real scalars, but their proofs remain valid for complex scalars as well (which means, in particular, that u is then complex-linear).

[4] To be precise, [13] and [14] assume a property which is formally stronger than the bounded extension property, but which (see Corollary 5.3) is actually equivalent to it.

D which are analytic on $D - T$), and S is a closed subset of T of Lebesgue measure 0. Here $(C(S), H)$ has the bounded extension property by E. Bishop's refinement [2] of the Rudin-Carleson extension theorem.[5]

The purpose of this paper is to give a new and simpler proof of Theorem 1.3, while at the same time generalizing it in several directions. In particular, we will show that $C(S)$ can be replaced by any subspace E of $C(S)$ satisfying the following mild (and possibly superfluous) condition.

DEFINITION 1.4. A separable Banach space E is a *π_1-space* if it has an increasing sequence $F_1 \subset F_2 \subset \cdots$ of finite-dimensional subspaces, whose union is dense in E, such that there is a projection of norm one from E onto each F_n.[6] Such a sequence is called a *π_1-sequence* for E.

Every finite-dimensional Banach space is obviously a π_1-space. More generally, if S is compact metric, then ([11] or [12]) $C(S)$ is a π_1-space. In fact, all the standard separable Banach spaces are π_1-spaces (see, for instance, [4]). However, an example constructed by V. Gurariĭ [8] shows that there are separable Banach spaces which are not π_1-spaces.

THEOREM 1.5. *Let T be any topological space. If E is a separable π_1-space, and if (E, H) has the bounded extension property, then there exists a linear extension $u : E \to H$ of norm one.*

The proof of Theorem 1.5, which is given in Sections 2 and 3, is entirely self-contained and elementary. Only when we want to deduce Theorem 1.3 from Theorem 1.5 do we need the fact, quoted above, that $C(S)$ is a π_1-space whenever S is compact metric.

In Sections 4–8 we obtain some refinements and extensions of Theorem 1.5. These sections are independent of each other, except that Corollary 5.3 is used in the proof of Lemma 8.3. Section 9 is devoted to examples.

Throughout the paper, T, $S \subset T$, $E \subset C(S)$ and $H \subset C(T)$ will retain the meaning they had in this introduction. If $J \subset C(S)$ and $K \subset C(T)$ are linear subspaces, $\Lambda(J, K)$ will denote the set of linear extensions $u : J \to K$.

2. Preliminary results

Throughout this section, we tacitly assume that (E, H) has the bounded extension property.

LEMMA 2.1. *Let $G \subset E$ be a finite-dimensional subspace, and $\delta > 0$. Then there exists a bounded family*

$$\{v_W : W \supset S, W \text{ open in } T\} \subset \Lambda(G, H)$$

such that $|(v_W g)(t)| \leq \delta$ *whenever* $t \in T - W$ *and* $\| g \| \leq 1$.

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[5] For a recent generalization of this theorem, see [9].

[6] Such spaces were first considered by J. Lindenstrauss [10], who called them "spaces with the 1-projective approximation property". The term "π_1-space" was introduced by F. Browder and D. G. de Figueiredo [4].

Proof. Let $g^1, \cdots, g^n$ be a base for G, so that each $g \in G$ has a unique expansion $g = \sum_{i=1}^n c_i(g)g^i$. For each i, let a_i be the norm of the projection map $g \to c_i(g)g^i$, let $a = \sum_{i=1}^n a_i$ and let $\varepsilon = \delta/a$. For each i and open $W \supset S$, let $g^i_{\varepsilon,W}$ be as in Definition 1.2, and let v_W be the unique element of $\Lambda(G, H)$ such that $v_W g^i = g^i_{\varepsilon,W}$ $(i = 1, \cdots, n)$. These v_W satisfy all our requirements.

PROPOSITION 2.2 *Let F, G be finite-dimensional subspaces of E, let $u \in \Lambda(F, C(T))$, and let $\gamma > 0$. Then there exists a $v \in \Lambda(G, H)$ such that*

$$\| uf + vg \| < 1 + \gamma$$

whenever $f \in F$, $g \in G$, $\| f + g \| \leq 1$, and $\| uf \| \leq 1$.

Proof. For each $v \in \Lambda(G, H)$, let

$$K_v = \{uf + vg : f \in F, g \in G, \| f + g \| \leq 1, \| uf \| \leq 1\},$$

and let

$$\alpha_v(t) = \sup \{| h(t) | : h \in K_v\} \qquad \text{for } t \in T.$$

Let us show that α_v is continuous.

Observe, first, that $K_v \subset C(T)$ is compact: Let

$$P = \{(f, g) \in F \times G : \| f + g \| \leq 1, \| uf \| \leq 1\}.$$

Then P is a bounded, closed subset of $F \times G$, and is thus compact. But $(f, g) \to uf + vg$ is a continuous map from P onto K_v, so K_v is also compact.

To see now that α_v is continuous, define $\phi : T \to C(K_v)$ by $(\phi t)(h) = h(t)$. Then ϕ is continuous (since K_v is compact), and $\alpha_v(t) = \| \phi(t) \|$, so α_v is continuous.

Let the family $\{v_W\}$ be as in Lemma 2.1 with $\delta = \frac{1}{4}\gamma$, and denote α_{v_W} by α_W. Since $\{v_W\}$ is bounded, there is an $M > 0$ such that $\| v_W \| < M$ for all open $W \supset S$. Now if $\| f + g \| \leq 1$ and $\| uf \| \leq 1$, then $\| f \| \leq 1$ and hence $\| g \| \leq 2$. Thus

$$\begin{aligned} \alpha_W(t) &= 1 && \text{if } t \in S, \\ \alpha_W(t) &\leq 1 + 2\delta && \text{if } t \in T - W, \\ \alpha_W(t) &\leq 1 + 2M && \text{if } t \in T. \end{aligned}$$

Let $W_1 = T$, and then define open $W_n \supset S$ $(n = 1, 2, \cdots)$ inductively by

$$W_{n+1} = W_n \cap \{t \in T : \alpha_{W_n}(t) < 1 + 2\delta\}.$$

Denote v_{W_n} by v_n, and α_{W_n} by α_n.

Our definitions imply that, if $t \in T$, then $\alpha_n(t) > 1 + 2\delta$ for at most one n: In fact, if there are such n, let n_0 be the smallest one. Then $t \notin W_n$ for any $n > n_0$, and hence $\alpha_n(t) \leq 1 + 2\delta$ for all $n > n_0$.

Pick an integer $N > 0$ such that $(1 + 2M)N^{-1} \leq \delta$. The preceding paragraph implies that, if $t \in T$, then

$$\frac{1}{N}\sum_{n=1}^{N}\alpha_n(t) \leq \frac{1}{N}\,(N(1 + 2\delta) + (1 + 2M)) \leq 1 + 3\delta.$$

Now let

$$v = \frac{1}{N}\sum_{n=1}^{N} v_n\,.$$

If $f \in F$, $g \in G$, $\| f + g \| \leq 1$, and $\| uf \| \leq 1$, then, for all $t \in T$,

$$\begin{aligned} | (uf + vg)(t) | &= \frac{1}{N}\left| \sum_{n=1}^{N} (uf + v_n g)(t) \right| \\ &\leq \frac{1}{N}\sum_{n=1}^{N} | (uf + v_n\, g)(t) | \\ &\leq \frac{1}{N}\sum_{n=1}^{N}\alpha_n(t) \leq 1 + 3\delta. \end{aligned}$$

Hence $\| uf + vg \| \leq 1 + 3\delta < 1 + \gamma$, and that completes the proof.

COROLLARY 2.3. *Let $F \subset G$ be finite-dimensional subspaces of E admitting a linear projection $\pi : G \to F$ of norm 1, and let $\varepsilon > 0$. Then any $u \in \Lambda(F, H)$, with $\| u \| < 1 + \varepsilon$, can be extended to a $u' \in \Lambda(G, H)$ with $\| u' \| < 1 + \varepsilon$.*

Proof. Pick $\gamma > 0$ so that

$$(1 + \gamma)\| u \| < 1 + \varepsilon.$$

Pick $v \in \Lambda(G, H)$ according to Proposition 2.2, and let

$$u'h = u(\pi h) + v(h - \pi h), \qquad h \in G.$$

Since $u \in \Lambda(F, H)$ and $v \in \Lambda(G, H)$, we have $u' \in \Lambda(G, H)$. Let $h \in G$ with $\| h \| \leq 1$, and let us show that $\| u'h \| < 1 + \varepsilon$.

Since $\| u \| \geq 1$, we may let $k = h/\| u \|$. It suffices to show that

$$\| u'k \| < 1 + \gamma.$$

Let $f = \pi k$ and $g = k - \pi k$, so that

$$u'k = uf + vg.$$

But

$$\| f + g \| = \| k \| \leq \| h \| \leq 1,$$

and

$$\| uf \| = \| u\pi k \| \leq \| u \|\, \| \pi \|\, \| k \| = \| \pi \|\, \| h \| \leq 1,$$

so the choice of v implies that $\| u'k \| < 1 + \gamma$. That completes the proof.

COROLLARY 2.4. *If F is a finite-dimensional subspace of E, if $\varepsilon > 0$, and if $u \in \Lambda(F, H)$ with $\| u \| < 1 + \varepsilon$, then there exists a $w \in \Lambda(F, H)$ with $\| w \| = 1$ and $\| u - w \| < \varepsilon$.*

Proof. It will suffice to construct a $w_1 \in \Lambda(F, H)$ such that, for some positive $\gamma < \frac{1}{2}\varepsilon$, we have $\| w_1 \| < 1 + \gamma$ and $\| u - w_1 \| < \varepsilon - \gamma$. For we could then inductively repeat the process, obtaining a Cauchy sequence $w_n \in \Lambda(F, H)$ whose limit w satisfies both our requirements.

Denote $\| u \|$ by λ, and pick a positive $\gamma < \frac{1}{2}\varepsilon$ such that

$$(\lambda - 1)(1 + \gamma) \leq \varepsilon - \gamma.$$

Now pick $v \in \Lambda(F, H)$ according to Proposition 2.2 (with $G = F$), and let

$$w_1 = \frac{1}{\lambda} u + \frac{\lambda - 1}{\lambda} v.$$

Clearly $w_1 \in \Lambda(F, H)$.

To show that $\| w_1 \| < 1 + \gamma$, let $h \in F$ with $\| h \| \leq 1$, and let us show that $\| w_1 h \| \leq 1 + \gamma$. Let $f = \lambda^{-1}h$ and $g = (\lambda - 1)\lambda^{-1}h$, so that

$$w_1 h = uf + vg.$$

Then

$$\| f + g \| = \lambda^{-1}(1 + (\lambda - 1))\| h \| \leq 1,$$

and

$$\| uf \| \leq \| u \| \, \| f \| = \lambda\lambda^{-1}\| h \| \leq 1,$$

so the choice of v implies that $\| w_1 h \| < 1 + \gamma$.

To show that $\| u - w_1 \| < \varepsilon - \gamma$, pick $h \in F$ with $\| h \| \leq 1$, and let us show that $\| (u - w_1)h \| < \varepsilon - \gamma$. Now note that

$$u - w_1 = \frac{\lambda - 1}{\lambda}(u - v).$$

Let $f = \lambda^{-1}h$, $g = -\lambda^{-1}h$. It will suffice to show that

$$\| uf + vg \| < 1 + \gamma,$$

because then

$$\| (u - w_1)h \| = (\lambda - 1)\| uf + vg \| < (\lambda - 1)(1 + \gamma) \leq \varepsilon - \gamma.$$

But

$$\| f + g \| = 0,$$

and

$$\| uf \| \leq \| u \| \, \| f \| = \lambda\lambda^{-1}\| h \| \leq 1,$$

so that the choice of v implies that $\| uf + vg \| < 1 + \gamma$. That completes the proof.

3. Proof of Theorem 1.5

We begin with the following lemma, which will also be used in the next section.

LEMMA 3.1. *Let $F_1 \subset F_2 \subset \cdots$ be a π_1-sequence for E, let $\lambda \geq 1$, and let $v_n \in \Lambda(F_n, H)$, with $\| v_n \| \leq \lambda$ and $v_{n+1} \mid F_n = v_n$ for all n. Then there exists a (unique) $u = \Lambda(E, H)$ such that $\| u \| \leq \lambda$ and $u \mid F_n = v_n$ for all n.*

Proof. Let $F = \bigcup_{n=1}^{\infty} F_n$, and define $v : F \to H$ by $v \mid F_n = v_n$. Clearly $v \in \Lambda(F, H)$, and $\| v \| \leq \lambda$. Since F is dense in E, v can be extended uniquely to a continuous linear $u : E \to H$, and $\| u \| \leq \lambda$. To see that $u \in \Lambda(E, H)$, define $r_S : C(T) \to C(S)$ by $r_S(f) = f \mid S$. Since r_S is continuous,

$$E' = \{f \in E : r_S(uf) = f\}$$

is closed in E. But $E' \supset F$ and F is dense in E, so $E' = E$. Hence $u \in \Lambda(E, H)$, and that completes the proof.

Proof of Theorem 1.5. Let $F_1 \subset F_2 \cdots$ be a π_1-sequence for E. Pick any $w_1 \in \Lambda(F_1, H)$ with $\| w_1 \| = 1$; such a w_1 exists by Corollary 2.4. By Corollaries 2.3 and 2.4, we can now inductively pick a sequence $w_n \in \Lambda(F_n, H)$ such that, for each $n \geq 1$, $\| w_n \| = 1$ and

$$\| w_{n+1} \mid F_n - w_n \| \leq 2^{-n}.$$

Now for each fixed n, the sequence $w_m \mid F_n \in \Lambda(F_n, H)$ $(m = n, n + 1, \cdots)$ is Cauchy, and thus has a limit $v_n \in \Lambda(F_n, H)$ with $\| v_n \| = 1$. Clearly $v_{n+1} \mid F_n = v_n$ for all n. Hence, by Lemma 3.1, there exists a $u \in \Lambda(E, H)$ with $\| u \| \leq 1$ and $u \mid F_n = v_n$ for all n. Since $\| u \| \geq 1$ for any linear extension, that completes the proof.

4. Extending linear extensions

An interesting feature of the Borsuk-Dugundji theorem (Theorem 1.1) is that the linear extension u can be chosen so that[7] $u1_S = 1_T$. Does this remain true for Theorem 1.3 if $1_T \in H$? Curiously, the answer is "yes" for real scalars (see Section 8), and "no" for complex scalars (Example 9.1). However, we will now prove a result which implies that it is "almost" true even in the complex case. (It implies that because, if S is compact metric, then $C(S)$ always has a π_1-sequence $F_1 \subset F_2 \subset \cdots$, with F_1 the one-dimensional subspace spanned by 1_S [11].)

Observe that part (a) of Theorem 4.1 actually sharpens Theorem 1.5.

THEOREM 4.1. *Suppose that T is any topological space, and that (E, H) has the bounded extension property. Suppose also that E is a separable π_1-space with π_1-sequence $F_1 \subset F_2 \subset \cdots$, and that $w : F_1 \to H$ is a linear extension of norm one. Then, for any $\varepsilon > 0$:*

(a) *There exists a linear extension $u : E \to H$, with $\| u \| = 1$, such that $\| w - u \mid F_1 \| \leq \varepsilon$.*

(b) *There exists a linear extension $v : E \to H$, with $\| v \| \leq 1 + \varepsilon$, such that $w = v \mid F_1$.*

Proof. (a) The proof proceeds precisely as the proof of Theorem 1.5 in Section 3, taking $w_1 = w$ and replacing 2^{-n} by $2^{-n}\varepsilon$. The linear extension

[7] If A is a set, 1_A denotes the function identically 1 on A.

$v : E \to H$ constructed in the proof of Theorem 1.5 will now satisfy all our requirements.

(b) By Corollary 2.3, there exist linear extensions $v_n : F_n \to H$ such that $v_1 = w$, and $\| v_n \| < 1 + \varepsilon$ and $v_{n+1} \mid F_n = v_n$ for all n. Our conclusion now follows from Lemma 3.1.

5. Dominated convergence

Let $\mathfrak{D}_T$ denote the set of all continuous bounded $\Delta : T \to \mathbf{R}$ with a positive lower bound. If $\Delta \in \mathfrak{D}_T$, define $\Delta^{-1} \in \mathfrak{D}_T$ by $\Delta^{-1}(t) = (\Delta(t))^{-1}$. If $\Delta \in \mathfrak{D}_T$, $Q \subset T$, and $f \in C(Q)$, define $\Delta f \in C(Q)$ by $(\Delta f)(t) = \Delta(t)f(t)$; if $A \subset C(Q)$, let $\Delta A = \{\Delta f : f \in A\}$.

The following result refines Theorem 1.5.

THEOREM 5.1. *Let T be a topological space, let $\Delta \in \mathfrak{D}_T$, and suppose that (E, H) has the bounded extension property and that $\Delta^{-1}E$ is a separable π_1-space. Then there exists a linear extension $u : E \to H$ such that, if $f \in E$ and $| f | \leq \Delta \mid S$, then $| uf | \leq \Delta$.*

Proof. The proof is almost shorter than the statement. First, it is easy to check that $(\Delta^{-1}E, \Delta^{-1}H)$ also has the bounded extension property, so by Theorem 1.5 there exists a linear extension $v : \Delta^{-1}E \to \Delta^{-1}H$ of norm one. If $u : E \to H$ is now defined by

$$u(f) = \Delta v(\Delta^{-1}f),$$

then u satisfies all our requirements. That completes the proof.

When applying Theorem 5.1, note that each of the following conditions implies that $\Delta^{-1}E$ is a separable π_1-space.

(5.1.1) E is finite-dimensional,

(5.2.2) S is compact metric, and $E = C(S)$,

(5.2.3) E is a separable π_1-space, and $\Delta(s) = 1$ for $s \in S$.

DEFINITION 5.2. (a) The pair (E, H) has the *dominated extension property*[8] if, for every $\Delta \in \mathfrak{D}_T$, any $f \in E$ with $|f(s)| \leq \Delta(s)$ for $s \in S$ can be extended to some $f' \in H$ with $|f'(t)| \leq \Delta(t)$ for all $t \in T$.

(b) The pair (E, H) has the *strict dominated extension property* if (a) is satisfied with $\leq$ everywhere replaced by $<$.

The following result now follows from Theorem 5.1, which is used to show that (c) $\to$ (a). There are more direct proofs of this implication, but they all require a fair amount of work.

COROLLARY 5.3. *If T is normal, then the following properties of a pair (E, H) are equivalent:*

[8] This term was coined by Semadeni.

(a) (E, H) *has the dominated extension property.*
(b) (E, H) *has the strict dominated extension property.*
(c) (E, H) *has the bounded extension property.*

Proof. (a) $\rightarrow$ (*b*). Let $\Delta \in \mathfrak{D}_T$ and let $f \in E$ with $|f(s)| < \Delta(s)$ for all $s \in S$. Using the normality of T, it is easy to construct a $\Delta_0 \in \mathfrak{D}_T$ such that $\Delta_0(t) < \Delta(t)$ for every $t \in T$, and $|f(s)| \leq \Delta_0(s)$ for all $s \in S$. By (a), f can be extended to some $f' \in H$ such that $|f'(t)| \leq \Delta_0(t) < \Delta(t)$ for all $t \in T$.

(b) $\rightarrow$ (*c*). Let $f \in E$ and $\varepsilon > 0$ be given. Let $M = \|f\| + 1$. For each open $W \supset S$, pick a continuous $\Delta_W : T \rightarrow [\varepsilon, M]$ such that $\Delta_W(S) = M$ and $\Delta_W(T - W) = \varepsilon$, and then use (b) to extend f to a continuous $f_{\varepsilon,W} \in H$ such that $|f_{\varepsilon,W}(t)| < \Delta_W(t)$ for every $t \in T$. These $f_{\varepsilon,W}$ satisfy the requirements of Definition 1.2.

(c) $\rightarrow$ (a). Suppose that $\Delta \in \mathfrak{D}_T$, $f \in E$, and $|f(s)| \leq \Delta(s)$ for every $s \in S$. Let E_f be the one-dimensional subspace of E spanned by f; then E_f is a separable π_1-space. Applying Theorem 5.1 to the pair (E_f, H), we obtain an extension of f to some $f' \in H$ such that $|f'(t)| \leq \Delta(t)$ for all $t \in T$. That completes the proof.

If T is compact metric, then the bounded extension property is equivalent to a remarkably weak condition:

PROPOSITION 5.4. *If T is compact metric, then the following properties of a pair (E, H) are equivalent.*

(a) (E, H) *has the bounded extension property.*

(b) *To every $f \in E$ there corresponds a bounded sequence $f_n \in H$ such that $f_n \mid S \in E$ for all n, $f_n(s) \rightarrow f(s)$ if $s \in S$, and $f_n(t) \rightarrow 0$ if $t \in T - S$.*

We omit the proof of Proposition 5.4. Note that it is not *obvious* from (b) that $f \in E$ has *any* extension $f' \in H$.

6. Banach space-valued functions

Let B be a Banach space, and let $C(X, B)$ denote the Banach space of bounded, continuous functions from X to B. It is a striking fact that, if E and H are assumed to be subspaces of $C(S, B)$ and $C(T, B)$, respectively, and if absolute values are suitably replaced by norms, then all our definitions remain meaningful, and Theorem 1.5 and its refinements in Sections 4 and 5 *remain true with exactly the same proofs.* In order to benefit from this observation, however, we must know something about what subspaces E of $C(S, B)$ are separable π_1-spaces. In particular, when is $C(S, B)$ itself a separable π_1-space? We can answer the latter question as follows:

PROPOSITION 6.1. *If S is compact metric, and if B is a separable π_1-space, then $C(S, B)$ is a separable π_1-space.*

Before proving this result, let us note that, in view of the observation in the first paragraph of this section, Proposition 6.1 implies the following generalization of Theorem 1.3.

THEOREM 6.2. *If T is compact metric, B a separable π_1-space, H a closed linear subspace of $C(T, B)$, and if $(C(S, B), H)$ has the bounded extension property, then there exists a linear extension $u : C(S, B) \to H$ of norm one.*

To prove Proposition 6.1, we will use the following result of Grothendieck [7, p. 90], where $A \otimes^\wedge B$ denotes the completion of the algebraic tensor product $A \otimes B$ in the norm given by

$$\| \textstyle\sum_{i=1}^{n} x_i \otimes y_i \| = \sup \{ | \textstyle\sum_{i=1}^{n} f_i(x_i) g_i(y_i) | : f_i \in S_A^*, g_i \in S_B^* \},$$

where S_A^* and S_B^* are the unit spheres of the dual spaces of A and B, respectively.

LEMMA 6.2. [7]. *If S is a compact Hausdorff space, and if B is a Banach space, then $C(S, B)$ is isometrically isomorphic to $C(S) \otimes^\wedge B$.*

In view of Lemma 6.2, and the fact that $C(S)$ is a separable π_1-space if S is compact metric ([11] or [12]), Proposition 6.1 is a special case of the following result.

PROPOSITION 6.3. *If A and B are separable π_1-spaces, so is $A \otimes^\wedge B$.*

Proof. Let $A_1 \subset A_2 \subset \cdots$ be a π_1-sequence for A, let $B_1 \subset B_2 \subset \cdots$ be a π_1-sequence for B, and let $C_n = A_n \otimes^\wedge B_n$ for all n. By a result of Schatten [16, Lemma 2.12], the norm on $A_n \otimes^\wedge B_n$ is the same as the one this space inherits as a subspace of $A \otimes^\wedge B$, so that $C_n \subset A \otimes^\wedge B$ as a normed linear space. Also $C_1 \subset C_2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} C_n$ is dense in $A \otimes^\wedge B$. It remains to find a projection of norm one from $A \otimes^\wedge B$ onto each C_n.

Let $\alpha_n : A \to A_n$ and $\beta_n : B \to B_n$ be projections of norm one. Define

$$\alpha_n \otimes^\wedge \beta_n : A \otimes^\wedge B \to A_n \otimes^\wedge B_n$$

by extending the algebraic tensor product

$$\alpha_n \otimes \beta_n : A \otimes B \to A_n \otimes B_n$$

over the completion $A \otimes^\wedge B$. This is possible because, as is easily checked from the definition of the norms, $\alpha_n \otimes \beta_n$ is bounded, with

$$\| \alpha_n \otimes \beta_n \| \leq \| \alpha_n \| \| \beta_n \| = 1.$$

Hence also $\| \alpha_n \otimes^\wedge \beta_n \| \leq 1$, and $\alpha_n \otimes^\wedge \beta_n$ is the required projection.

7. Two special cases

If the bounded extension property is eliminated from Theorems 1.3 and 1.5, then, as we shall show in Examples 9.2 and 9.5, there may exist no linear extension $u : E \to H$ at all (even when every $f \in E$ can be extended to some $f' \in H$). There are, however, two special cases (which are, in a sense, dual to each other) where the situation is different. We define $r_S : H \to C(S)$ by $r_S f = f \mid S$, and *isomorphic* means linearly homeomorphic.

PROPOSITION 7.1. *If either of the following two conditions is satisfied, there exists a linear extension $u : E \to H$ (although not necessarily of norm one).*

(a) *$r_S H \supset E$, and E is isomorphic to l_1.*

(b) *H is separable, $r_S H = E$, and $r_S^{-1}(0)$ is isomorphic to c_0.*

Proof. (a). Let $H_0 = r_S^{-1}(E)$, and let $\pi = r_S \mid H_0$. Since π is a continuous linear map from H_0 onto E, and E is isomorphic to l_1, there exists a continuous linear inverse $u : E \to H_0$ for π (i.e. $\pi uf = f$ for all $f \in E$) [5, *p.* 31 (12)]. But that means that u is a linear extension.

(b) Let $K = r_S^{-1}(0)$. Since K is isomorphic to c_0, a theorem of Sobczyk [17] [15; Theorem 4] implies that there exists a projection p from H onto K. Hence there is an isomorphism v from $p^{-1}(0)$ onto the quotient space H/K, defined by $v(g) = g + K$. Since r_S maps H onto E, the open mapping theorem implies that r_S is a quotient map, so there is an isomorphism w from E onto H/K, defined by $w(f) = r_S^{-1}(f) + K$. Then $u : E \to p^{-1}(0) \subset H$, defined by $u = v^{-1} \circ w$, is a linear extension.

8. Linear extensions u with $u(1_S) = 1_T$

As observed in Section 4, the Borsuk-Dugundji theorem (Theorem 1.1) always permits the linear extension u to be chosen so that $u(1_S) = 1_T$. The purpose of this section is to prove that this remains true for Theorem 1.3, provided we either use real scalars, or use complex scalars and assume that $H \subset C(T)$ is self-adjoint (i.e. $f \in H$ implies $\bar{f} \in H$). (Example 9.1 shows that this may be false with complex scalars if H is not self-adjoint.)

THEOREM 8.1. *Suppose that S is compact metric, that $(C(S), H)$ has the bounded extension property, and that $1_T \in H$. Then:*

(a) *With real scalars, there exists a linear extension $u : C(S) \to H$ of norm 1 with $u(1_S) = 1_T$.*

(b) *With complex scalars, and with H self-adjoint, there exists a linear extension $u : C(S) \to H$ of norm 1 with $u(1_S) = 1_T$. Moreover, $u\bar{f} = \overline{uf}$ for every $f \in C(S)$.*

Proof that 8.1(a) *implies* 8.1(b). Let $C(S)$ and H be as in 8.1(b), and let $C_R(S)$ and H_R denote the spaces of real-valued functions in $C(S)$ and H, respectively. Then $1_T \in H_R$, and it is easily checked that $(C_R(S), H_R)$ has the bounded extension property. Hence, by 8.1(a), there exists a real-linear extension $u_R : C_R(S) \to H_R$ of norm 1 with $u(1_S) = 1_T$. Now define $u : C(S) \to H$ by

$$uf = u_R \operatorname{Re} f + iu_R \operatorname{Im} f.$$

It is easy to check that u is a complex-linear extension, that $u1_S = u1_T$, and that $u\bar{f} = \overline{uf}$ for every $f \in C(S)$. It remains to verify that $\| u \| \leq 1$.

Let $f \in C(S)$ with $\| f \| \leq 1$, let $t \in T$, and let us show that $| (uf)(t) | \leq 1$. Pick a complex scalar α, with $| \alpha | = 1$, so that $(\alpha uf)(t)$ is real. Letting $f' = \alpha f$, it follows that

$$\begin{aligned} |(uf)(t)| &= |(uf')(t)| = |(u_R \operatorname{Re} f')(t)| \leq \| u_R \operatorname{Re} f' \| \\ &= \| \operatorname{Re} f' \| \leq \| f' \| = \| f \| \leq 1. \end{aligned}$$

That completes the proof that 8.1(a) implies 8.1(b).

The remainder of this section will be devoted to proving Theorem 8.1(a). We begin with several preliminary observations and results. The hypotheses of Theorem 8.1(a) will always be tacitly assumed.

It was noted in the introduction that $C(S)$ is a π_1-space, but we can be more precise: Call $\Phi = \{\phi_1, \cdots, \phi_n\}$ a *peaked partition of unity* on S if it is a partition of unity and if, for each $i = 1, \cdots, n$, there is an $s_i \in S$ such that $\phi_i(s_i) = 1$. The linear space $[\Phi]$ spanned by such a Φ is called a *peaked partition subspace* of $C(S)$, and the map $\pi : C(S) \to [\Phi]$, defined by

$$(*) \qquad \pi f = \sum_{i=1}^{n} f(s_i)\phi_i,$$

is a projection of norm 1 onto $[\Phi]$ (see [11]). It was proved in [11] that, for compact metric S, the space $C(S)$ always has a π_1-sequence $F_1 \subset F_2 \subset \cdots$ consisting of peaked partition subspaces, and with $F_1 = [1_S]$.

If $F \subset C(S)$ and $u \in \Lambda(F, H)$, let us say that $u \geq 0$ if $uf \geq 0$ whenever $f \geq 0$. The following lemma follows from $(*)$.

LEMMA 8.2. *If $\Phi = \{\phi_1, \cdots, \phi_n\}$ is a peaked partition of unity on S, if $w \in \Lambda([\Phi], H)$, and if $w\phi_i \geq 0$ for $i = 1, \cdots, n$, then $w \geq 0$.*

LEMMA 8.3. *Let $f \in C(S)$, $f \geq 0$, let $\varepsilon > 0$, and let $f' \in H$ be an extension of f with $f' \geq -\varepsilon$. Then f can be extended to some $f^* \in H$ with $f^* \geq 0$ and $\| f^* - f' \| \leq 4\varepsilon$.*

Proof. Let $g = \inf(f', 0)$. Then $g \mid S = 0$. Let

$$\Delta = 2\varepsilon 1_T + g.$$

Then $\Delta \in \mathfrak{D}_T$ (see Section 5), and $\Delta \mid S = 2\varepsilon 1_S$. Hence, by Corollary 5.3, there is an extension of $2\varepsilon 1_S$ to some $h \in H$ with $| h | \leq \Delta$, and clearly $\| h \| = \| \Delta \| = 2\varepsilon$. If we now let

$$f^* = f' + 2\varepsilon 1_T - h,$$

then it is easy to verify that f^* satisfies all our requirements. That completes the proof.

COROLLARY 8.4. *Let G be a peaked partition subspace of $C(S)$. Let $v \in \Lambda(G, H)$, let $\varepsilon > 0$, and suppose that $vg \geq -\varepsilon$ whenever $g \in G$ and $0 \leq g \leq 1$. Then there exists $w \in \Lambda(G, H)$ with $w \geq 0$ and $\| w - v \| \leq 4n\varepsilon$, where $n = \dim G$.*

Proof. Let $G = [\Phi]$, where $\Phi = \{\phi_1, \cdots, \phi_n\}$ is a peaked partition of unity on S. Then $0 \leq \phi_i \leq 1$ for $i = 1, \cdots, n$, so $v\phi_i \geq -\varepsilon$ for all i. By Lemma 8.3, each ϕ_i can be extended to some $\phi_i^* \in H$ with $\phi_i^* \geq 0$ and $\| v\phi_i - \phi_i^* \| \leq 4\varepsilon$. Let w be the unique element of $\Lambda(G, H)$ such that

$w\phi_i = \phi_i^*$ for $i = 1, \cdots, n$. Then $w \geq 0$ by Lemma 8.2, and $\| (v - w)\phi_i \| \leq 4\varepsilon$ for $i = 1, \cdots, n$. Hence $\| v - w \| \leq 4n\varepsilon$ by $(*)$, and that completes the proof.

PROPOSITION 8.5. *Let F, G be peaked partition subspaces of $C(S)$, $u \in \Lambda(F, H)$, and $\beta > 0$. Then there exists $w \in \Lambda(G, H)$ such that $w \geq 0$ and*

$$\| uf + wg \| < 1 + \beta$$

whenever $f \in F$, $g \in G$, $\| f + g \| \leq 1$, and $\| uf \| \leq 1$.

Proof. Let us first observe that Proposition 2.2 remains true if we are given finite-dimensional subspaces $F_i \subset C(S)$ $(i = 1, \cdots, n)$, and $u_i \in \Lambda(F_i, C(T))$ for each i, and require that v behave properly with respect to each u_i. The proof is the same, except that now K_v and α_v must be defined with consideration for all the u_i.

Let us now apply the previous paragraph with $n = 2$, $F_1 = F$, $u_1 = u$, $F_2 = [1_S]$, $u_2(\lambda 1_S) = \lambda 1_T$, and $\gamma = \beta(1 + 8n)^{-1}$, where $n = \dim G$, yielding a suitable $v \in \Lambda(G, H)$. This v satisfies the hypothesis of Corollary 8.4, with ε replace by γ, for if $g \in G$ and $0 \leq g \leq 1$, then $\| 1_S + (-g) \| \leq 1$ and $\| u_2 1_S \| = 1$, so $\| u_2 1_S + v(-g) \| < 1 + \gamma$, whence $\| 1_T - vg \| < 1 + \gamma$, and thus $vg > -\gamma$. By Corollary 8.4, there is thus a $w \in \Lambda(G, H)$ with $w \geq 0$ and $\| w - v \| < 4n\gamma$. Now if $f \in F$, $g \in G$, $\| f + g \| \leq 1$, and $\| uf \| \leq 1$, then $\| g \| \leq 2$, and hence

$$\| uf + wg \| \leq \| uf + vg \| + \| vg - wg \| < (1 + \gamma) + 8n\gamma = 1 + \beta.$$

That completes the proof.

COROLLARY 8.6. *If F is a peaked partition subspace of $C(S)$, if $\varepsilon > 0$, and if $u \in \Lambda(F, H)$ with $\| u \| < 1 + \varepsilon$ and $u \geq 0$, then there exists $w \in \Lambda(F, H)$ with $\| w \| = 1$, $w \geq 0$, and $\| u - w \| < \varepsilon$.*

Proof. This follows from Proposition 8.5 precisely as Corollary 2.4 followed from Proposition 2.2. That completes the proof.

Let $s_0 \in S$ be fixed. Then clearly any $f \in C(S)$ has a unique decomposition

$$f = f_0 + \lambda 1_S,$$

where $f_0(s_0) = 0$ and λ is real (in fact, $\lambda = f(s_0)$).

The proof of the following lemma can be left to the reader.

LEMMA 8.7. *Suppose that $F \subset C(S)$, $1_S \in F$, $1_T \in H$, and $u \in \Lambda(F, H)$. Define $u' \in \Lambda(F, H)$ by*

$$u'(f_0 + \lambda 1_S) = uf_0 + \lambda 1_T.$$

If $u \geq 0$ and $\| u \| = 1$, then $u' \geq 0$ and $\| u' \| = 1$. Moreover,

$$\| u - u' \| = \| u1_S - 1_T \|.$$

LEMMA 8.8. *Let $F \subset G$ be peaked partition subspaces of $C(S)$, let $u \in \Lambda(F, H)$*

with $\| u \| = 1$ *and* $u1_S = 1_T$, *and let* $\delta > 0$. *Then there exists a* $u' \in \Lambda(G, H)$, *with* $\| u' \| = 1$ *and* $u'1_S = 1_T$, *such that* $\| u - u' \mid F \| < \delta$.

Proof. Since G is a peaked partition subspace of $C(S)$, there exists a projection $\pi : G \to F$ of norm 1. We can therefore apply Corollary 2.3, with $\varepsilon = \delta(2 + 16n)^{-1}$, where $n = \dim G$, to extend u to some $v \in \Lambda(G, H)$ with $\| v \| < 1 + \varepsilon$. Now if $g \in G$ and $0 \le g \le 1$, *then* $\| 1_S - g \| \le 1$, so

$$\| 1_T - vg \| = \| v(1_S - g) \| < 1 + \varepsilon,$$

and hence $vg \ge -\varepsilon$. We can therefore apply Corollary 8.4 to obtain $w \in \Lambda(G, H)$ with $w \ge 0$ and $\| w - v \| < 4n\varepsilon$. Hence $\| w \| < 1 + (1 + 4n)\varepsilon$.

Next, we apply Corollary 8.6 to obtain $w' \in \Lambda(G, H)$ with $\| w' \| = 1$, $w' \ge 0$, and $\| w - w' \| < (1 + 4n)\varepsilon$. Finally, we apply Lemma 8.7 to find $u' \in \Lambda(G, H)$ with $\| u' \| = 1$ and

$$\| w' - u' \| = \| w'1_S - 1_T \| = \| (w' - v)1_S \| \le \| w' - v \|.$$

Hence

$$\begin{aligned} \| u - u' \mid F \| &\le \| v - u' \| \le \| v - w' \| + \| w' - u' \| \\ &\le 2 \| w' - v \| \le 2 (\| w' - w \| + \| w - v \|) \\ &\le 2 (1 + 8n)\varepsilon = \delta. \end{aligned}$$

That completes the proof.

Proof of Theorem 8.1(a). *Let* $F_1 \subset F_2 \subset \cdots$ be a π_1-sequence for $C(S)$ consisting of peaked partition subspaces, with $F_1 = [1_S]$. Define $u_1 \in \Lambda(F_1, H)$ by $u_1(\lambda 1_S) = \lambda 1_T$. By Lemma 8.8 we can now inductively define $u_n \in \Lambda(F_n, H)$, with $\| u_n \| = 1$ and $u_n 1_S = 1_T$, so that $\|u_{n+1} \mid F_n - u_n \| < 2^{-n}$ for all n. We now proceed exactly as the proof of Theorem 1.5 to obtain a $u \in \Lambda(C(S), H)$ with $\| u \| = 1$ and $u1_S = 1_T$, and that completes the proof.

9. Examples

Our first example shows that, for complex scalars, Theorem 1.3—and hence also Theorem 1.5—cannot be strengthened by asserting that there exists a linear extension $u : E \to H$ of norm one such that $u1_S = u1_T$. (See, however, Theorems 4.1 and 8.1.) (The disc algebra is defined in the remark following Theorem 1.3.)

EXAMPLE 9.1. *Let* T *be the unit circle, and let* $S \subset T$ *contain at least two points and have Lebesgue measure zero. Let* $E = C(S)$ *and let* $H \subset C(T)$ *be the disc algebra. Then there is no linear extension* $u : C(S) \to H$ *of norm one such that* $u1_S = 1_T$.

Proof. Let s_1 and s_2 be two different points in S. Choose f_1 in $C(S)$ such that

$$1 = f_1(s_1) \ge f(s) \ge f_1(s_2) = 0 \qquad \text{for } s \in S,$$

and let $f_2 = 1_S - f_1$. Then $\| f_i \| = f_i(s_i) = 1$ $(i = 1, 2)$, $f_1 + f_2 = 1_S$, and $f_i(s_j) = 0$ for $j \neq i$ $(i, j = 1, 2)$. Hence, for arbitrary complex numbers a_1 and a_2,

$$\max_{i=1,2} | a_i | \geq \| a_1 f_1 + a_2 f_2 \| \\ \geq \max_{i=1,2} | (a_1 f_1 + a_2 f_2)(s_i) | = \max_{i=1,2} | a_i |.$$

Now let $u : C(S) \to H$ be a linear extension of norm 1. Clearly u is a linear isometry from $C(S)$ into H. Therefore, for arbitrary complex numbers a_1 and a_2,

$$\| a_1 uf_1 + a_2 uf_2 \| = \| a_1 f_1 + a_2 f_2 \| = \max_{i=1,2} | a_i |.$$

This implies (by taking $\bar{a}_i = (uf_i)(t)$) that

$$| (uf_1)(t) | + | (uf_2)(t) | \leq 1 \qquad \text{for } t \in T. \tag{1}$$

Now if $u1_S = 1_T$, then $uf_1 + uf_2 = 1_T$. Hence

$$uf_1(t) + uf_2(t) = 1 \qquad \text{for } t \in T. \tag{2}$$

It follows from (1) and (2) that uf_1 and uf_2 are non-negative—and hence real—functions, and must therefore be constant because they belong to the disc algebra. But this leads to a contradiction, because

$$uf_1(s_1) = f_1(s_1) \neq f_1(s_2) = uf_1(s_2).$$

That completes the proof.

Our last three examples show how Theorems 1.3 and 1.5 can become false if the bounded extension property is omitted from the hypotheses, or even if it is weakened by omitting the word "bounded" in Definition 1.2.

The following example provides a converse to Proposition 7.1(a).

EXAMPLE 9.2. *Let S be any infinite, closed subset of the interval $[-1, 0]$, and let $T = S \cup [1, 2]$. Let $E = C(S)$ or, more generally, any infinite-dimensional closed linear subspace of $C(S)$ which is not isomorphic to l_1. Then there is a subspace H of $C(T)$ such that:*

(a) *Every $f \in E$ has an extension $f' \in H$.*

(b) *There is no linear extension $u : E \to H$.*

Proof. By [1, p. 111], the separable Banach space E is the image of l_1 under a linear map $\alpha : l_1 \to E$ with $\| \alpha \| = 1$. By [5, p. 93], there is a linear isometry β from l_1 into $C([1, 2])$. Let us define an isometric isomorphism γ from l_1 into $C(T)$ by

$$(\gamma x)(t) = (\alpha x)(t) \quad \text{if} \quad t \in S,$$

$$(\gamma x)(t) = (\beta x)(t) \quad \text{if} \quad t \in T - S,$$

and let $H = \gamma l_1$.

Clearly (a) is satisfied, because $\alpha l_1 = E$. To prove (b), suppose there were a linear extension $u : E \to H$. Then $u(E)$ is infinite-dimensional and compleplemented in H (with complement $\{f \in H : f \mid S = 0\}$); hence, since H is isomorphic to l_1, so is $u(E)$ by [15, Theorem 1]. But $u(E)$ is isomorphic to E (since $\| f \| \leq \| uf \| \leq \| u \| \| f \|$ for every $f \in E$), so E is isomorphic to l_1, contrary to our assumptions. That completes the proof.

Remark. In certain cases, such as when S is a convergent sequence and $E = C(S)$, it can be shown that 9.2(a) can be strengthened by choosing f' so that $\| f' \| = \| f \|$.

The verification of the following simple example can be left to the reader. Note that here there are linear extensions, but not of norm one.

EXAMPLE 9.3. *Let* $T = [0, 2]$, $S = [0, 1]$, *and*

$$H = \left\{ f \in C(T) : \int_0^2 f(t)\, dt = 0 \right\}.$$

Then

(a) *If* $W \supset S$ *is open in* T, *then every* $f \in E$ *has an extension* $f' \in H$ *such that* $f'(t) = 0$ *when* $t \in T - W$.

(b) *There is no linear extension* $u : C(S) \to H$ *of norm* 1 (*in fact,* 1_S *has no extension in* H *of norm* 1).

In the proof of Example 9.5, we will need the following lemma, which seems to be known among Banach space specialists, but for which we have found no reference in the literature; the proof, which is somewhat complicated, is omitted.

LEMMA 9.4. *Let* $L = (l_1 \times l_1 \times \cdots)_{c_0}$ *be the Banach space of sequences* $x = (x_n)$, *with* $x_n \in l_1$ *for all* n *and* $\| x_n \| \to 0$, *and with* $\| x \|$ *defined by* $\| x \| = \sup_{n=1}^{\infty} \| x_n \|$. *Then* L *has no infinite-dimensional reflexive subspaces.*

EXAMPLE 9.5. *There exists a compact metric space* T, *closed* $S \subset T$, *and closed subspaces* $E \subset C(S)$ *and* $H \subset C(T)$ *such that:*

(a) *If* $W \supset S$ *is open in* T, *then every* $f \in E$ *has an extension* $f' \in H$ *such that* $f'(t) = 0$ *if* $t \in T - W$.

(b) *There is no linear extension* $u : E \to H$.

Proof. Let S be the interval $[-1, 0]$, let

$$I_n = [2^{-2n}, 2^{-2n+1}],$$

and let

$$T = S \cup \bigcup_{n=1}^{\infty} I_n .$$

Let

$$E = C_0(S) = \{f \in C(S) : f(0) = 0\}$$

or, more generally, any subspace of $C_0(S)$ which contains an infinite-dimensional reflexive subspace.[9] By [1, p. 111], the separable Banach space E is the

[9] Since every separable Banach space is isomorphic to a subspace of $C_0(S)$, E can thus be chosen "almost arbitrarily".

image of l_1 under a linear map $\alpha : l_1 \to E$ with $\|\alpha\| = 1$. By [5, p. 93], there are linear isometries $\beta_n : l_1 \to C(I_n)$ (into) for all n. Let L be the space of Lemma 9.4, and define the linear map $\gamma : L \to C(T)$ by

$$(\gamma x)(t) = (\beta_n x_n)(t) \qquad \text{if} \quad t \in I_n,$$
$$(\gamma x)(t) = \textstyle\sum_{n=1}^{\infty} 2^{-n}(\alpha x_n)(t) \quad \text{if} \quad t \in S.$$

It is easily checked that indeed $\gamma x \in C(T)$ for all $x \in L$. Let $H = \gamma(L)$. To verify (a), observe first that $W \supset I_m$ for some m; if $f \in E$, we pick $x \in L$ such that $\alpha x_m = f$ and $x_n = 0$ for $n \neq m$, and then take $f' = \gamma x$. It remains to verify (b) and the fact that H is closed in $C(T)$.

Suppose there were a linear extension $u : E \to H$. Then u is an isomorphism from E into H (since $\|f\| \leq \|uf\| \leq \|u\|\,\|f\|$ for all $f \in E$), so H has an infinite-dimensional reflexive subspace. By Lemma 9.4, we can now obtain a contradiction by showing that γ is an isometry; that will also prove that H is closed in $C(T)$.

Let $x \in L$. Then

$$\|\gamma x\| \geq \sup_{n=1}^{\infty} \|\beta_n x_n\| = \sup_{n=1}^{\infty} \|x_n\| = \|x\|.$$

To check that also $\|\gamma x\| \leq \|x\|$, we will show that $|(\gamma x)(t)| \leq \|x\|$ for every $t \in T$. If $t \in I_n$, then

$$|(\gamma x)(t)| = |(\beta_n x_n)(t)| \leq \|\beta_n\|\,\|x_n\| = \|x_n\| \leq \|x\|.$$

If $t \in S$, then

$$|(\gamma x)(t)| \leq \textstyle\sum_{n=1}^{\infty} 2^{-n}\|\alpha x_n\| \leq \sum_{n=1}^{\infty} 2^{-n}\|x\| = \|x\|.$$

That completes the proof.

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