ALGEBRAICALLY TRIVIAL DECOMPOSITIONS OF HOMOTOPY 3-SPHERES[1]

BY

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Every compact 3-manifold M^3 without boundary possesses a cell-decomposition Ψ that contains just one vertex, say O, (see for instance [3, Sec. 5]). From Ψ we may read by a well-known procedure (see [7, §62]) a *"corresponding" presentation*

$$\mathfrak{P}(\Psi) = (\{g_1, \cdots, g_a\}, \{\mathfrak{r}_1, \cdots, \mathfrak{r}_b\})$$

of the fundamental group $\pi_1(M^3)$ where the generators $g_1, \cdots, g_a$ are in 1-1 correspondence with the (oriented) 1-dimensional elements $E_1^1, \cdots, E_a^1$ of Ψ and the relators $\mathfrak{r}_1, \cdots, \mathfrak{r}_b$ are in 1-1 correspondence with the 2-dimensional elements $E_1^2, \cdots, E_b^2$ of Ψ, i.e., $\mathfrak{r}_j$ is a word in the $g_i^{\pm 1}$'s obtained by running once around the boundary of E_j^2. In this way the relators $\mathfrak{r}_j$ are uniquely defined up to cyclic permutations and inversions, i.e., if we denote by $\langle \mathfrak{r}_j \rangle$ the set of all cyclic permutations of $\mathfrak{r}_j$ and of $\mathfrak{r}_j^{-1}$ then the $\langle \mathfrak{r}_j \rangle$'s are uniquely defined.

In the special case that M^3 is a homotopy 3-sphere, $\mathfrak{P}(\Psi)$ is a presentation of the trivial group. However, it is—in general—an unsolved problem to recognize whether or not a given presentation $\mathfrak{P}(\Psi)$ presents the trivial group; this problem seems to be extremely difficult and it may be unsolvaable, since the triviality problem of group theory is unsolvable (see [1], [6]). One might expect that these group theoretic difficulties are also the reason for the difficulties of the Poincaré problem. But the result of this paper shows that this is not so: We shall prove that every homotopy 3-sphere M^3 possesses a cell-decomposition Ψ such that the corresponding presentation

$$\mathfrak{P}(\Psi) = (\{g_1, \cdots, g_a\}, \{\mathfrak{r}_1, \cdots, \mathfrak{r}_b\})$$

is *obviously trivial*, i.e., such that $\mathfrak{P}(\Psi)$ can be transformed by simple cancellation operations (without changing the generators g_i and the number b of relators) into the *"standard trivial presentation"*

$$\mathfrak{O} = (\{g_1, \cdots, g_a\}, \{g_1, \cdots, g_a, \overset{b-a}{*}\})$$

where $\overset{b-a}{*}$ means that $\mathfrak{O}$ contains $b - a$ times the empty relator (i.e., the relations of $\mathfrak{O}$ are[2] $g_1 = 1, \cdots, g_a = 1$, and $b - a$ times the trivial relation $1 = 1$). To make this precise we say that a presentation $\mathfrak{P}''$ is obtained from

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[2] Here the equality sign means that both sides of the equation represent the same group element; but in general, if not stated otherwise, we call two words equal if and only if they read, letter by letter, in the same way.

a presentation $\mathfrak{P}' = (\{g_1, \cdots, g_a\}, \{\mathfrak{r}'_1, \cdots, \mathfrak{r}'_b\})$ by a *cancellation operation* of Type 1 or 2, respectively, if the following holds:

Type 1. (Cancelling a syllable $g_i^{-1} g_i$.) For some i, j ($i = 1, \cdots, a$; $j = 1, \cdots, b$) there is a word $\mathfrak{r}''$ such that $g_i^{-1} g_i \mathfrak{r}'' \in \langle \mathfrak{r}'_j \rangle$ and $\mathfrak{P}''$ is obtained by replacing $\mathfrak{r}'_j$ by $\mathfrak{r}''$. (Note that this operation does not in general allow cancelling a syllable $g_i g_i^{-1}$.)

Type 2. (Cancelling a syllable which is itself a relator.) For some j, k ($j, k = 1, \cdots, b; j \neq k$) there are words $\mathfrak{r}'^*_k$, $\mathfrak{r}''$ such that $\mathfrak{r}'^*_k \in \langle \mathfrak{r}'_k \rangle$, $\mathfrak{r}'^*_k \mathfrak{r}'' \in \langle \mathfrak{r}'_j \rangle$, and $\mathfrak{P}''$ is obtained by replacing $\mathfrak{r}'_j$ by $\mathfrak{r}''$; the length l of $\mathfrak{r}'_k$ is called the *length* of the cancellation operation.

We shall prove the following

THEOREM. *If M^3 is a homotopy 3-sphere then there exists a cell-decomposition Ψ of M^3, containing just one vertex O, such that a corresponding presentation*

$$\mathfrak{P}(\Psi) = (\{g_1, \cdots, g_a\}, \{\mathfrak{r}_1, \cdots, \mathfrak{r}_b\})$$

of the fundamental group $\pi_1(M^3)$ with generators g and relators $\mathfrak{r}$ can be transformed into the standard trivial presentation

$$\mathfrak{O} = (\{g_1, \cdots, g_a\}, g_1, \cdots, g_a, \overset{b-a}{*}\})$$

by means of a finite sequence of cancellation operations of Type 1 *and a subsequent finite sequence of cancellation operations of Type* 2 *with lengths not exceeding* 3.

One might call a cell-decomposition Ψ with the above properties "*algebraically trivial*". I hope that the above theorem will be useful for deriving a proof of the Poincaré conjecture. However, this remains a difficult problem. A reason for the difficulty is the lack of correspondence between Tietze transformations of the group presentation $\mathfrak{P}(\Psi)$ and transformations of the cell-decomposition Ψ. If a presentation $\mathfrak{Q}$ is derived from $\mathfrak{P}(\Psi)$ by a Tietze transformation then we may ask the question: does there exist a cell-decomposition Ω of M^3 such that $\mathfrak{Q}$ corresponds to Ω? Let us call the Tietze transformation *good* if the answer to the question is "yes", and *bad* if the answer is "no". Unfortunately, most Tietze transformations are bad from this point of view. The only large class of good and simple Tietze transformations I know are the *eliminations*: If $\mathfrak{P}(\Psi) = (\{g_1, \cdots, g_a\}, \{\mathfrak{r}_1, \cdots, \mathfrak{r}_b\})$ contains a relator, say $\mathfrak{r}_l$, such that for some k $g_k \mathfrak{w}^{-1} \in \langle \mathfrak{r}_l \rangle$ where $\mathfrak{w}$ is a word in the g_i's not containing $g_k^{\pm 1}$, and if $\mathfrak{Q}$ is obtained from $\mathfrak{P}(\Psi)$ by deleting g_k and $\mathfrak{r}_l$ and replacing in all relators $\mathfrak{r}_j$ ($j \neq l$) the letter $g_k^{\pm 1}$ by the word $\mathfrak{w}^{\pm 1}$, then the Tietze transformation $\mathfrak{P}(\Psi) \to \mathfrak{Q}$ is good. Moreover, I would like to remark without proof: If it were possible to restrict the lengths of the cancellation operations in our theorem to 2 instead of 3 then the sequence of cancellation operations could be changed into a sequence of good Tietze transformations. This would mean a proof of the Poincaré conjecture since it is easy to show that M^3 is a 3-sphere if it possesses a cell-decomposition Ω such that the standard trivial presentation $\mathfrak{O}$ corresponds to Ω.

Proof of the theorem

1. Preliminaries. Let M^3 be a *homotopy* 3-*sphere*, i.e., a compact, simply connected 3-manifold without boundary.

We choose the semilinear standpoint as described in [4, Sec. 3]; i.e., we assume that all point sets, denoted by *capital roman letters*, are *piecewise rectilinear polyhedral point sets* in a euclidean space $\mathfrak{E}^n$ of sufficiently large dimension n, etc. We denote the *closure, boundary*, and *interior* of a point set X by $\overline{X}$, $^{\cdot}X$, $^{\circ}X$, respectively.

2. The idea of the proof. First we consider (as in [3], [5]) a cell-decomposition Γ of M^3 into one vertex E^0, r open arcs E_i^1, r open disks E_i^2, and one open 3-cell, and a singular fan V^2 corresponding to Γ (i.e., a wedge of r singular disks V_i^2 with $^{\cdot}V_i^2 = \bar{E}_i^1$ where the V_i^2's may intersect themselves and each other in double arcs; for details see [3, Sec. 5, 6]). Let T^3 be a small neighborhood of $^{\cdot}V^2$ in M^3. Now we consider the "middle parts," A_{*j}^1, of the double arcs A_j^1 $(j = 1, \cdots, s)$ of V^2 that lie outside of $^{\circ}T^3$ (see Fig. 1) and the "middle part" V_2^* of V^2 obtained from V^2 by removing its boundary $^{\cdot}V^2$ and the open annuli that lie in the $V_i^2 \cap T^3$'s between $^{\cdot}V_i^2$ and $V_i^2 \cap {}^{\cdot}T^3$. Since T^3 is a Heegaard-handlebody in M^3 we can "project" the A_{*j}^1's into $^{\cdot}T^3$ (in the same way we projected the arcs B in [5, Sec. 3]) so that we obtain a projection cylinder K_j^2 for each arc A_{*j}^1. Now we "thicken" V_*^2 and we obtain by this a 3-dimensional polyhedron V_*^3 where $V_*^3 + T^3$ is obviously a handlebody with s handles corresponding to the A_{*j}^1's. Moreover, one can show that $V_*^3 + T^3$ is a Heegaard-handlebody in M^3, and that those parts, say K_{*h}^2 $(h = 1, \cdots, b)$, of the projection cylinders K_j^2 that lie outside of $^{\circ}(V_*^3 + T^3)$ contain a complete system of meridian disks of $M^3 - {}^{\circ}(V_*^3 + T^3)$.

Now one may expect to obtain an especially simple Heegaard-diagram of M^3 (and a corresponding cell-decomposition Ψ; compare [5, Sec. 8]) from the handlebody $V_*^3 + T^3$ and the outer meridian disks K_{*h}^2. It remains to select inner meridian disks X_j^2 $(j = 1, \cdots, s)$ of $V_*^3 + T^3$ in a suitable way. This can be done as indicated in Fig. 1: The polyehdron X_j in Fig. 1 consists of two disks in T^3, parallel to the disk $V_{X''j}^2 \subset V_*^2 \cap T^3$, and one arc outside of $^{\circ}T^3$ joining these disks in V_*^2 (encircling A_{*j}^1 and the disk $V_{X'j}^2 \subset V_*^2 + T^3$). If V_*^2 is thickened to V_*^3 then the joining arc of X_j may be thickened to a disk which (together with the two disks in T^3) yieldsa meridian disk X_j^2 of $V_*^3 + T^3$.

First let us discuss the pleasant case that the arcs A_{*j}^1 are unknotted and unlinked over $^{\cdot}T^3$, i.e., that there can be found projection cylinders K_j^2 which are nonsingular and pairwise disjoint. In this case we obtain a Heegaard-diagram[3] which is so simple that it is fairly easy to show that M^3 is a 3-sphere:

[3] Here we admit the case that the number of "outer" meridian circles $^{\cdot}K_{*h}^2$ is greater than the genus s of the Heegaard-surface $^{\cdot}(V_*^3 + T^3)$.

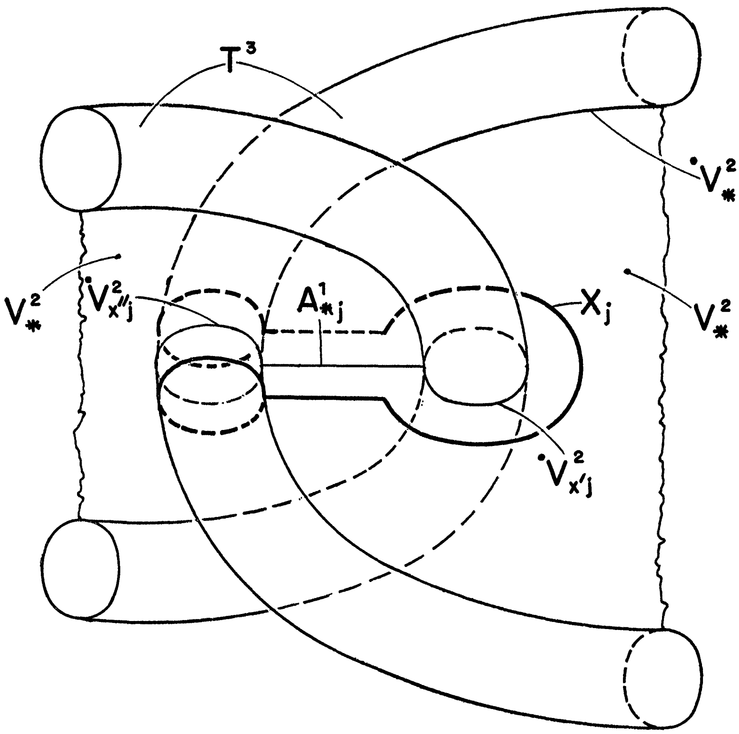


FIG. I

The "projection arc" ${}^{\cdot}K_j^2 \cap {}^{\cdot}T^3$ of A^1_{*j} (see Fig. 2) intersects the meridian circle ${}^{\cdot}X_j^2$ in a point near the point ${}^{\cdot}A^1_{*j} \cap {}^{\cdot}V^2_{X''j}$; it may have further intersections with the circles ${}^{\cdot}V^2_{X'k}$, ${}^{\cdot}V^2_{X''k}$ ($k = 1, \cdots, s$) and with ${}^{\cdot}V^2_*$ where the intersections with the ${}^{\cdot}V^2_{X'k}$'s and ${}^{\cdot}V^2_{X''k}$'s lie close to intersections of K_j^2 with the X_k^2's. (In Fig. 2 it is assumed that the projection arc, from left to right, intersects the circles ${}^{\cdot}V^2_{X''p}$, ${}^{\cdot}V^2_{X'f}$, ${}^{\cdot}V^2_{X'l}$, ${}^{\cdot}V^2_*$, ${}^{\cdot}V^2_{X''m}$, ${}^{\cdot}V^2_{X''q}$.) We may easily achieve that no connected component of ${}^{\cdot}V^3_* \cap {}^{\circ}K_j^2$ is a closed curve but that all these connected components are open arcs with end points in ${}^{\circ}({}^{\cdot}K_j^2 \cap {}^{\cdot}T^3)$. Now, ${}^{-}(K_j^2 - V^3_*)$ contains at least one "inner" component, say K^2_{*1}, that borders on just one connected component of $V^3_* \cap K_j^2$. (In Fig. 2 the components K^2_{*1} and K^2_{*3} are inner ones.) This disk K^2_{*1} corresponds to a relator $\mathfrak{r}_1$ of the group presentation $\mathfrak{P}$ corresponding to our Heegaard-diagram (the generators g_k corresponding to the oriented meridian[4] circles ${}^{\cdot}X_k^2$) where the

[4] By this we mean that the generator g_k may be represented by an oriented simple closed curve in ${}^{\cdot}(T^3 + V^3_*)$ that pierces ${}^{\cdot}X_k^2$ in just one point, in the positive sense, and is disjoint from ${}^{\cdot}X_1^2, \cdots, {}^{\cdot}X_{k-1}^2, {}^{\cdot}X_{k+1}^2, \cdots, {}^{\cdot}X_s^2$. Each intersection point in ${}^{\cdot}K^2_{*1} \cap {}^{\cdot}X_k^2$ corresponds to a letter $g_k^{\pm 1}$ in the word $\mathfrak{r}_1$.

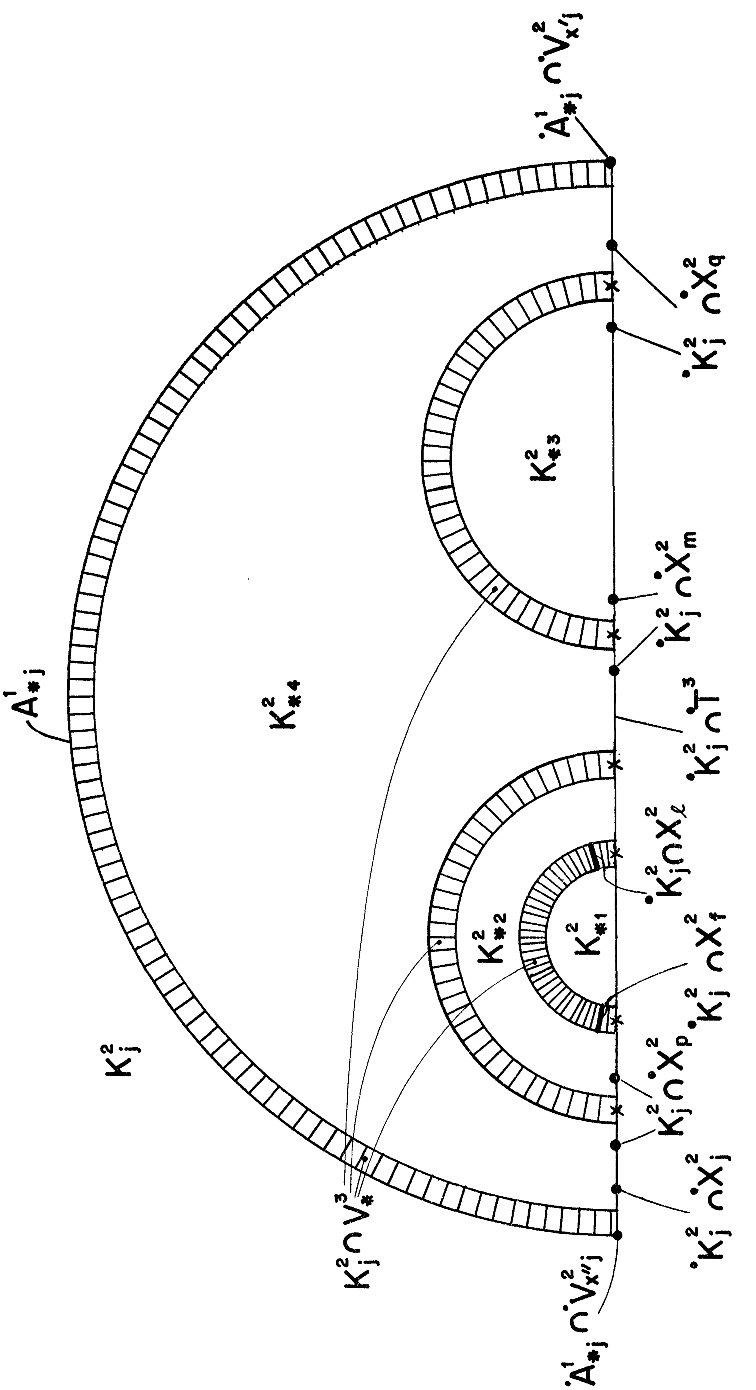


FIG.2 The points on $\dot{K}^2_j \cap \dot{T}^3$ marked by x mean from left to right intersections with $\dot{V}^2_{x''p}, \dot{V}^2_{x'f}, \dot{V}^2_{x'\ell}, \dot{V}^2_*, \dot{V}^2_{x''m}, \dot{V}^2_{x''q}$

length of $\mathfrak{r}_1$ is at most 2. Moreover, if $K^2_{*1} \neq {}^-(K^2_j - V^3_*)$, then there is another connected component, say K^2_{*2}, of ${}^-(K^2_j - V^3_*)$ that corresponds to a relator $\mathfrak{r}_2$ which contains $\mathfrak{r}_1$ as a syllable. That means that we can simplify $\mathfrak{P}$ by a cancellation operation of Type 2 whose length is not greater than 2. Then, if

$$K^2_{*1} + K^2_{*2} \neq {}^-(K^2_j - V^3_*),$$

we can carry out another cancellation operation of that type, and so on, until we obtain a relator equal to g_j. This can be done for all $j = 1, \cdots, s$, yielding a standard trivial presentation $\mathfrak{D}$. Now it is not difficult to show that this sequence of cancellation operations can be replaced by a sequence of good Tietze transformations since no cancellation operation is of length greater than 2. (A cancellation operation of length 2 can be replaced by an elimination such that in all relators a certain generator g_f is replaced by another generator $g_l^{\pm 1}$, and by certain subsequent operations which can be arranged to be good Tietze transformations; the essential point is that the lengths of the relators do not increase under these eliminations.) Hence there is a cell-decomposition Ψ_0 of M^3 that is obviously a cell-decomposition of a 3-sphere.

Of course, one may try to find Γ and V^2 so that the A^1_{*j}'s are unknotted and unlinked over ${}^{\cdot}T^3$. This would prove the Poincaré conjecture. But my attempts in this direction failed. (It was possible to achieve the unknottedness but not the unlinkedness.)

Now we are left with the general case, namely that the arcs A^1_{*j} may be knotted and linked over ${}^{\cdot}T^3$. We may apply a cheap trick: We consider the double arcs, say $C^1_1, \cdots, C^1_u$, of the projection cylinder $K^2 = \bigcup_{j=1}^s K^2_j$ (compare Fig. 5, Case 2, in Sec. 4) and we add small neighborhoods $C^3_1, \cdots, C^3_u$ of them (in $M^3 - {}^{\circ}T^3$) to T^3, obtaining an expanded handlebody $T^3_\vee$. Now we have enforced that those pieces, say A^1_{jk}, of the A^1_{*j}'s that lie outside of ${}^{\circ}T^3_\wedge$ are unknotted and unlinked over ${}^{\cdot}T^3_\vee$, where we simply take $K^2 - {}^{\circ}T^3_\wedge$ for the projection cylinder. Then the connected components of the projection cylinder (i.e., the projection cylinders of the A^1_{jk}'s into ${}^{\cdot}T^3_\wedge$) yield diagrams very similar to Fig. 2. But the essential difference is that the handlebody $V^3_* + T^3_\wedge$ has more handles than $V^3_* + T^3$ (corresponding to the connected components of the $C^3_m \cap V^3_*$'s); therefore we need additional meridian disks in $V^3_* + T^3_\wedge$ (which we shall construct in detail in Sec. 7). These additional meridian disks intersect the projection cylinder, with the result that the "inner disks" correspond to relators which may contain "cancellation syllables" $g_i^{-1} g_i$ and which may remain of length 3 even after the cancellation syllables are deleted. That is the reason why this attempt yields a proof of our theorem but not a proof of the Poincaré conjecture.

3. Projecting the 1-skeleton of a cell-decomposition Δ of the singular fan V^2_* into the Heegaard-surface ${}^{\cdot}T^3$. We consider a cell-decomposition Γ of M^3 that contains just one vertex E^0, just r elements $E^1_1, \cdots, E^1_r, E^2_1, \cdots,$

E_r^2, of each dimension 1 and 2 ($r \neq 0$), and just one 3-dimensional element E^3 (see [3, Sec. 5]). Further we consider a singular fan V^2, defined by a map $\zeta : V'^2 \to M^3$, (as in [3, Sec. 6]), such that the only singularities of V^2 are double arcs $A_1^1, \cdots, A_s^1$ ($s \neq 0$) with inverse images $A_j'^1, A_j''^1$ ($j = 1, \cdots, s$) as in Fig. 3, and such that $\dot{V}^2 = \bigcup_{i=1}^r \bar{E}_i^1$, where V'^2 consists of disks $V_i'^2$ ($i = 1, \cdots, r$) with just one common vertex E'^0 in their boundaries and with $\zeta(\dot{V}_i'^2) + \bar{E}_i^1$. We choose a small neighborhood T^3 of $\bigcup_{i=1}^r \bar{E}_i^1$ in M^3 (as in [5, Sec. 2]) which is a Heegaard-handlebody in M^3.

Notation. (See Fig. 3.) We denote the connected components of $\zeta^{-1}(V^2 \cap T^3)$ by $V_{\mathrm{T}}'^2, V_{\mathrm{X}'j}'^2, V_{\mathrm{X}''j}'^2$ ($j = 1, \cdots, s$) such that $V_{\mathrm{T}}'^2, V_{\mathrm{X}'j}'^2, V_{\mathrm{X}''j}'^2$ are neighborhoods of $\dot{V}'^2$, $\dot{A}_j'^1 \cap {}^\circ V'^2$, $\dot{A}_j''^1 \cap {}^\circ V'^2$, respectively, in V'^2. Further we denote

$$^-(V'^2 - V_{\mathrm{T}}'^2), \quad ^-(V_i'^2 - V_{\mathrm{T}}'^2), \quad ^-[A_j'^1 - (V_{\mathrm{T}}'^2 + V_{\mathrm{X}'j}'^2)],$$
$$^-[A_j''^1 - (V_{\mathrm{T}}'^2 + V_{\mathrm{X}''j}'^2)]$$

by $V_*'^2, V_{*i}'^2, A_{*j}'^1, A_{*j}''^1$, respectively. *We denote the images under ζ by omitting upper primes*, i.e., $\zeta(V_{\mathrm{T}}'^2)$, $\zeta(V_{\mathrm{T}j}'^2)$, $\zeta(A_{*j}'^1) = \zeta(A_{*j}''^1)$, etc., are denoted by V_{T}^2, $V_{\mathrm{X}'j}^2$, A_{*j}^1, etc., respectively. Finally we denote $\bigcup_{j=1}^s A_{*j}^1$ by A_*^1.

We choose a coherent orientation ω_{M} of M^3 and an orientation ω_{V} of V_*^2 that is carried over by ζ from a coherent orientation ω_{V}' of $V_*'^2$; now, if an oriented arc O^1 intersects V_*^2 in a piercing point, not in A_*^1, then we call this intersection *positive* or *negative* according to whether the corresponding intersection number (see [7, §73]) is positive or negative.

We choose a cell-decomposition $\Gamma_\#$ of M^3 which is dual to Γ (compare [5, Sec. 3]) such that the 1-skeleton $G_\#^1$ of $\Gamma_\#$ is disjoint from $T^3 + A_*^1$, such that the vertex of $\Gamma_\#$ does not lie in V_*^2, and such that the 1-dimensional elements of $\Gamma_\#$ intersect V_*^2 at most in isolated piercing points. Then we choose a small neighborhood $T_\#^3$ of $G_\#^1$ in M^3 and we denote the "handle-shell" $M^3 - {}^\circ(T^3 + T_\#^3)$ by F^3 and $V_*^2 \cap F^3$ by V_{F}^2.

Now we can project V_{F}^2 "nicely" into the handle-surface $\dot{T}^3$:

Our main objective is, of course, to nicely project the double arcs $A_{*j}'^1$ of V_*^2. But the double arcs of the projection cylinders, corresponding to the overcrossings[5] of A_*^1, will pierce V_*^2 in points that do not lie in A_*^1. These piercing points will correspond to certain handles of the Heegaard-handlebody we shall construct. Therefore we shall also need arcs in V_*^2 which join the piercing points to points in $\dot{V}_*^2$, and we shall have to consider projection cylinders of these arcs; the additional projection cylinders so obtained will contain additional double arcs, yielding additional piercing points with V_*^2, and so on. For this reason it seems convenient to consider a cell-decomposition Δ of V_{F}^2 and to demand that all its elements project in a nice way:

LEMMA 1. *F^3 can be represented as cartesian product $\dot{T}^3 \times I_{\mathrm{F}}^1$, where I_{F}^1*

[5] We use the expressions "over"- and "under-crossingpoint", "projection cylinder", etc., as in [5].

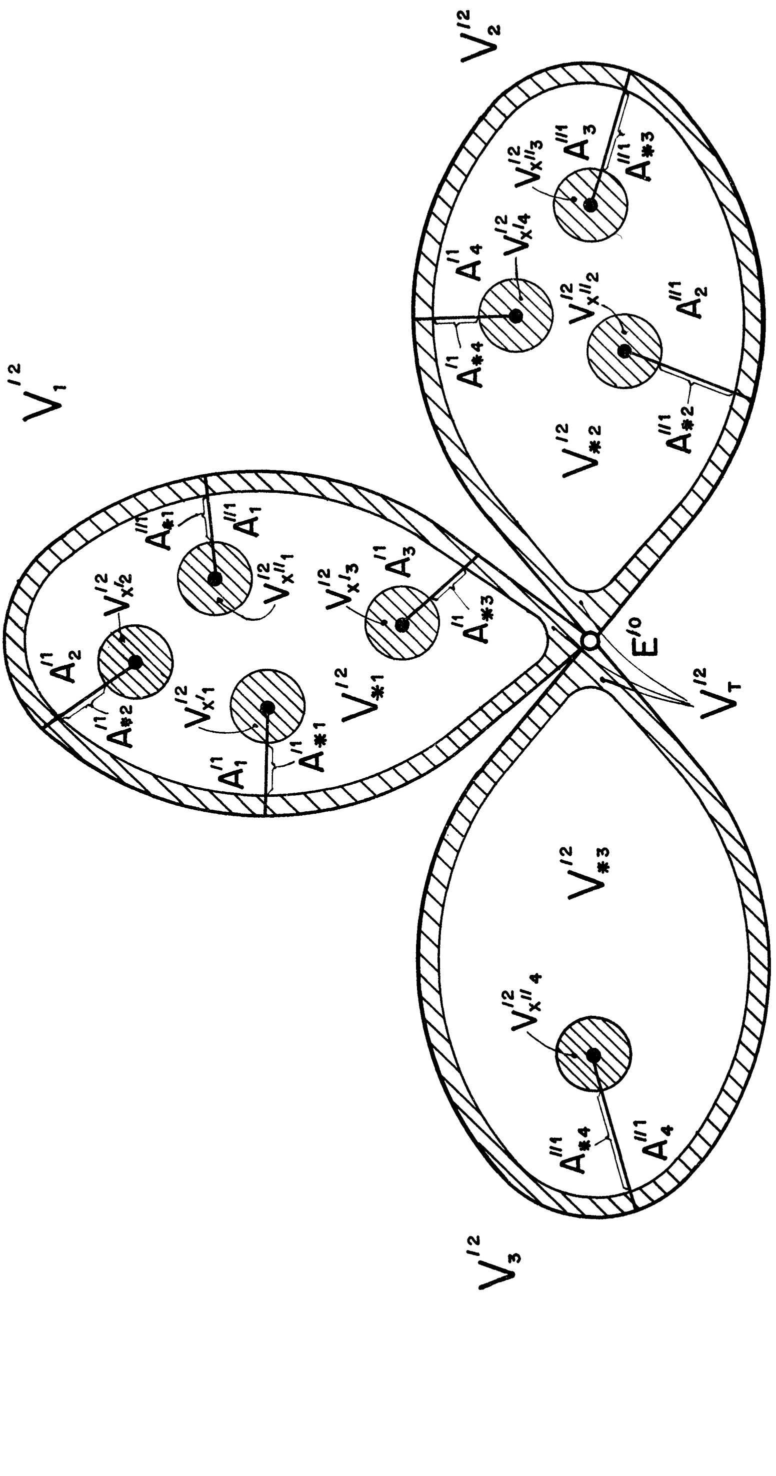

V'^2_1
V'^2_2
V'^2_3
V'^2_T
E'^0
V'^2_{*1}
V'^2_{*2}
V'^2_{*3}
A'^1_1
A'^1_2
A'^1_3
A'^1_4
A'^1_{*1}
A'^1_{*2}
A'^1_{*3}
A'^1_{*4}
A''^1_1
A''^1_2
A''^1_3
A''^1_4
A''^1_{*1}
A''^1_{*2}
A''^1_{*3}
A''^1_{*4}
$V'^2_{x'1}$
$V'^2_{x'2}$
$V'^2_{x'3}$
$V'^2_{x'4}$
$V'^2_{x''1}$
$V'^2_{x''2}$
$V'^2_{x''3}$
$V'^2_{x''4}$

FIG. 3

means a unit interval $0 \leqq x_F \leqq 1$, *such that* $p \times 0 = p$ *for all* $p \in {}^{\cdot}T^3$, *and such that there exists a cell-decomposition* Δ *of* V_F^2 *with the following properties:*

(1) Δ *projects normally into* ${}^{\cdot}T^3$, *i.e.*,

(1a) *if* N *is an element of* Δ *and if* $p \in {}^{\cdot}T^3$ *then* $p \times I_F^1$ *intersects* N *in at most one point;*

(1b) *if* N_1, N_2 *are elements of* Δ *and if* D *is the union of all points* $p \in {}^{\cdot}T^3$ *such that* $p \times I_F^1$ *intersects both* $\bar{N}_1$ *and* $\bar{N}_2$ *then* D *is a cell or is empty;*

(1c) *if in* (1b) *the dimensions of* N_1, N_2 *are* d_1, d_2, *respectively, and if* D *is not empty then the dimension of* D *is* $d_1 + d_2 - 2$;

(1d) *if* $p \in {}^{\cdot}T^3$ *then* $p \times I_F^1$ *intersects the* 1*-skeleton of* Δ *in at most two points;*

(1e) *if* $p \times I_F^1$ $(p \in {}^{\cdot}T^3)$ *intersects two edges* N_1^1, N_2^1 *of* Δ *in the points* $p \times a_1, p \times a_2$, *respectively,* $(0 \leqq a_2 < a_1 \leqq 1)$ *then* N_1^1 *overcrosses* N_2^1 (*i.e.*, N_2^1 *pierces the projection cylinder*[5] *of* N_1^1 *in* $p \times a_2$, *see* [5, *Sec.* 3]; *if* $a_2 = 0$ *then* N_2^1 *pierces the projection arc of* N_1^1 *in* ${}^{\cdot}T^3$).

(2) Δ *is sufficiently fine, i.e.*,

(2a) *if* q *is a vertex of* Δ *that lies in* ${}^{\circ}V_F^2 - A_*^1$ *then* q *can be joined to a point* q_0 *in* ${}^{\cdot}V_*^2 - {}^{\cdot}A_*^1$ *by an arc* Q^1 *that lies in the* 1*-skeleton of* Δ *so that* ${}^{\circ}Q^1$ *lies in* ${}^{\circ}V_F^2 - A_*^1$;

(2b) *if* N^1 *is an edge of* Δ *then* N^1 *overcrosses* $A_*^1 + {}^{\cdot}V_F^2$ *at most once.*

(3) V_F^2 *is not folded and not twisted along* A_*^1, *i.e., there exists a small neighborhood* U_A^3 *of* A_*^1 *in* F^3 *such that the following holds:*

if O^1 *is an oriented interval in* U_A^3, *in the* x_F*-direction* (*i.e., an arc in* U_A^3 *that projects into one point in* ${}^{\cdot}T^3$ *and that is oriented in the direction of increasing* x_F) *then* $\zeta^{-1}(O^1 \cap V_*^2)$ *consists of at most two points and all piercings of* O^1 *through* $V_*^2 - A_*^1$ *are positive.*

Proof. I. Let $\mathfrak{E}^{\vee 4}$ be a euclidean 4-space and let us denote one of its coordinates by x_F and the unit interval of the x_F-axis by I_F^1. We choose a (semilinear) homeomorphism η_T of T^3 into the 3-dimensional subspace $x_F = 0$ of $\mathfrak{E}^{\vee 4}$ and we denote $\eta_T(T^3)$ by $T^{\vee 3}$. We denote the handle shell ${}^{\cdot}T^{\vee 3} \times I_F^1$ by $F^{\vee 3}$ and we associate with any point $q \in F^{\vee 3}$ the coordinates (p, a) so that p is the projection of q into ${}^{\cdot}T^{\vee 3}$ in the x_F-direction and a is the x_F-coordinate of q. We can extend η_T to a (semilinear) homeomorphism η of $T^3 + F^3$ onto $T^{\vee 3} + F^{\vee 3}$. We denote $\eta(V_F^2)$ by $V_I^{\vee 2}$ and $\eta(A_*^1)$ by $A_{*I}^{\vee 1}$. We choose a *rectilinear* triangulation Δ_T of ${}^{\cdot}T^{\vee 3}$ and a corresponding "prismatical" decomposition Λ of $F^{\vee 3}$: For the elements of Λ we take $W \times 0$, $W \times {}^{\circ}I_F^1$, and $W \times 1$ for all $W \in \Lambda_T$.

II. We can transform $V_I^{\vee 2}$ by a "small isotopic deformation" into a polyhedron $V_{II}^{\vee 2}$ such that the transform $A_*^{\vee 1}$ of $A_{*I}^{\vee 1}$ projects normally into ${}^{\cdot}T^{\vee 3}$ and is in "normal position" with respect to Λ; by this we mean: There exists a self-homeomorphism ϑ_I of $F^{\vee 3}$ which is the identity outside of a small neighborhood of $A_{*I}^{\vee 1}$ in $F^{\vee 3}$ such that $\vartheta_I({}^{\cdot}F^{\vee 3}) = {}^{\cdot}F^{\vee 3}$, and such that with the notation $\vartheta_I(V_I^{\vee 2}) = V_{II}^{\vee 2}$, $\vartheta_I(A_{*I}^{\vee 1}) = A_*^{\vee 1}$ the following holds: (II.i) if $p \in {}^{\cdot}T^{\vee 3}$

then $p \times I_F^1$ intersects $\check{A}_*^1$ in at most two points; (II.ii) if $p \in {}^{\cdot}\check{A}_*^1$ then $(p \times {}^{\circ}I_F^1) \cap \check{A}_*^1$ is empty; (II.iii) if $p \times I_F^1$ intersects $\check{A}_*^1$ in two points $p \times a_1$, $p \times a_2$ ($p \in {}^{\cdot}\check{T}^3$, $0 < a_2 < a_1 < 1$) then p lies in an open triangle of Λ_T and there are small neighborhoods N_1^1, N_2^1 of $p \times a_1$, $p \times a_2$, respectively, in $\check{A}_*^1$ which are straight line segments such that N_1^1 overcrosses N_2^1; (II.iv) $\check{A}_*^1$ is disjoint from the 1-skeleton of Λ, and ${}^{\circ}\check{A}_*^1$ intersects the 2-dimensional elements of Λ at most in isolated piercing points.

III. We choose a small neighborhood U_*^3 of $\check{A}_*^1$ in $\check{F}^3$, and we can find a small neighborhood U^3 of $\check{A}_*^1$ in U_*^3 such that (see Fig. 4) the following holds: (0) if $\check{O}^1$ is an interval in U_*^3, in the x_F-direction, then $\check{O}^1 \cap U^3$ is connected (or empty) and $\check{O}^1 \cap {}^{\cdot}U^3$ consists of at most two points.

To obtain U^3 we may choose a rectilinear triangulation Ξ of $\check{A}_*^1$ which contains all intersection points of $\check{A}_*^1$ with 2-elements of Λ as vertices, but

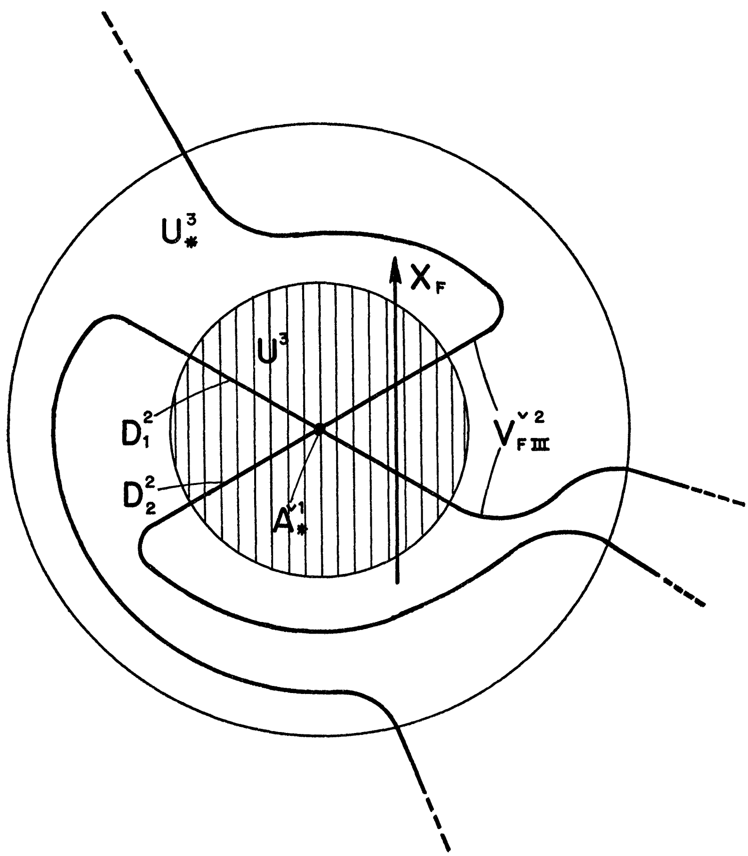
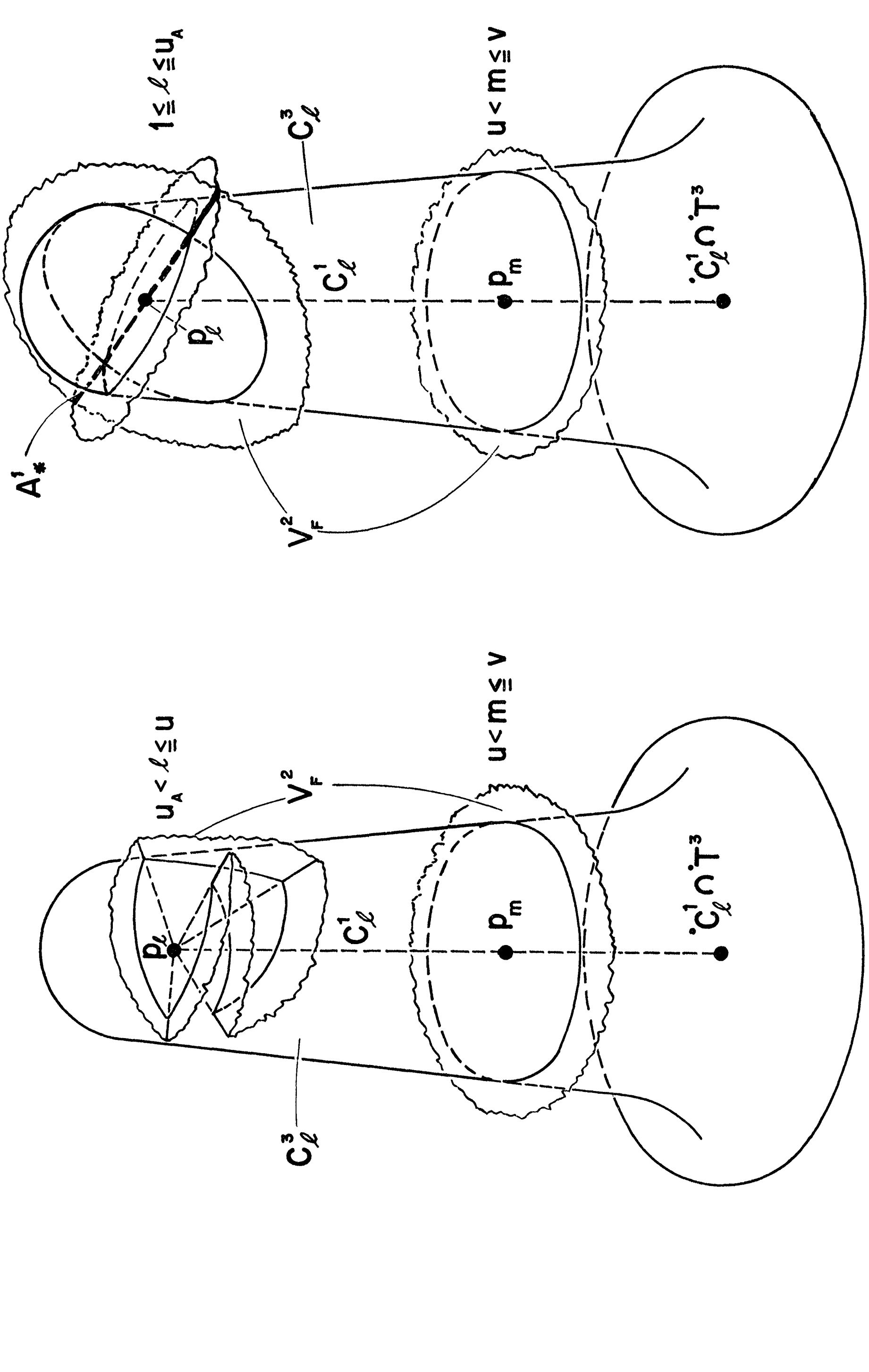


FIG. 4 (Cross section)



Case 1: p_ℓ is a vertex of Δ

Case 2: p_ℓ lies in A^1_*

FIG.5

does not contain any over- or under-crossing point of $A_*^{\vee 1}$ as a vertex. Then we can find a rectilinear sub-triangulation Λ_T^* of Λ_T and a corresponding prismatical subdivision Λ^* of Λ such that every vertex of Ξ lies in a 2-element of Λ^* and such that Λ^*, $A_*^{\vee 1}$ have the properties (II.iii) and (II.iv). We denote the open, rectilinear intervals in which $A_*^{\vee 1}$ intersects the 3-elements of Λ^* by $D_1^1, \cdots, D_w^1$ such that the D_i^1's lie in the order of the enumeration in $A_*^{\vee 1}$, and we assume that the neighborhood U_*^3 is small with respect to Λ^*; further we denote the 3-element of Λ^* that contains D_i^1 by P_i^3. Now we can find small, cylindrical neighborhoods $U_i^3, \cdots, U_w^3$ of $\bar{D}_1^1, \cdots, \bar{D}_w^1$, respectively, in $U_*^3 \cap \bar{P}_1^3, \cdots, U_*^3 \cap \bar{P}_w^3$, respectively, that have the property (0) such that ${}^{\cdot}U_i^3 \cap {}^{\cdot}U_{i-1}^3$ is either empty (if $\bar{D}_{i-1}^1$ and $\bar{D}_i^1$ are "end pieces" of connected components of $A_*^{\vee 1}$) or is a connected component of ${}^{\cdot}U_i^3 \cap {}^{\cdot}P_i^3$ and also a connected component of ${}^{\cdot}U_{i-1}^3 \cap {}^{\cdot}P_{i-1}^3$ (and such that $U_j^3 \cap U_i^3$ is empty whenever $|i - j| > 1$). Then $U^3 = \bigcup_{i=1}^{w} U_i^3$ has the demanded properties. Now we can 'isotopically smooth out" $V_{II}^{\vee 2}$ in the neighborhood U^3 of $A_*^{\vee 1}$ and "wind it about $A_*^{\vee 1}$" so that it is pierced by the intervals $O^{\vee 1}$ in the demanded way. By this we mean: We can find a self-homeomorphism ϑ_{II} of $F^{\vee 3}$ with $\vartheta_{II}(U_i^3) = U_i^3$ and $\vartheta_{II}(P_i^3) = P_i^3$ (for all $i = 1, \cdots, w$) which is the identity on ${}^{-}(F^{\vee 3} - U_*^3)$ and on $A_*^{\vee 1}$ such that the image $V_{III}^{\vee 2}$ of $V_{II}^{\vee 2}$ under ϑ_{II} has the following properties: (III.i) if U_0^3 is a connected component of U^3 then $V_{III}^{\vee 2} \cap U_0^3$ consists of two disks D_1^2, D_2^2, piercing each other in $A_*^{\vee 1} \cap U_0^3$, such that every interval $O^{\vee 1} \subset U_*^3$ in x_F-direction intersects each disk D_1^2, D_2^2 in at most one point; (III.ii) if an interval $O^{\vee 1}$ in x_F-direction pierces D_1^2 or D_2^2 then the intersection number is positive when $O^{\vee 1}$ is oriented in the direction of increasing x_F and $D_1^2, D_2^2, F^{\vee 3}$ are oriented according to ω_V, ω_M, respectively, carried over by $\vartheta_{II}\, \vartheta_I\, \eta$; (III.iii) there exists a *rectilinear* triangulation $\Delta_0^\vee$ of $V_{III}^{\vee 2}$ such that no vertex of $\Delta_0^\vee$ is an over- or under-crossing point of $A_*^{\vee 1}$.

To obtain ϑ_{II} we first deform $V_{II}^{\vee 2} \cap U_1^3$ in a suitable way, i.e. we can find a self-homeomorphism ϑ_1 of $F^{\vee 3}$ with $\vartheta_1(P_1^3) = P_1^3$ which is the identity outside of a small neighborhood of U_1^3 in $F^{\vee 3}$ and on $A_*^{\vee 1}$ such that the conditions (III.i, ii, iii) hold with U^3 replaced by U_1^3 and $V_{III}^{\vee 2}$ replaced by $\vartheta_1(V_{II}^{\vee 2})$. Then we can find, step by step, self-homeomorphisms $\vartheta_2, \cdots, \vartheta_w$ of $F^{\vee 3}$ with $\vartheta_i(P_i^3) = P_i^3$ such that ϑ_i is the identity on $A_*^{\vee 1}$, on U_{i-1}^3, and outside of a small neighborhood of U_i^3, and such that (III.i, ii, iii) hold with U^3 replaced by $U_1^3 + \cdots + U_i^3$ and $V_{III}^{\vee 2}$ replaced by $\vartheta_i\, \vartheta_{i-1} \cdots \vartheta_1(V_{II}^{\vee 2})$. Then we may take $\vartheta_w \cdots \vartheta_1$ for ϑ_{II}.

$V_{III}^{\vee 2}$ and $\Delta_0^\vee$ fulfill the conditions corresponding to (3) and (1a), (1b). Moreover, each connected component of $V_{III}^{\vee 2} - A_*^{\vee 1}$ contains in its boundary arcs of $\vartheta_{II}\, \vartheta_I\, \eta({}^{\cdot}V_*^2 - {}^{\cdot}A_*^1)$. So if $\Delta_{III}^\vee$ is a regular subdivision of $\Delta_0^\vee$ (obtained by starring each edge and each triangle of $\Delta_0^\vee$) then each vertex q of $\Delta_{III}^\vee$ can be joined to a vertex in $\vartheta_{II}\, \vartheta_I\, \eta({}^{\cdot}V_*^2 - {}^{\cdot}A_*^1)$ by an edge path in the 1-skeleton of $\Delta_{III}^\vee$ whose interior lies in ${}^{\circ}V_{III}^{\vee 2} - A_*^{\vee 1}$, i.e., the condition corresponding to (2a) is fulfilled by $V_{III}^{\vee 2}$ and $\Delta_{III}^\vee$. We choose $\Delta_{III}^\vee$ so that it fulfills condition

(III.iii) (i.e., none of the starring points is an over- or under-crossing point of $A_*^{\vee 1}$).

IV. Now we can deform $V_{\mathrm{III}}^{\vee 2}$ and $\Delta_{\mathrm{III}}^{\vee}$ by a small isotopic deformation, leaving $A_*^{\vee 1}$ pointwise fixed, into a polyhedron $V^{\vee 2}$ and a triangulation $\Delta_{\mathrm{IV}}^{\vee}$ of $V^{\vee 2}$, respectively, such that the rectilinearity of the triangulation may be destroyed, but (IV.i) the conditions (1a), (1b), (3) are preserved, and (IV.ii) the conditions corresponding to (1c, d, e) are also fulfilled. We denote the corresponding self-homeomorphism of $F^{\vee 3}$ by ϑ_{III}.

V. We can subdivide the edges of Δ_{IV} by new vertices in such a way that the condition corresponding to (2b) is fulfilled and such that all the other conditions are preserved. We call the cell-decomposition so obtained $\Delta^{\vee}$.

VI. Now we carry over the product representation of $F^{\vee 3}$ and the decomposition $\Delta^{\vee}$ from $F^{\vee 3}$ to F^3 by means of the homeomorphism $(\eta \mid F^3)^{-1}\vartheta_{\mathrm{I}}^{-1}\vartheta_{\mathrm{II}}^{-1}\vartheta_{\mathrm{III}}^{-1} : F^{\vee 3} \to F^3$, denoted by κ. In other words, we associate with each point $q \in F^3$ the coordinates $(\kappa(p^{\vee}), a)$ where $(p^{\vee}, a)$ are the coordinates of $\kappa^{-1}(q)$ in $F^{\vee 3}$; by this we define the product representation $F^3 = {}^{\cdot}T^3 \times I_{\mathrm{F}}^1$ of F^3; further we denote by Δ the cell-decomposition of $V_{\mathrm{F}}^2 = \kappa(V^{\vee 2})$ whose elements are the images under κ of the elements of $\Delta^{\vee}$. Then all conditions of Lemma 1 are fulfilled (where we choose for U_{A}^3 a small neighborhood of A_*^1 in $\kappa(U^3)$). This proves Lemma 1.

4. Expanding the handlebody T^3 into $T_{\wedge}^3$. By $\Delta({}^{\circ}V_{\mathrm{F}}^2)$ we mean the set of those elements of Δ that lie in ${}^{\circ}V_{\mathrm{F}}^2$. We consider the set $\{p_1, \cdots, p_u\}$ of all points that are either vertices of $\Delta({}^{\circ}V_{\mathrm{F}}^2)$ or undercrossing points[5] of the 1-skeleton of $\Delta({}^{\circ}V_{\mathrm{F}}^2)$. Let $C_1^1, \cdots, C_u^1$ be the projection intervals of $p_1, \cdots, p_u$, respectively (i.e., the arcs in F^3, in x_{F}-direction, joining the p's to points in ${}^{\cdot}T^3$). Then we choose small, pairwise disjoint neighborhoods $C_1^3, \cdots, C_u^3$ of $C_1^1, \cdots, C_u^1$, respectively, in F^3 (see Fig. 5 which shows the two most complicated cases) and we denote the handlebody $T^3 + \bigcup_{i=1}^{u} C_i^3$ by $T_{\wedge}^3$. We choose the C_i^3's so that each interval $p \times I_{\mathrm{F}}^1$ with $p \in {}^{\cdot}C_i^3 \cap {}^{\cdot}T^3$ intersects ${}^{-}({}^{\cdot}C_i^3 - {}^{\cdot}T^3)$ in just one point (which is a piercing point if $p \in {}^{\circ}({}^{\cdot}C_i^3 \cap {}^{\cdot}T^3)$).

Notation. (See Fig. 6.) We choose an orientation ω_{A} of A_*^1 such that the arc A_{*j}^1 ($j = 1, \cdots, s$) is oriented from its boundary point in ${}^{\cdot}V_{\mathrm{X}'j}^2$ to its boundary point in ${}^{\cdot}V_{\mathrm{X}''j}^2$. We arrange the enumeration of the points $p_1, \cdots, p_u$ so that $p_1, \cdots, p_{u_{\mathrm{A}}}$ lie in A_*^1 and that $p_{u_{\mathrm{A}}+1}, \cdots, p_u$ do not lie in A_*^1; moreover, if a point p runs through $A_{*1}^1, \cdots, A_{*s}^1$ in the order of the enumeration and in the direction of ω_{A} then we assume that p meets the points $p_1, \cdots, p_{u_{\mathrm{A}}}$ in the order of the enumeration. For convenience we denote the points $p_1, \cdots, p_{u_{\mathrm{A}}}$ also by

$$p_{11}, \cdots, p_{1t_1}, p_{21}, \cdots, p_{2t_2}, \cdots, p_{s1}, \cdots, p_{st_s}$$

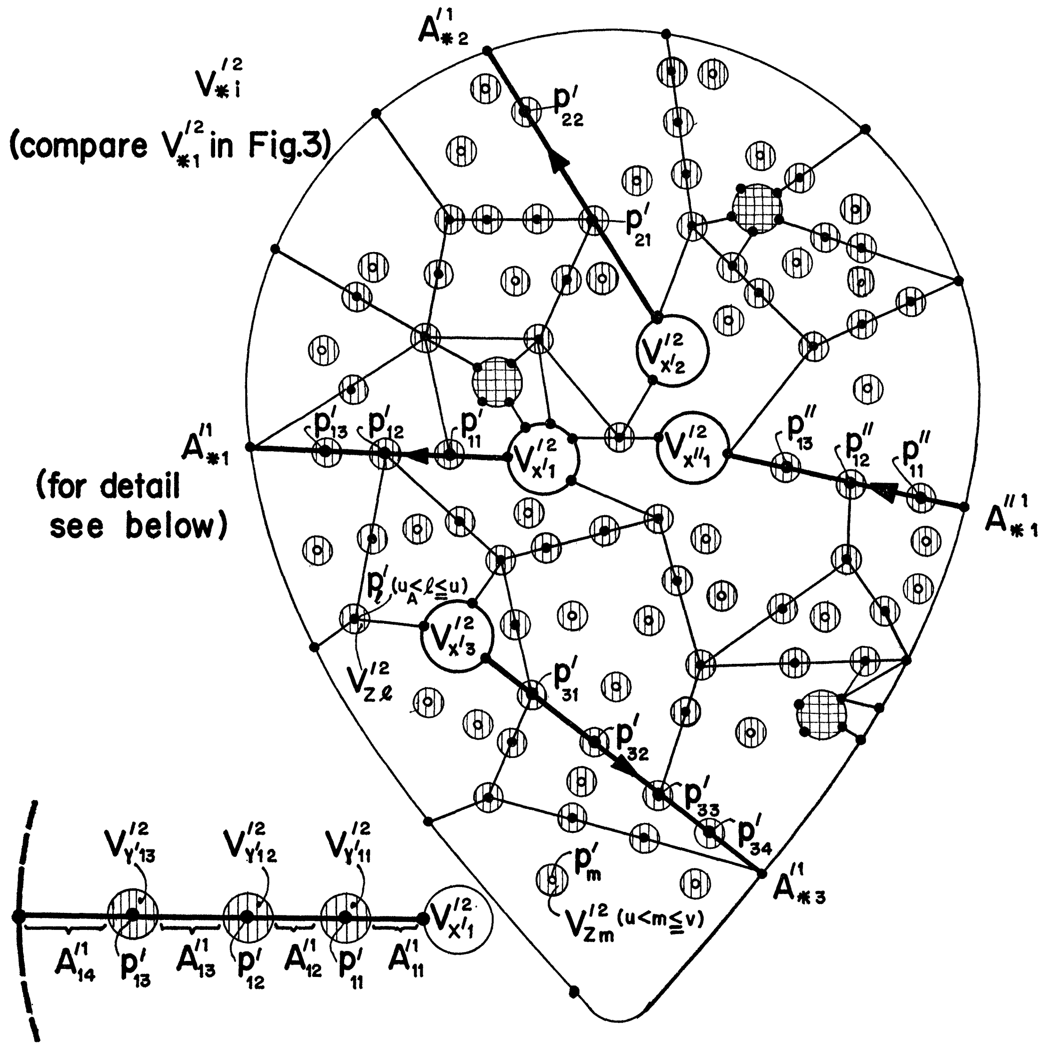


FIG. 6 The arrows correspond to ω_A;
the arcs and intervals map into the 1-skeleton of Δ;
the points $\circ$ map into $\bigcup_{\ell=1}^{u} {}^{\circ}C^1_\ell \cap V^2_F$;
$\zeta^{-1}(\bigcup_{\ell=1}^{u} C^3_\ell \cap V^2_F)$ is indicated by hatching,
$\zeta^{-1}(T^3_\# \cap V^2_*)$ by double hatching.

so that $p_{j1}, \cdots, p_{jt_j}$ lie in A^1_{*j} in the order of the second index, between $^\circ V^2_{X'j}$ and $^\cdot V^2_{X''j}$ ($u_A = \sum_{j=1}^{s} t_j$). We denote the points in $\bigcup_{l=1}^{u} {}^\circ C^1_l \cap V^2_F$ by $p_{u+1}, \cdots, p_v$. The inverse images $\zeta^{-1}(p_{jk})$ of p_{jk} ($j = 1, \cdots, s$; $k = 1, \cdots, t_j$) are denoted by p'_{jk} and p''_{jk} so that $p'_{jk} \in A'^1_{*j}$ and $p''_{jk} \in A''^1_{*j}$; the point $\zeta^{-1}(p_m)$ ($m = u_A + 1, \cdots, v$) is denoted by p'_m. Further we denote that connected component of $\zeta^{-1}(\bigcup_{l=1}^{u} C^3_l \cap V^2_F)$ that is a neighborhood of p'_{jk}, p''_{jk}, p'_m, respectively, ($j = 1, \cdots, s$; $k = 1, \cdots, t_j$; $m = u_A + 1, \cdots, v$) by $V'^2_{Y'jk}$, $V'^2_{Y''jk}$, V'^2_{Zm}, respectively; and we denote $\zeta(V'^2_{Y'jk})$, $\zeta(V'^2_{Y''jk})$, $\zeta(V'^2_{Zm})$ by $V^2_{Y'jk}$, $V^2_{Y''jk}$, V^2_{Zm}, respectively. Finally we denote the connected components of $A^1_{*j} - \bigcup_{l=1}^{u} {}^\circ C^3_l$ by $A^1_{j1}, \cdots, A^1_{jt_j+1}$ so that the A^1_{jk}'s lie in A^1_{*j} in the order of the index k (in the sense of the orientation ω_A); and we denote the connected components of $\zeta^{-1}(A^1_{jk})$ by A'^1_{jk} and A''^1_{jk} so that $A'^1_{jk} \subset A'^1_{*j}$ and $A''^1_{jk} \subset A''^1_{*j}$.

5. Trees in V^2_F. The intersections V^2_{Zl} ($l = u_A + 1, \cdots, v$) of $\bigcup_{h=1}^{u} C^3_h$ with V^2_F correspond to certain handles of the handlebody H^3 composed of $T^3_\wedge$ and a polyhedron V^3_* obtained from V^2_* by thickening. We shall need disks in M^3 with boundaries in $^\cdot H^3$ that correspond to these handles in the following way: The boundary of the first disk, say $K^2_{u_A+1}$, runs just once over the handle corresponding to $V^2_{Zu_A+1}$ (under proper notation) and over no other handles that correspond to V^2_{Zl}'s. The boundary of the $(m - u_A)^{\text{th}}$ disk ($u_A < m \leqq v$), K^2_m, runs just once over the handle corresponding to V^2_{Zm} but not over handles that correspond to V^2_{Zl}'s with $l > m$. We can find such disks K^2_m in a convenient way in the projection cylinder of some polyhedron J^1 (see Fig. 7) in the 1-skeleton of Δ that contains all the points $p_{u_A+1}, \cdots, p_u$ and that consists of trees each of which contains just one point in $^\cdot V^2_* - {}^\cdot A^1_*$.

LEMMA 2. *In the* 1-*skeleton of* Δ *there exists a* 1-*dimensional polyhedron* J^1 *with the following properties:*

(i) *every connected component of* J^1 *is a tree (i.e., simply connected) that contains just one point in* $^\cdot V^2_* - {}^\cdot A^1_*$, *the so-called base point, and otherwise lies in* $^\circ V^2_F - A^1_*$;

(ii) J^1 *contains all the points* $p_{u_A+1}, \cdots, p_u$;

(iii) *if* p *is an end point of* J^1 *(i.e., a point in* $^\cdot J^1 \cap {}^\cdot V^2_F$ *from which just one edge of* J^1 *originates) then* p *is one of the points* $p_{u_A+1}, \cdots, p_u$.

Proof. Let p_l be an arbitrary point with $u_A < l \leqq u$. Then, because of property (2a) of Δ (in Lemma 1), there is an arc, say Q^1_l, that lies in the 1-skeleton of Δ so that $^\circ Q^1_l \subset {}^\circ V^2_F - A^1_*$, and $^\cdot Q^1_l = p_l + q_l$ where q_l is a point in $^\cdot V^2_* - {}^\cdot A^1_*$.

Now we consider the following sequence $J^1_{(1)}, \cdots, J^1_{(u-u_A)}$ of 1-dimensional polyhedra: $J^1_{(1)} = Q^1_{u_A+1}$. If $p_{u_A+h+1} \in J^1_{(h)}$ ($1 \leqq h < u - u_A$) then we take $J^1_{(h+1)} = J^1_{(h)}$. If $p_{u_A+h+1} \notin J^1_{(h)}$ then $Q^1_{u_A+h+1}$ contains an arc, say $Q^1_\sim$, such that $^\cdot Q^1_\sim = p_{u_A+h+1} + q_\sim$ where $q_\sim \in J^1_{(h)}$ or $q_\sim = q_{u_A+h+1}$ (in which case $Q^1_\sim = Q^1_{u_A+h+1}$) and such that $^\circ Q^1_\sim \cap J^1_{(h)} = \emptyset$; then we take $J^1_{(h+1)} = J^1_{(h)} + Q^1_\sim$.

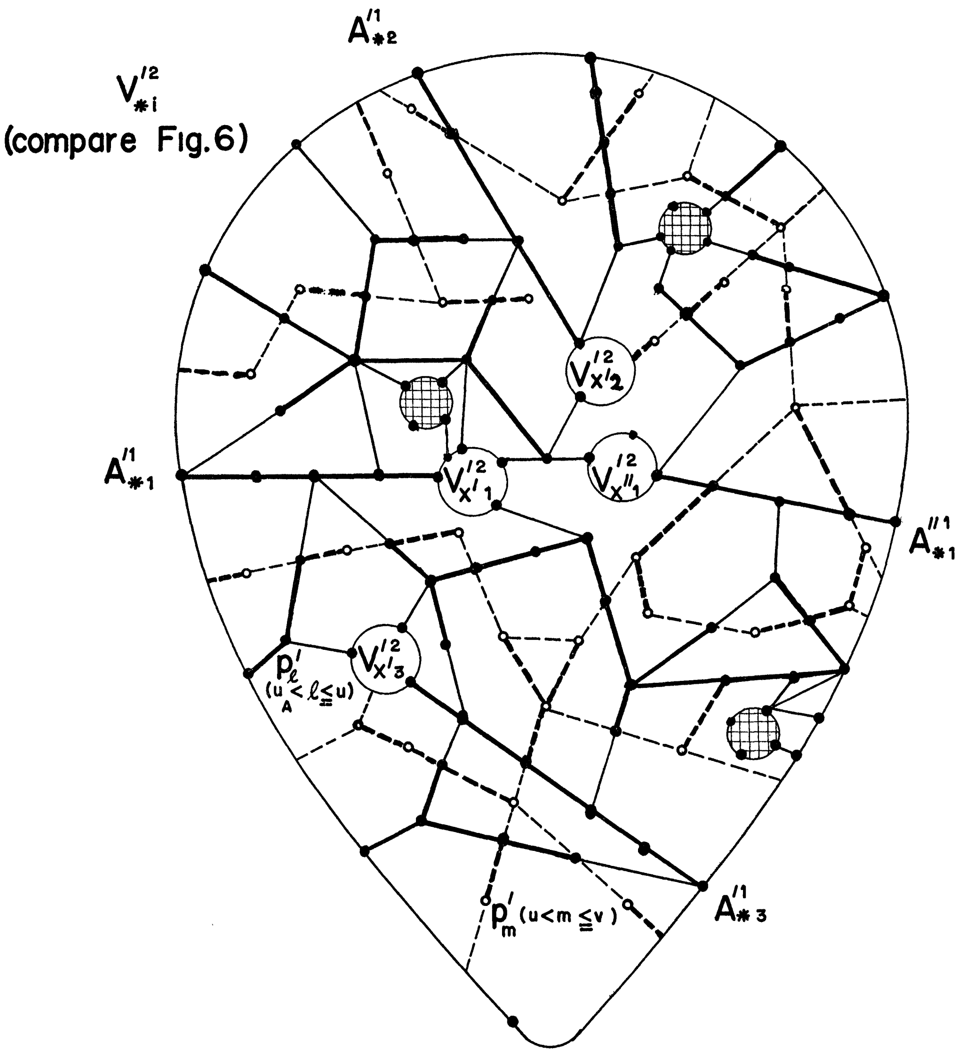


FIG. 7 The heavy segments mean $\zeta^{-1}(J^1 + A^1_*)$;
the dotted segments mean $\zeta^{-1}(\bar{[}V^2_F \cap {}^{\circ}K^2])$;
the heavy dotted segments mean $\zeta^{-1}(J^1_\#)$.

Each $J^1_{(h)}$ has properties (i) and (iii); the last element, $J^1_{(u-u_A)}$, has all three properties demanded for J^1 which proves Lemma 2.

Notation. We denote the projection cylinder[5] of $J^1 + A^1_*$ by K^2. (We assume that the neighborhoods C^3_l of Sec. 4 are small also with respect to K^2.)

LEMMA $2_{\#}$. *In* $^{-}(V_F^2 \cap {}^{\circ}K^2)$ *there exists a 1-dimensional polyhedron* $J_{\#}^1$ *(see Fig. 7) with the following properties:*

($i_{\#}$) *every connected component of* $J_{\#}^1$ *is a tree that contains just one point in the 1-skeleton of* Δ, *the "base point" (it follows that a base point of* $J_{\#}^1$ *is either one of the points* $p_1, \cdots, p_u$ *or lies in* ${}^{\cdot}V_F^2 \cap {}^{\cdot}T^3 - {}^{\cdot}A_*^1$*);*

($ii_{\#}$) $J_{\#}^1$ *contains all the points* $p_{u+1}, \cdots, p_v$*;*

($iii_{\#}$) *if* p *is an end point of* $J_{\#}^1$ ($p \in {}^{\cdot}J_{\#}^1$; *p is not a base point) then p is one of the points* $p_{u+1}, \cdots, p_v$.

Proof. Let p_m be an arbitrary point with $u < m \leqq v$. Then p_m lies in a 2-dimensional element, say N_m^2, of Δ (see Fig. 8); moreover, p_m lies under a point $p_{\mu(m)}$ with $u_A < \mu(m) \leqq u$. Now let $J_\natural^1$ be that connected component of J^1 that contains $p_{\mu(m)}$ and let $q_\natural$ be its base point. Then $q_\natural$ does not lie over or in N_m^2, and hence the projection cylinder $K_\natural^2$ of $J_\natural^1$ intersects $\bar{N}_m^2$ in a 1-dimensional polyhedron that contains an arc, say Q_m^1, so that ${}^{\circ}Q_m^1 \subset N_m^2$ and ${}^{\cdot}Q_m^1 = p_m + q_m$ where q_m is a point in ${}^{\cdot}N_m^2$ (see Fig. 8). Now we may continue as in the second paragraph of the proof of Lemma 2 (replacing l by m, $J_{(h)}^1$ by $J_{\#(h)}^1$, u by v, and u_A by u). This proves Lemma $2_{\#}$.

Notation. We arrange the enumeration of the points $p_{u_A+1}, \cdots, p_v$ so that for each $m = u_A + 1, \cdots, v$ $J^1 + J_{\#}^1$ contains an arc, denoted by J_m^1, that joins p_m either to a point in $({}^{\cdot}V_F^2 \cap {}^{\cdot}T^3 - {}^{\cdot}A_*^1) + {}^{\circ}A_*^1$ or to a point $p_{\lambda(m)}$ with $u_A < \lambda(m) < m$ so that ${}^{\circ}J_m^1$ does not contain any point p_l ($l = 1, \cdots, v$).

6. A prismatical neighborhood V_*^3 of V_*^2. We "thicken" V_*^2. First we choose a "prismatic neighborhood" $V_*'^3$ of $V_*'^2$, i.e., a polyhedron containing $V_*'^2$ (and consisting of r pairwise disjoint 3-cells, disjoint from M^3) that can be represented as cartesian product $V_*'^2 \times I_V^1$, where I_V^1 means an interval $-1 \leqq x_V \leqq +1$, with $p' \times 0 = p'$ for all $p' \in V_*'^2$. Then we extend the map $\zeta \mid V_*'^2$ to a map $\tilde{\zeta}: V_*'^3 \to M^3$ such that the following holds:

Notation. $V_{*i}'^3$ means $V_{*i}'^2 \times I_V^1$; V_*^3, V_{*i}^3 mean $\tilde{\zeta}(V_*'^3)$, $\tilde{\zeta}(V_{*i}'^3)$, respectively.

(1) $V_*^3 - {}^{\circ}T^3$ is a small neighborhood of $V_*^2 - {}^{\circ}T^3$ in $M^3 - {}^{\circ}T^3$.

Notation. (See Fig. 9.) Let $K_\dagger^2$ by the projection cylinder of A_{jk}^1 ($j = 1, \cdots, s; k = 1, \cdots, t_j + 1$) or of J_m^1 ($m = u_A + 1, \cdots, v$), and let $K_{\#}^2$ be that connected component of $^{-}(K_\dagger^2 - \bigcup_{l=1}^{u} C_l^3) \cap V_*^3$ that contains A_{jk}^1 or $^{-}(J_m^1 - \bigcup_{l=1}^{u} C_l^3)$, respectively; then we denote $^{-}[(K_\dagger^2 - \bigcup_{l=1}^{u} C_l^3) - K_{\#}^2]$ by K_{jk}^2 or K_m^2, respectively.[6] Further we denote ${}^{\cdot}K_{\#}^2 \cap {}^{\cdot}K_{jk}^2$, ${}^{\cdot}K_{\#}^2 \cap {}^{\cdot}K_m^2$ by K_{Vjk}^1, K_{Vm}^1, respectively, and ${}^{\cdot}K_{jk}^2 - {}^{\circ}K_{Vjk}^1$, ${}^{\cdot}K_m^2 - {}^{\circ}K_{Vm}^1$ by K_{Tjk}^1, K_{Tm}^1, respectively.

[6] It is essential to remark that $K_{\#}^2$ is a neighborhood of A_{jk}^1 or $^{-}(J_m^1 - \bigcup_{l=1}^{u} C_l^3)$, respectively, in $^{-}(K_\dagger^2 - \bigcup_{l=1}^{u} C_l^3)$, and that consequently the K_{jk}^2's and K_m's are disks. This holds since none of those 2-elements of Δ that are incident to A_{jk}^1 or J_m^1, respectively, intersects $K_\dagger^2$ [because of (1a) in Lemma 1].

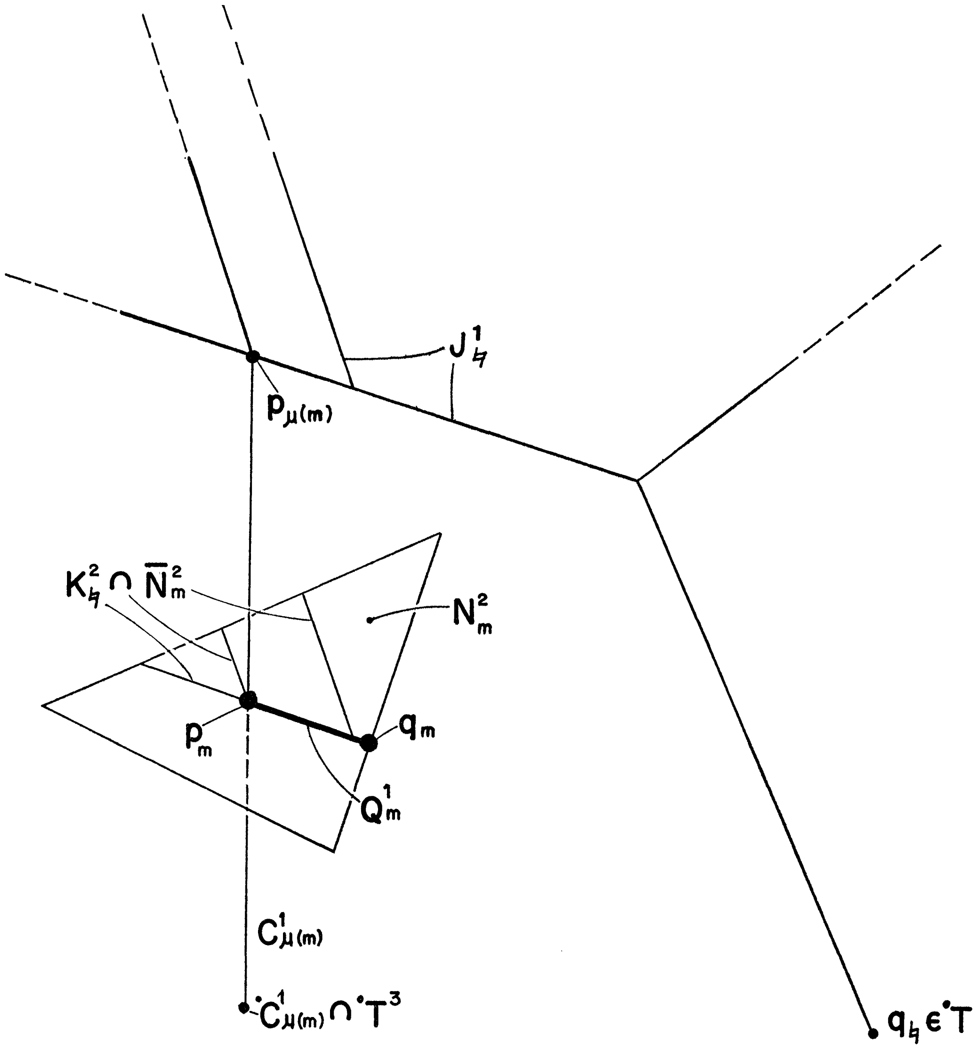


FIG. 8

(2) T^3, $T^3_\wedge$, F^3, $\cdot V^2$, $^\circ V^2_T$, $K^2_{jk} - K^1_{Vjk}$, and $K^2_m - K^1_{Vm}$ ($j = 1, \cdots, s$; $k = 1, \cdots, t_j + 1$; $m = u_A + 1, \cdots, v$) *intersect* V^3_* *prismatically with respect to* x_V, $\tilde{\zeta}$; i.e.,

$$\tilde{\zeta}^{-1}(T^3 \cap V^3_*) = [\zeta^{-1}(T^3 \cap V^2_*)] \times I^1_V$$

and correspondingly for $T^3_\wedge$, F^3, etc.

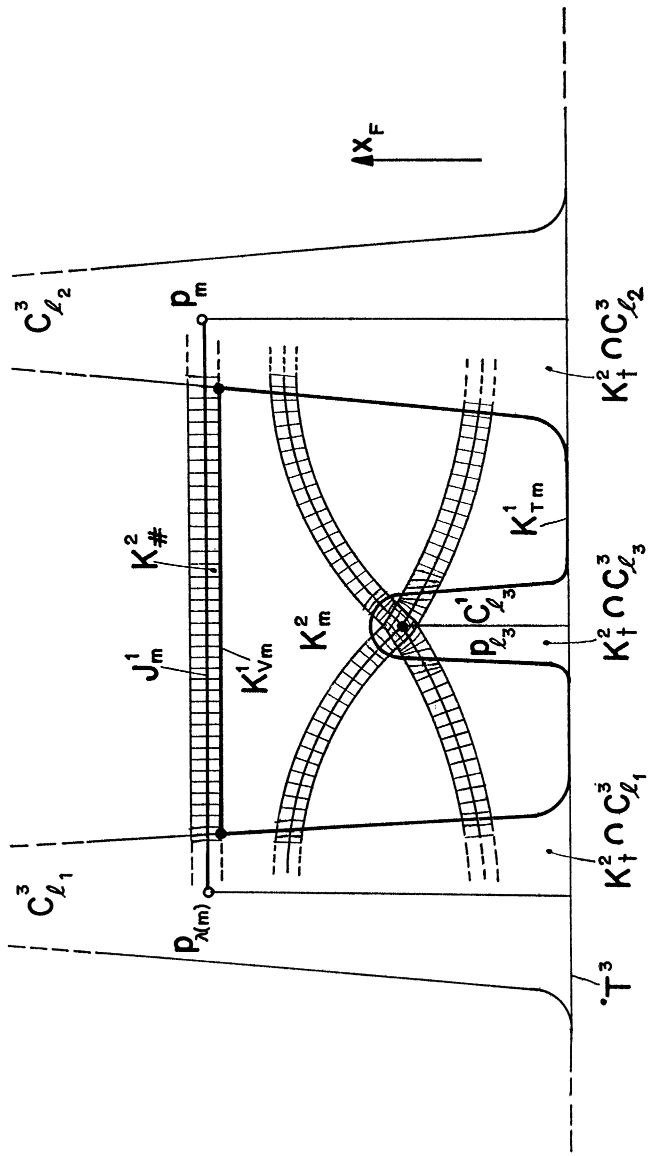


FIG. 9 V^3_* is indicated by hatching.

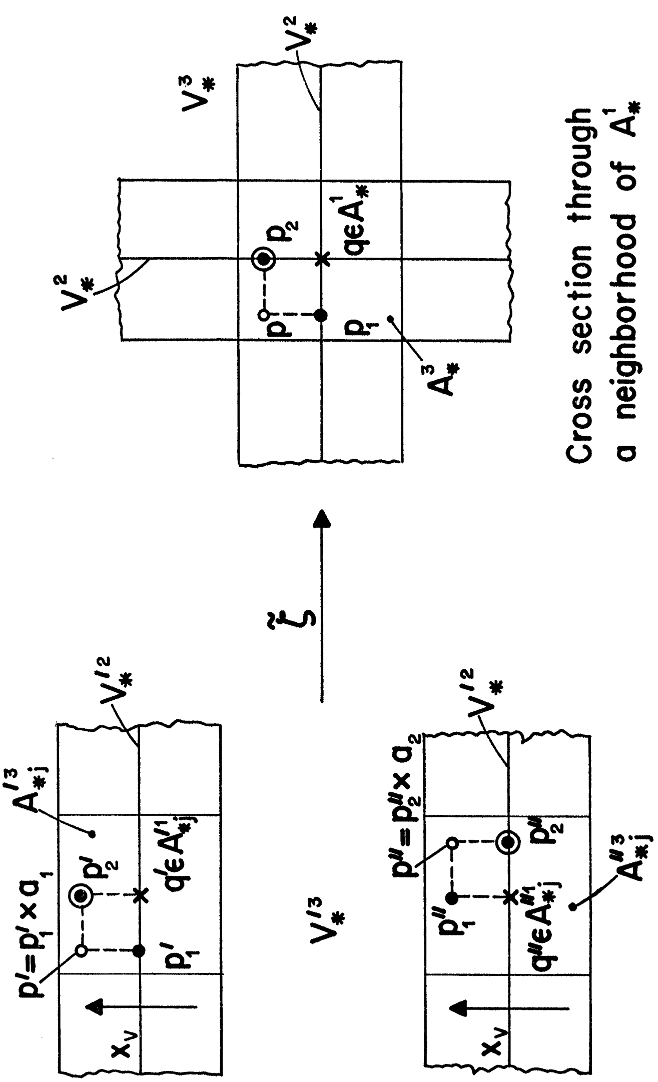

Cross section through a neighborhood of A¹*
V³*
V²*
p
p₂
q∈A¹*
p₁
A³*
ζ̃
V′³*
V′²*
A′³*j
p′=p′₁×a₁
p′₂
q′∈A′¹*j
p′₁
xᵥ
p″=p″₂×a₂
p″₂
p″₁
q″∈A″¹*j
A″³*j
xᵥ

FIG. 10

(3) The singularities of V^3_* are *orthogonal with respect to* x_V, $\tilde{\zeta}$; by this we mean the following (see Fig. 10):

(3.1) The set of all singular points of V^3_* with respect to $\tilde{\zeta}$ is a neighborhood, denoted by A^3_*, of A^1_* in $M^3 - {}^\circ T^3$ which is small with respect to V^2_* and intersects V^3_* prismatically with respect to x_V, $\tilde{\zeta}$.

(3.2) Let A'^3_{*j} and A''^3_{*j} ($j = 1, \cdots, s$) be connected components of $\tilde{\zeta}^{-1}(A^3_*)$ such that $A'^1_{*j} \subset A'^3_{*j}$ and $A''^1_{*j} \subset A''^3_{*j}$, then $\tilde{\zeta} \mid A'^3_{*j}$, $\tilde{\zeta} \mid A''^3_{*j}$, and $\tilde{\zeta} \mid [V'^3_* - \tilde{\zeta}^{-1}(A^3_*)]$ are homeomorphisms.

(3.3) Let p be an arbitrary point of A^3_* and let p', p'' be the two points of $\tilde{\zeta}^{-1}(p)$, $p' = p_1' \times a_1$, $p'' = p_2'' \times a_2$ (p_1', $p_2'' \in V'^2_*$; a_1, $a_2 \in [-1, 1]$); denote $\zeta(p_1')$, $\zeta(p_2'')$ by p_1, p_2, respectively; now let p_1'', p_2' be that point of $\tilde{\zeta}^{-1}(p_1)$, $\tilde{\zeta}^{-1}(p_2)$, respectively, that is different from p_1', p_2'', respectively. Then there is a point $q \in A^1_*$ with $\zeta^{-1}(q) = q' + q''$ such that $p_1'' = q'' \times a_2$ and $p_2' = q' \times a_1$.

(4) If $p' \in V'^2_*$ such that $\zeta(p) \notin A^1_*$, and if (see Fig. 11) the interval $\tilde{\zeta}(p' \times I^1_V)$ is oriented according to increasing x_V, then the intersection of $\tilde{\zeta}(p' \times I^1_V)$ and V^2_* is positive (with respect to the orientations ω_V and ω_M introduced in Sec. 3).

(5) (See Fig. 11.) Let A^3_{jk} ($j = 1, \cdots, s$; $k = 1, \cdots, t_j + 1$) be that connected component of ${}^-(A^3_* - \bigcup_{l=1}^{u} C^3_l)$ that contains A^1_{jk}; then K^1_{Vjk} lies in ${}^\cdot A^3_{jk}$ so that

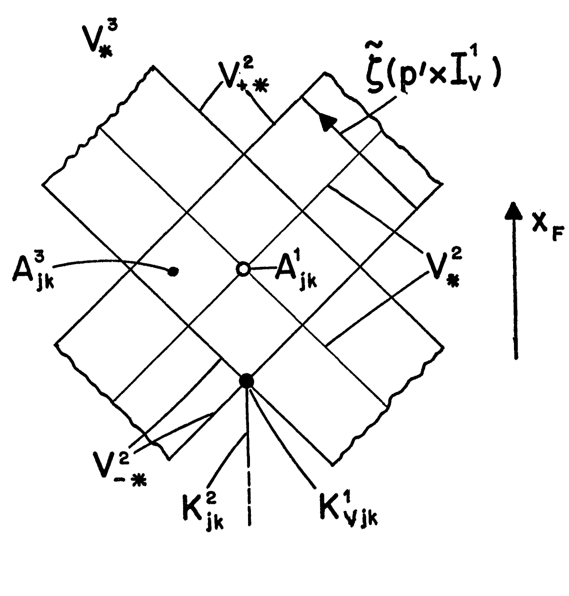


FIG. 11 (Cross section)

$$\bar{\zeta}^{-1}(K^1_{\mathrm{V}jk}) \subset V'^2_* \times -1.$$

Further

$$\zeta^{-1}(K^1_{\mathrm{V}m}) \subset \zeta^{-1}(J^1_m) \times \pm 1$$

(for all $m = u_{\mathrm{A}} + 1, \cdots, v$).

Notation. We denote the connected components of $\bar{\zeta}^{-1}(^-[V^3_* \cap {}^\circ T^3_\wedge])$ by $V'^3_{\mathrm{X}'j}$, $V'^3_{\mathrm{X}''j}$, $V'^3_{\mathrm{Y}'jk}$, $V'^3_{\mathrm{Y}''jk}$, $V'^3_{\mathrm{Z}m}$ so that

$$V'^3_{\mathrm{X}'j} = V'^2_{\mathrm{X}'j} \times I^1_{\mathrm{V}}, \cdots, V'^3_{\mathrm{Z}m} = V'^2_{\mathrm{Z}m} \times I^1_{\mathrm{V}};$$

correspondingly we denote $\bar{\zeta}(V'^3_{\mathrm{X}'j}), \cdots, \bar{\zeta}(V'^3_{\mathrm{Z}m})$ by $V^3_{\mathrm{X}'j}, \cdots, V^3_{\mathrm{Z}m}$, respectively. Let A^3_*, A'^3_{*j}, A''^3_{*j}, A^3_{jk} be as introduced in (3.1), (3.2), (5), respectively, and correspondingly

$$A'^3_{jk} = \bar{\zeta}^{-1}(A^3_{jk}) \cap A'^3_{*j}, \qquad A''^3_{jk} = \bar{\zeta}^{-1}(A^3_{jk}) \cap A''^3_{*j};$$

further we denote $A'^3_{*j} \cap V'^2_*$, $A''^3_{jk} \cap V'^2_*$, etc., by A'^2_{*j}, A''^2_{jk}, etc. Finally we denote $V'^2_{\mathrm{X}'j} \times +1$, $V'^2_{\mathrm{X}'j} \times -1$ by $V'^2_{+\mathrm{X}'j}$, $V'^2_{-\mathrm{X}'j}$, respectively, and correspondingly $V'^2_{\mathrm{X}''j} \times \pm 1$, $V'^2_{\mathrm{Y}'jk} \times \pm 1$, etc., by $V'^2_{\pm\mathrm{X}''j}$, $V'^2_{\pm\mathrm{Y}'jk}$, etc., and $\bar{\zeta}(V'^2_{\pm\mathrm{X}'j})$, etc., by $V^2_{\pm\mathrm{X}'j}$, etc.

7. Constructing meridian disks X^2_j, Y^2_{jk}, Z^2_m in $H^3 = T^3_\wedge + V^3_*$. We denote the handlebody $T^3_\wedge + V^3_*$ by H^3. For the following construction see Fig. 12.

We choose pairwise disjoint, small neighborhoods $U'^2_{\mathrm{A}j}$ $(j = 1, \cdots, s)$ of $A'^2_{*j} + V'^2_{\mathrm{X}'j} + \bigcup_{k=1}^{t_j} V'^2_{\mathrm{Y}'jk}$ in V'^2_*.

$^-({}^\cdot U'^2_{\mathrm{A}j} - {}^\cdot V'^2_*)$ is an arc, denoted by $X'^1_{\mathrm{V}j}$, with boundary points in ${}^\cdot V'^2_*$. (We denote $X'^1_{\mathrm{V}j} \times I^1_{\mathrm{V}}$, $\zeta(X'^1_{\mathrm{V}j})$, $\bar{\zeta}(X'^1_{\mathrm{V}j} \times I^1_{\mathrm{V}})$ by $X'^2_{\mathrm{V}j}$, $X^1_{\mathrm{V}j}$, $X^2_{\mathrm{V}j}$, respectively.)

Then we choose pairwise disjoint disks $X^2_{(j}$, $X^2_{)j}$ which are topologically parallel to $V^2_{-\mathrm{X}''j}$, $V^2_{+\mathrm{X}''j}$, respectively, in T^3, such that

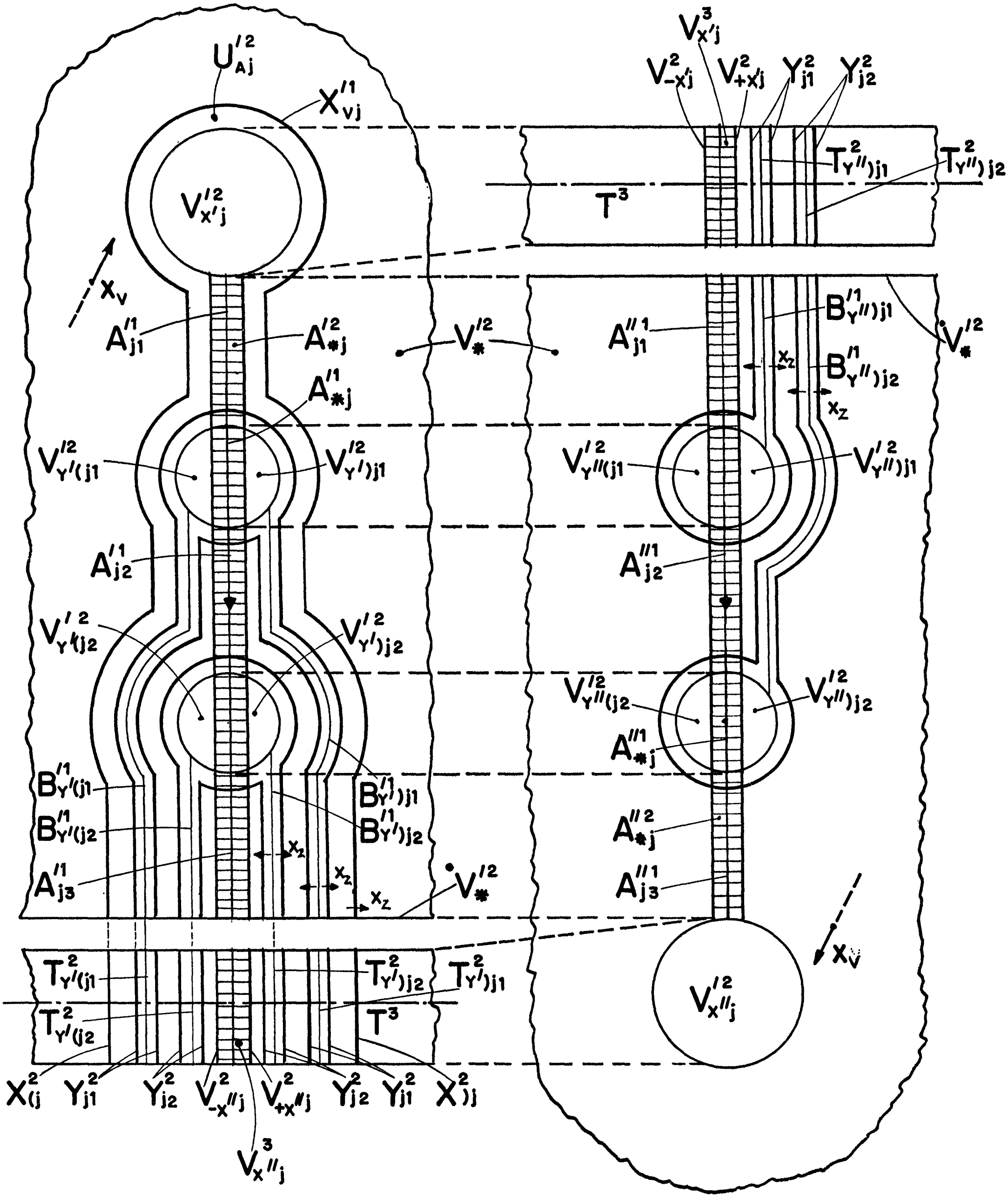
$$({}^\cdot X^2_{(j} + {}^\cdot X^2_{)j}) \cap V^3_* = {}^\cdot X^2_{\mathrm{V}j} \cap {}^\cdot T^3$$

and such that the parallelism is with respect to V^2, V^3_*, ${}^\cdot K^2$, $T^3_\wedge$ (as defined in [4, Sec. 3]).

Now we denote the disks $X^2_{\mathrm{V}j} + X^2_{(j} + X^2_{)j}$ by X^2_j.

Each $^-(V^2_{\mathrm{Y}'jk} - A^3_*)$ and $^-(V^2_{\mathrm{Y}''jk} - A^3_*)$ $(j = 1, \cdots, s;\ k = 1, \cdots, t_j)$ consists of two connected components $V^2_{\mathrm{Y}'(jk}$, $V^2_{\mathrm{Y}')jk}$ and $V^2_{\mathrm{Y}''(jk}$, $V^2_{\mathrm{Y}'')jk}$, respectively; we arrange the notation so that ${}^\cdot V^2_{\mathrm{Y}'(jk}$, ${}^\cdot V^2_{\mathrm{Y}''(jk}$ intersect V^2_{-*} (in one arc each) and that ${}^\cdot V^2_{\mathrm{Y}')jk}$, ${}^\cdot V^2_{\mathrm{Y}'')jk}$ intersect V^2_{+*} (in one arc each).

We choose pairwise disjoint arcs $B^1_{\mathrm{Y}'(jk}$, $B^1_{\mathrm{Y}')jk}$, $B^1_{\mathrm{Y}'')jk}$, $B^1_{\mathrm{Z}m}$ $(j = 1, \cdots, s;$ $k = 1, \cdots, t_j;\ m = u_{\mathrm{A}} + 1, \cdots, v)$ in $V^2_{\mathrm{F}} - (A^3_* + {}^\circ T^3_\wedge)$ that join points in ${}^\cdot V^2_{\mathrm{Y}'(jk}$, ${}^\cdot V^2_{\mathrm{Y}')jk}$, ${}^\cdot V^2_{\mathrm{Y}'')jk}$, ${}^\cdot V^2_{\mathrm{Z}m}$, respectively, to points in ${}^\cdot V^2_*$, such that the following holds (Fig. 12 shows the inverse images of the B's marked by upper primes):



U'^2_{Aj}
X'^1_{vj}
$V'^2_{x'j}$
x_v
A'^1_{j1}
A'^2_{*j}
A'^1_{*j}
V'^2_*
$V'^2_{Y'(j1}$
$V'^2_{Y')j1}$
A'^1_{j2}
$V'^2_{Y'(j2}$
$V'^2_{Y')j2}$
$B'^1_{Y'(j1}$
$B'^1_{Y')j1}$
$B'^1_{Y'(j2}$
$B'^1_{Y')j2}$
A'^1_{j3}
x_z
$T^2_{Y'(j1}$
$T^2_{Y')j2}$
$T^2_{Y')j1}$
$T^2_{Y'(j2}$
T^3
$X^2_{(j}$ Y^2_{j1} Y^2_{j2} $V^2_{-x''j}$ $V^2_{+x''j}$ Y^2_{j2} Y^2_{j1} $X^2_{)j}$
$V^3_{x''j}$
$V^3_{x'j}$
$V^2_{-x'j}$ $V^2_{+x'j}$ Y^2_{j1} Y^2_{j2}
$T^2_{Y'')j1}$
$T^2_{Y'')j2}$
T^3
A''^1_{j1}
$B'^1_{Y'')j1}$
$B'^1_{Y'')j2}$
V'^2_*
$V'^2_{Y''(j1}$
$V'^2_{Y'')j1}$
A''^1_{j2}
$V'^2_{Y''(j2}$
$V'^2_{Y'')j2}$
A''^1_{*j}
A''^2_{*j}
A''^1_{j3}
x_v
$V'^2_{x''j}$

FIG. 12

(1) $B^1_{Y'(jk}$ and $B^1_{Y')jk}$ lie in the boundary of a small neighborhood, say $U^3_{A'jk}$, of

$$A^3_{jk+1} + V^3_{Y'jk+1} + \cdots + A^3_{jt_j} + V^3_{Y'jt_j} + A^3_{jt_j+1}$$

in M^3 (with $U^3_{A'jk} \subset {}^{\circ}U^3_{A'jk-1}$ if $k > 1$);

(2) $B^1_{Y'')jk}$ lies in the boundary of a small neighborhood, say $U^3_{A''jk}$, of

$$A^3_{j1} + V^3_{Y''j1} + \cdots + A^3_{jk-1} + V^3_{Y''jk-1} + A^3_{jk}$$

in M^3 (with $U^3_{A''jk} \subset {}^{\circ}U^3_{A''jk+1}$ if $k < t_j$);

(3) B^1_{Zm} is disjoint from ${}^{\cdot}K^2$ and from the X^2_j's; ${}^{\circ}B^1_{Zm}$ is disjoint from ${}^{\cdot}T^3_{\wedge}$; B^1_{Zm} intersects K^2 at most in isolated piercing points.

We denote $\tilde{\zeta}(\zeta^{-1}(B^1_{Y'(jk}) \times I^1_V)$ by $B^2_{Y'(jk}$; etc.

We choose pairwise disjoint meridian disks $T^2_{Y'(jk}$, $T^2_{Y')jk}$, $T^2_{Y'')jk}$, T^2_{Zm} of T^3 which are disjoint from $\bigcup_{h=1}^{s} X^2_h$ such that

(a) $T^2_{Y'(jk} \cap V^3_* = {}^{\cdot}T^2_{Y'(jk} \cap {}^{\cdot}V^3_* = {}^{\cdot}B^2_{Y'(jk} \cap {}^{\cdot}T^3$, etc.,
$T^2_{Zm} \cap V^3_* = {}^{\cdot}T^2_{Zm} \cap {}^{\cdot}V^3_* = {}^{\cdot}B^2_{Zm} \cap {}^{\cdot}T^3$;

(b) $T^2_{Y'(jk}$, $T^2_{Y')jk}$, $T^2_{Y'')jk}$ are topologically parallel to $V^2_{-X''j}$, $V^2_{+X''j}$, $V^2_{+X'j}$, respectively, in T^3, with respect to V^2, ${}^{\cdot}K^2$, $T^3_{\wedge}$;

(c) ${}^{\cdot}T^2_{Zm}$ is disjoint from the ${}^{\cdot}C^3_l$'s (i.e., ${}^{\cdot}T^2_{Zm} \subset {}^{\circ}[{}^{\cdot}T^3 \cap {}^{\cdot}T^3_{\wedge}]$) and intersects ${}^{\cdot}K^2 \cap {}^{\cdot}T^3$ at most in isolated piercing points;

(d) T^2_{Zm} intersects ${}^{\cdot}V^2$ in just one point, different from E^0, and intersects V^2 in just one arc which is a piercing arc.

Now we choose pairwise disjoint, small neighborhoods U^3_{Yjk} and U^3_{Zm} of

$$V^3_{Y'jk} + V^3_{Y''jk} + B^2_{Y'(jk} + B^2_{Y')jk} + B^2_{Y'')jk} + T^2_{Y'(jk} + T^2_{Y')jk} + T^2_{Y'')jk}$$

and

$$V^3_{Zm} + B^2_{Zm} + T^2_{Zm},$$

respectively, in M^3, which intersect V^3_* prismatically with respect[7] to x_V, $\tilde{\zeta}$. Then we denote the disks ${}^{\cdot}U^3_{Yjk} \cap H^3$, ${}^{\cdot}U^3_{Zm} \cap H^3$ by Y^2_{jk}, Z^2_m, respectively.

We have ${}^{\cdot}X^2_j$, ${}^{\cdot}Y^2_{jk}$, ${}^{\cdot}Z^2_m \subset {}^{\cdot}H^3$ and ${}^{\circ}X^2_j$, ${}^{\circ}Y^2_{jk}$, ${}^{\circ}Z^2_m \subset {}^{\circ}H^3$; hence the disks X^2_j, Y^2_{jk}, Z^2_m are meridian disks of H^3.

Thickening the meridian disks. Let X^3_j, Y^3_{jk}, Z^3_m be pairwise disjoint, small neighborhoods of X^2_j, Y^2_{jk}, Z^2_m, respectively, in H^3 which intersect V^3_* prismatically with respect[7] to x_V, $\tilde{\zeta}$; we can represent them as cartesian products $X^2_j \times I^1_Z$, $Y^2_{jk} \times I^1_Z$, $Z^2_m \times I^1_Z$, respectively, where I^1_Z is an interval $-1 \leqq x_Z \leqq +1$, such that the following holds:

(α) $p \times 0 = p$ for all $p \in X^2_j$, Y^2_{jk}, Z^2_m;

(β) the top and bottom disks

$$X^2_j \times \pm 1, \quad Y^2_{jk} \times \pm 1, \quad Z^2_m \times \pm 1,$$

[7] This is possible because of the orthogonality condition (3) in Sec. 6.

denoted by $X^2_{\pm j}$, $Y^2_{\pm jk}$, $Z^2_{\pm m}$, respectively, are the connected components of

$$^{-}(^{\cdot}X^3_j \cap {}^{\circ}H^3), \quad ^{-}(^{\cdot}Y^3_{jk} \cap {}^{\circ}H^3), \quad ^{-}(^{\cdot}Z^3_m \cap {}^{\circ}H^3),$$

respectively; (these disks are not indicated in Fig. 12, but the x_Z-direction is indicated by small arrows);

(γ) $\zeta^{-1}(X^2_{-j} \cap V^2_*)$ separates $\zeta^{-1}(X^2_{+j} \cap V^2_*)$ from $V'^2_{X'j}$ in V'^2_*,
$Y^2_{-jk} \cap V^2_*$ separates $Y^2_{+jk} \cap V^2_*$ from $V^2_{Y'jk} + V^2_{Y''jk}$ in V^2_*,
$Z^2_{-m} \cap V^2_*$ separates $Z^2_{+m} \cap V^2_*$ from V^2_{Zm} in V^2_*;

(δ) T^3, V^2, A^3_*, K^2 intersect X^3_j, Y^3_{jk}, Z^3_m prismatically with respect to x_Z;

(ε) the intersections $X^3_j \cap V^3_*$, $Y^3_{jk} \cap V^3_*$, $Z^3_m \cap V^3_*$ are *orthogonal with respect to* x_Z, x_V, $\tilde{\zeta}$, i.e., the following condition is fulfilled which is completely analogous to (3.3) in Sec. 6 (compare Fig. 10):

Let p be an arbitrary point of $X^3_j \cap V^3_*$, $Y^3_{jk} \cap V^3_*$, or $Z^3_m \cap V^3_*$ and let p'' be a point in $\tilde{\zeta}^{-1}(p)$ where $p = p_1 \times_Z a_1$ and $p'' = p''_2 \times_V a_2$ (we use the symbols $\times_Z$ and $\times_V$ to distinguish the product representation of the X^3_j, Y^3_{jk}, Z^3_m's from that of V'^3_*); denote $\zeta(p''_2)$ by p_2; now, if $p \notin A^3_*$, let $p''_1 = \tilde{\zeta}^{-1}(p_1)$, and if $p \in A^3_*$ let p''_1 be that point in $\tilde{\zeta}^{-1}(p_1)$ that lies in the same connected component of $\tilde{\zeta}^{-1}(A^3_*)$ as p''. Then there is a point q in $X^2_j \cap V^2_*$, $Y^2_{jk} \cap V^2_*$, or $Z^2_m \cap V^2_*$, respectively, and there is a point q'' in $\zeta^{-1}(q)$ such that $p''_1 = q'' \times_V a_2$ and $p_2 = q \times_Z a_1$.

8. H^3 is a Heegaard-handlebody in M^3. We denote the connected components of $K^2 - {}^{\circ}H^3$ by $K^2_{*1}, \cdots, K^2_{*b}$. Note that these are disks (because of (1) in Lemma 1).

LEMMA 3. *H^3 is a Heegaard-handlebody in M^3, and more in detail we have:*
(a) $^{-}[H^3 - (\bigcup_{j=1}^{s} X^3_j + \bigcup_{j,k=1}^{s,t_j} Y^3_{jk} + \bigcup_{m=u_A+1}^{v} Z^3_m)]$ *is a 3-cell, say* W^3, *i.e., H^3 is a handlebody, the disks X^2_j, Y^2_{jk}, Z^2_m form a complete system of meridian disks of H^3, and the genus a of H^3 is equal to*

$$s + \textstyle\sum_{j=1}^{s} t_j + (v - u_A) = v + s;$$

(b) *the connected components of $M^3 - (H^3 + K^2)$ are open 3-cells, i.e., $M^3 - {}^{\circ}H^3$ is a handlebody, the disks $K^2_{*1}, \cdots, K^2_{*b}$ contain a complete system of meridian disks of $M^3 - {}^{\circ}H^3$, and $b \geqq a$.*

Proof of (a). Let $T'^3_\wedge$ be a handlebody of genus r, disjoint from M^3, such that

$$T'^3_\wedge \cap V'^3_\wedge = {}^{\cdot}T'^3_\wedge \cap {}^{\cdot}V'_* = {}^{\cdot}V'^2_* \times I^1_V,$$

such that $T'^3_\wedge + V'^3_*$ is a 3-cell, denoted by H'^3, and such that there is a map $\xi : H'^3 \to H^3$ of H'^3 onto H^3 with $\xi \mid V'^3_* = \tilde{\zeta}$ and with $\xi \mid T'^3_\wedge$ a homeomorphism of $T'^3_\wedge$ onto $T^3_\wedge$.

The connected components of

$$^{-}[H'^3 - \xi^{-1}(\bigcup_{j=1}^{s} X^3_j + \bigcup_{j,k=1}^{s,t_j} Y^3_{jk} + \bigcup_{m=u_A+1}^{v} Z^3_m)]$$

are 3-cells; we may denote them by $H_0'^3$, $H_{A'jl}'^3$, $H_{Y'jk}'^3$, $H_{Y''jk}'^3$, $H_{Zm}'^3$ ($j = 1, \cdots, s$; $k = 1, \cdots, t_j$; $l = 1, \cdots, t_j + 1$; $m = u_A + 1, \cdots, v$) such that (compare Fig. 12) $H_{A'jl}'^3 \cap A_{*j}'^1$ is an arc in $A_{jl}'^1$, $H_{Y'jk}'^3$ contains $V_{Y'jk}'^2$, $H_{Y''jk}'^3$ contains $V_{Y''jk}'^2$, and $H_{Zm}'^3$ contains $V_{Zm}'^2$. The restrictions

$$\xi \mid H_0'^3, \quad \xi \mid \bigcup_{j,k=1}^{s,t_j} H_{Y'jk}'^3, \quad \xi \mid \bigcup_{j,k=1}^{s,t_j} H_{Y''jk}'^3,$$

$$\xi \mid \left(\bigcup_{j,l=1}^{s,t_j+1} H_{A'jl}'^3 \quad + \quad \bigcup_{m=u_A+1}^{v} H_{Zm}'^3 \right)$$

are homeomorphisms.

Now $\xi(H_{Y'jk}'^3) + \xi(H_{Y''jk}'^3)$ ($j = 1, \cdots, s$; $k = 1, \cdots, t_j$) are pairwise disjoint 3-cells, say H_{Yjk}^3, and

$$\xi(H_0'^3) + \bigcup_{j,l=1}^{s,t_j+1} \xi(H_{A'jl}'^3) + \bigcup_{m=u_A+1}^{v} \xi(H_{Zm}'^3)$$

is a 3-cell, say H_{00}^3, where

$$H_{00}^3 \cap H_{Yjk}^3 = V_{Y'jk}^3 + V_{Y''jk}^3 .$$

Now $H_{00}^3 + \bigcup_{j,k=1}^{s,t_j} H_{Yjk}^3$ is a 3-cell and is equal to

$$^{-}[H^3 - (\bigcup_{j=1}^{s} X_j^3 + \bigcup_{j,k=1}^{s,t_j} Y_{jk}^3 + \bigcup_{m=u_A+1}^{v} Z_m^3)]$$

which proves (a).

Proof of (b). First we prove that the first homology group $\mathcal{H}_1(H^3 + K^2)$ is trivial: We denote by $\mathfrak{a}_{Xj}$, $\mathfrak{a}_{Yjk}$, $\mathfrak{a}_{Zm}$ ($j = 1, \cdots, s$; $k = 1, \cdots, t_j$; $m = u_A + 1, \cdots, v$) those elements of $\mathcal{H}_1(H^3)$ which correspond to piercings of X_j^2, Y_{jk}^2, Z_m^2, respectively (i.e., $\mathfrak{a}_{X1}$ may be represented by an oriented simple closed curve in H^3 that intersects X_1^3 in just one prismatical arc with induced orientation in the direction of increasing x_Z, and that is disjoint from $X_2^2, \cdots, X_s^2$, and from the Y_{jk}^3's and Z_m^3's; etc.). The $\mathfrak{a}$'s form a basis of $\mathcal{H}_1(H^3)$. Let α be the inclusion map $H^3 \subset H^3 + K^2$ and let

$$\alpha_* : \mathcal{H}_1(H^3) \to \mathcal{H}_1(H^3 + K^2)$$

be induced by α, then the $\alpha_*(\mathfrak{a})$'s form a basis of $\mathcal{H}_1(H^3 + K^2)$. Now the properly oriented boundary $\dot{K}_{j1}^{\to 2}$ of K_{j1}^2 (Sec. 6) belongs to $\mathfrak{a}_{Yj1}$, further $\dot{K}_{jk}^{\to 2}$ belongs to $\mathfrak{a}_{Yjk} - \mathfrak{a}_{Yjk-1}$ for all $k = 2, \cdots, t_j$, and finally $\dot{K}_{jt_j+1}^{\to 2}$ belongs to $\mathfrak{a}_{Xj} + \mathfrak{a}_{Yjt_j}$ (compare the more detailed discussion of the $\dot{K}_{jk}^2$'s in Sec. 10.1); hence

$$\alpha_*(\mathfrak{a}_{Yj1}) = \cdots = \alpha_*(\mathfrak{a}_{Yjt_j}) = \alpha_*(\mathfrak{a}_{Xj}) = \mathfrak{o}$$

where $\mathfrak{o}$ means the zero-element of $\mathcal{H}_1(H^3 + K^2)$. Similarly, $\dot{K}_m^{\to 2}$ ($m = u_A + 1, \cdots, v$) belongs (compare Fig. 9) either to $\mathfrak{a}_{Zm} - \mathfrak{a}_{Z\lambda(m)}$ (where $u_A < \lambda(m) < m$, see Sec. 5) or to $\mathfrak{a}_{Zm} + \mathfrak{b}$ with $\alpha_*(\mathfrak{b}) = \mathfrak{o}$; hence

$$\alpha_*(\mathfrak{a}_{Zu_A+1}) = \cdots = \alpha_*(\mathfrak{a}_{Zv}) = \mathfrak{o},$$

i.e., $\mathcal{H}_1(H^3 + K^2)$ is trivial, Q.E.D.

Let $K_{*1}^3, \cdots, K_{*b}^3$ be pairwise disjoint, small neighborhoods of $K_{*1}^2, \cdots,$

K^2_{*b}, respectively, in $M^3 - {}^\circ H^3$. Then $H^3 + \bigcup_{l=1}^b K^3_{*l}$ is a 3-manifold with trivial first homology group (since $H^3 + \bigcup_{l=1}^b K^3_{*l}$ collapses to $H^3 + K^2$), hence ${}^\cdot(H^3 + \bigcup_{l=1}^b K^3_{*l})$ consists of 2-spheres only (see [7, §64]); but these 2-spheres lie in the handlebody $M^3 - {}^\circ T^3$, and therefore, as a consequence of the Alexander theorem [2], bound 3-cells in $M^3 - {}^\circ T^3$. Therefore $M^3 - (H^3 + \bigcup_{l=1}^b K^3_{*l})$ consists of open 3-cells, and hence $M^3 - (H^3 + K^2)$ consists of open 3-cells. This proves (b).

9. Constructing the cell-decomposition Ψ of M^3. We take for Ψ a cell-decomposition of M^3, corresponding to the Heegaard-handlebody H^3 with the two systems

$$\{X^2_j, Y^2_{jk}, Z^2_m\}, \quad \{K^2_{*i}\}$$

of meridian disks (compare [5, Sec. 8]):

For the only vertex of Ψ we choose a point O in ${}^\circ W^3$. For the 1-dimensional elements of Ψ we choose pairwise disjoint, open arcs E^{*1}_{Xj}, E^{*1}_{Yjk}, E^{*1}_{Zm} in ${}^\circ H^3$ with common boundary O such that H^3 is a neighborhood of the 1-skeleton G^{*1} of Ψ, and such that E^{*1}_{Xj} intersects X^3_j in just one prismatical arc and is disjoint from $X^3_1, \cdots, X^3_{j-1}, X^3_{j+1}, \cdots, X^3_s$, from the Y^3's, and from the Z^3's; etc. For the 2-dimensional elements of Ψ we choose pairwise disjoint, open disks $E^{*2}_1, \cdots, E^{*2}_b$ in $M^3 - G^{*1}$ such that

$$E^{*2}_i \cap (M^3 - {}^\circ H^3) = K^2_{*i}, \quad {}^\cdot E^{*2}_i \subset G^{*1},$$

and such that $E^{*2}_i \cap {}^\circ H^3$ is an open annulus, say E^{*2}_{Hi} which intersects X^3_j, Y^3_{jk}, Z^3_m prismatically with respect to x_Z so that X^2_j, Y^2_{jk}, Z^2_m are intersected (at most) in open arcs each of which joins G^{*1} to ${}^\cdot H^3$. For the 3-dimensional elements of Ψ we take the connected components of $M^3 - \bigcup_{i=1}^b \bar{E}^{*2}_i$.

We choose a coherent orientation of G^{*1} so that in $E^{*1}_{Xj} \cap X^3_j$, $E^{*1}_{Yjk} \cap Y^3_{jk}$, $E^{*1}_m \cap Z^3_m$ the direction of E^{*1}_{Xj}, E^{*1}_{Yjk}, E^{*1}_{Zm}, respectively, coincides with the direction of increasing x_Z; then we associate generators g_{Xj}, g_{Yjk}, g_{Zm} of $\pi_1(M^3)$ with the so oriented 1-spheres $\bar{E}^{*1}_{Xj}$, $\bar{E}^{*1}_{Yjk}$, $\bar{E}^{*1}_{Zm}$, respectively, (with base point O). Now we may read relators $\mathfrak{r}_1, \cdots, \mathfrak{r}_b$ from the 2-elements $E^{*2}_1, \cdots, E^{*2}_b$, respectively, and we denote the presentation

$$(\{g_{Xj}, g_{Yjk}, g_{Zm}\}, \{\mathfrak{r}_i\})$$

of $\pi_1(M^3)$ by $\mathfrak{P}(\Psi)$.

10. Relator-diagrams corresponding to the presentation $\mathfrak{P}(\Psi)$ of $\pi_1(M^3)$. We map the disks K^2_{jk}, K^2_m ($j = 1, \cdots, s; k = 1, \cdots, t_j; m = u_A + 1, \cdots, v$; see Sec. 5 and Fig. 9) onto pairwise disjoint disks R^2_{jk}, R^2_m, respectively, (see Fig. 13 which corresponds to Fig. 9 if one assumes that $m > u$, $p_{l_3} = p_{j2}$, and $t_j = 2$, compare Fig. 12), by means of maps

$$\varkappa_{jk} : K^2_{jk} \to R^2_{jk}, \qquad \varkappa_m : K^2_m \to R^2_m,$$

respectively[8], such that:

(i) the restrictions of $\varkappa_{jk}$, $\varkappa_m$ to the open disks $\mathring{K}^2_{*i}$ $(i = 1, \cdots, b)$ are homeomorphisms;

(ii) $\varkappa_{jk}$, $\varkappa_m$ map each connected component of $K^2_{jk} \cap W^3$, $K^2_m \cap W^3$, respectively, into a single point;

(iii) if L is a connected component of the intersection of K^2_{jk} or K^2_m with $X^3_h - {}^{\cdot}W^3$, $Y^3_{hl} - {}^{\cdot}W^3$, or $Z^3_q - {}^{\cdot}W^3$, then $\varkappa_{jk}$ or $\varkappa_m$, respectively, maps L onto an open arc in such a way that all points with the same x_{Z}-coordinate have the same image point (but points with different x_{Z}-coordinates map always into different points).

If L as in (iii) then we orientate the image $\varkappa_{jk}(L)$ or $\varkappa_m(L)$, respectively, according to the direction of increasing x_{Z}, and we associate it with the generator $g_{\mathrm{X}h}$, $g_{\mathrm{Y}jk}$, $g_{\mathrm{Z}q}$, respectively.

We consider the cell-decompositions Θ_{jk}, Θ_m of R^2_{jk}, R^2_m, respectively, into the connected components of the images of $K^2_{jk} \cap \mathring{K}^2_{*i}$, $K^2_{jk} \cap W^3$, $K^2_{jk} \cap (X^3_h - {}^{\cdot}W^3)$, etc., etc. From each 2-dimensional element of Θ_{jk} or Θ_m we may read the relator $\mathfrak{r}_i$ that is associated with the inverse image disk K^2_{*i}. We call the decomposition Θ_{jk} or Θ_m, together with the association of its oriented edges to the generators g and of its 2-elements to the relators $\mathfrak{r}$ (see Fig. 13) a *relator-diagram corresponding to* $\mathfrak{P}(\Psi)$ and we denote it by $\mathfrak{R}_{jk}$ or $\mathfrak{R}_m$, respectively. From the boundary of R^2_{jk} or R^2_m we may read a word $\mathfrak{r}^{\#}_{jk}$ or $\mathfrak{r}^{\#}_m$, respectively, in the generators g (where all members of the cyclic class $\langle \mathfrak{r}^{\#}_{jk} \rangle$ or $\langle \mathfrak{r}^{\#}_m \rangle$, respectively, are equivalent). Now the relator-diagram $\mathfrak{R}_{jk}$ shows that[2] $\mathfrak{r}^{\#}_{jk} = 1$ is a true relation in the group $\pi_1(M^3)$; etc. Diagrams like these have been used by E. R. Van Kampen and other authors; see for instance [8].

For the proof of the theorem we shall need some special properties of our relator-diagrams $\mathfrak{R}$:

(10.1) By inspection of the curves ${}^{\cdot}K^2_{jk}$ and ${}^{\cdot}K^2_m$ we see that we can write for all $j = 1, \cdots, s$; $m = u_{\mathrm{A}} + 1, \cdots, v$ (compare Fig. 12):

$$\begin{aligned}
&\mathfrak{r}^{\#}_{j1} = g_{\mathrm{Y}j1}\, \mathfrak{e}_{j1} \\
&\mathfrak{r}^{\#}_{jk} = g_{\mathrm{Y}jk}\, g^{-1}_{\mathrm{Y}jk-1}\, \mathfrak{e}_{jk} \quad \text{(for all } k = 2, \cdots, t_j) \\
&\mathfrak{r}^{\#}_{jt_j+1} = g_{\mathrm{X}j}\, \mathfrak{e}_{jt_j+1}\, g_{\mathrm{Y}jt_j}\, \mathfrak{e}^{\#}_{jt_j+1} \\
&\mathfrak{r}^{\#}_m = g_{\mathrm{Z}m}\, \mathfrak{e}_m \quad \text{if } J^1_m \text{ joins } p_m \text{ to } {}^{\cdot}V^2_* \\
(\#) \quad &\phantom{\mathfrak{r}^{\#}_m} = g_{\mathrm{Z}m}\, \mathfrak{e}^{\#}_m\, g^{-1}_{\mathrm{X}j_m}\, \mathfrak{e}_m \quad \text{if } J^1_m \text{ joins } p_m \text{ to } {}^{\cdot}V^2_{\mathrm{X}'j_m} \\
&\phantom{\mathfrak{r}^{\#}_m} = g_{\mathrm{Z}m}\, \mathfrak{e}^{\#}_m\, g_{\mathrm{X}j_m}\, \mathfrak{e}_m \quad \text{if } J^1_m \text{ joins } p_m \text{ to } {}^{\cdot}V^2_{\mathrm{X}''j_m} \\
&\phantom{\mathfrak{r}^{\#}_m} = g_{\mathrm{Z}m}\, \mathfrak{e}^{\#}_m\, g^{-1}_{\mathrm{X}j_m}\, \mathfrak{e}^{\#\#}_m\, g^{-1}_{\mathrm{Y}j_mk_m}\, \mathfrak{e}_m \quad \text{if } J^1_m \text{ joins } p_m \text{ to } p_{j_mk_m} \text{ and meets } {}^{\cdot}V^2_{\mathrm{Y}'j_mk_m}
\end{aligned}$$

[8] These maps are not semilinear, but can be taken piecewise algebraic.

$= g_{Zm}\, \mathfrak{e}_m^{\#}\, g_{Yj_mk_m}^{-1}\, \mathfrak{e}_m$ if J_m^1 joins p_m to $p_{j_mk_m}$ and meets $^{\cdot}V^2_{Y''j_mk_m}$

$= g_{Zm}\, \mathfrak{e}_m^{\#}\, g_{Z\lambda(m)}^{-1}\, \mathfrak{e}_m$ if J_m^1 joins p_m to $p_{\lambda(m)}$ (case of Fig. 13),

where the $\mathfrak{e}$'s are words in the g's that are either empty or cancel to the empty word by repeated deleting of syllables $g_{\vee}^{-1} g_{\vee}$ (where the subscript $^{\vee}$ stands for Xc, Ycd, or Ze; $c = 1, \cdots, s$; $d = 1, \cdots, t_c$; $e = u_A + 1, \cdots, v$).

More in detail: The $\mathfrak{e}$'s are products of syllables of

Type a.

$$g_{Xh}^{-1}\, g_{Yh1}^{-1}\, g_{Yh1} \cdots g_{Yht_h}^{-1}\, g_{Yht_h}\, g_{Yht_h}^{-1}\, g_{Yht_h} \cdots g_{Yh1}^{-1}\, g_{Yh1}\, g_{Xh}$$

which occur in $\mathfrak{e}_\sim$ ($\sim$ stands for a pair of indices jk or a single index m), corresponding to the intersections $K^1_{T\sim} \cap {}^{\cdot}V^2_{X''h}$ (compare Fig. 12) and of

Type b. $g_{\vee}^{-1} g_{\vee}$, but not $g_{Xc}^{-1} g_{Xc}$ for any $c = 1, \cdots, s$, which occur in $\mathfrak{e}_\sim$, $\mathfrak{e}_\sim^{\#}$, $\mathfrak{e}_\sim^{\#\#}$ corresponding to the intersections $K^1_{T\sim} \cap {}^{\cdot}T^2_{\vee}$ and $K^1_{V\sim} \cap {}^{\cdot}B^2_{\vee}$ (where T^2_{Ycd}, B^2_{Ycd} stand for

$$T^2_{Y'(cd} + T^2_{Y')cd} + T^2_{Y'')cd}, \qquad B^2_{Y'(cd} + B^2_{Y')cd} + B^2_{Y'')cd},$$

respectively).

We always have

$$\mathfrak{e}^{\#}_{jt_j+1} = g_{Yjt_j}^{-1}\, g_{Yjt_j} \cdots g_{Yj1}^{-1}\, g_{Yj1}$$

and

$$\mathfrak{e}_m^{\#\#} = g_{Yj_m1}^{-1}\, g_{Yj_m1} \cdots g_{Yj_mk_m-1}^{-1}\, g_{Yj_mk_m-1}.$$

The relations[2] $\mathfrak{r}_\sim^{\#} = 1$ show obviously that $\pi_1(M^3)$ is the trivial group.

(10.2) It is essential that the decompositions $\Theta_\sim$ are especially simple: The 1-skeleton of $\Theta_\sim$ intersects $^{\circ}R^2_\sim$ in pairwise disjoint open arcs with boundaries in $^{\cdot}R^2_\sim$ (see Fig. 13); we denote these open arcs (in all the $^{\circ}R^2_\sim$'s) by $Q_1^1, \cdots, Q_w^1$; the $\bar{Q}_f^1$'s ($f = 1, \cdots, w$) are the images of the connected components of $^{\circ}K^2_\sim \cap V^2_*$ under $\varkappa_\sim$ (where these components are open arcs, say $P_1^1, \cdots, P_w^1$, with boundaries in $K^1_{T\sim}$ such that either

$$\bar{P}^1_{f_1} = \bar{P}^1_{f_2} \quad \text{or} \quad \bar{P}^1_{f_1} \cap \bar{P}^1_{f_2} = \emptyset$$

if $f_1, f_2 \in \{1, \cdots, w\}$). We denote the words corresponding to $Q_1^1, \cdots, Q_w^1$ by $\mathfrak{q}_1, \cdots, \mathfrak{q}_w$, respectively. In detail, we have the following five types of words $\mathfrak{q}_f$ (corresponding to six types of arcs $\bar{P}_f^1$; $f = 1, \cdots, w$):

(Type 1) $\mathfrak{q}_f = g_{Zl}\, \mathfrak{e}_{Qf}\, g_{Zm}^{-1}$ if P_f^1 joins $^{\cdot}V^2_{Zl}$ to $^{\cdot}V^2_{Zm}$ ($m \neq 1$)

(Type 2) $\mathfrak{q}_f = g_{Xj}\, \mathfrak{e}_{Qf}\, g_{Zm}^{-1}$ if P_f^1 joins $^{\cdot}V^2_{X'j}$ to $^{\cdot}V^2_{Zm}$

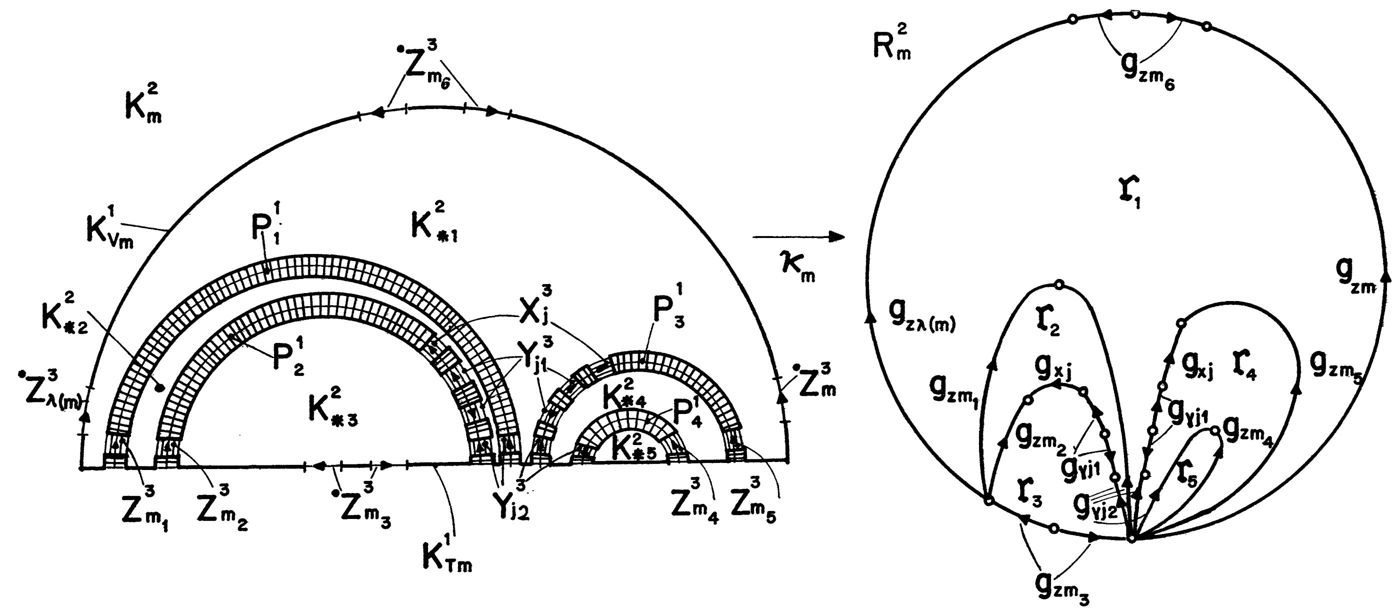


FIG. 13 $K^2_m \cap {}^{\bullet}W^3$ is drawn heavy; the arrows indicate x_z;

$\mathfrak{q}_1 = g_{\gamma j2}\, g^{-1}_{zm_1}$ ($\mathfrak{e}_{q1}$ empty)

$\mathfrak{q}_2 = g_{\gamma j2}\, g^{-1}_{\gamma j1}\, g_{\gamma j1}\, g_{xj}\, g^{-1}_{zm_2}$

$\mathfrak{q}_3 = g_{\gamma j2}\, g^{-1}_{\gamma j1}\, g_{\gamma j1}\, g_{xj}\, g^{-1}_{zm_5}$

$\mathfrak{q}_4 = g_{\gamma j2}\, g^{-1}_{zm_4}$

$\mathfrak{r}_1 = g_{zm}\, g^{-1}_{zm_6}\, g_{zm_6}\, g^{-1}_{z\lambda(m)}\, g_{zm_1}\, g^{-1}_{\gamma j2}\, g_{\gamma j2}\, g^{-1}_{\gamma j1}\, g_{\gamma j1}\, g_{xj}\, g^{-1}_{zm_5}$

$\mathfrak{r}_2 = g_{zm_2}\, g^{-1}_{xj}\, g^{-1}_{\gamma j1}\, g_{\gamma j1}\, g^{-1}_{\gamma j2}\, g_{\gamma j2}\, g^{-1}_{zm_1}$

$\mathfrak{r}_3 = g_{\gamma j2}\, g^{-1}_{\gamma j1}\, g_{\gamma j1}\, g_{xj}\, g^{-1}_{zm_2}\, g^{-1}_{zm_3}\, g_{zm_3}$

$\mathfrak{r}_4 = g_{zm_5}\, g^{-1}_{xj}\, g^{-1}_{\gamma j1}\, g_{\gamma j1}\, g^{-1}_{\gamma j2}\, g_{\gamma j2}\, g^{-1}_{zm_4}$

$\mathfrak{r}_5 = g_{zm_4}\, g^{-1}_{\gamma j2}$

$\mathfrak{r}^{\#}_m = g_{zm}\, g^{-1}_{zm_6}\, g_{zm_6}\, g^{-1}_{z\lambda(m)}\, g^{-1}_{zm_3}\, g_{zm_3}$

(Type 3) $\mathfrak{q}_f = \quad \mathfrak{e}_{Qf}\, g_{Zm}^{-1}$ if P_f^1 joins $\dot{V}^2_{X''j}$ to $\dot{V}^2_{Zm}$ (for some $j = 1, \cdots, s$)

or if P_f^1 joins $\dot{V}^2_*$ to $\dot{V}^2_{Zm}$

(Type 4) $\mathfrak{q}_f = g_{Yjk}\, \mathfrak{e}_{Qf}\, g_{Zm}^{-1}$ if P_f^1 joins $\dot{V}^2_{Y''k}$ to $\dot{V}^2_{Zm}$(P_1^1, P_4^1 in Fig. 13)

(Type 5) $\mathfrak{q}_f = g_{Yjk}\, \mathfrak{e}^{\#}_{Qf}\, g_{Xj}\, \mathfrak{e}_{Qf}\, g_{Zm}^{-1}$ if P_f^1 joins $\dot{V}^2_{Y'jk}$ to $\dot{V}^2_{Zm}$(P_2^1, P_3^1 in Fig. 13)

where $\mathfrak{e}_{Qf}$ and $\mathfrak{e}^{\#}_{Qf}$ are products of syllables $g_{\vee}^{-1}\, g_{\vee}$ which correspond to the intersections of P_f^1 with $B_{\vee}^1$. We do not have more than these six types of arcs $\bar{P}_f^1$ since it follows from the property (2b) of Δ (Lemma 1, Sec. 3) that at least one boundary point of P_f^1 lies in the boundary of a disk V^2_{Zm}. It is remarkable that the word $\mathfrak{q}_f$ cancels down to a word of length either 1 (Type 3), or 2 (Types 1, 2, 4), or 3 (Type 5).

(10.3) If there are two (or three) edges of $\Theta_{\sim}$ in $\dot{R}^2_{\sim}$ that do not correspond to parts of $\mathfrak{e}_{\sim}$, $\mathfrak{e}^{\#}_{\sim}$, or $\mathfrak{e}^{\#\#}_{\sim}$ in ($\#$) then (any two of) these edges are not separated by the $\dot{Q}_f^1$'s in $\dot{R}^2_{\sim}$. Similarly, if two edges in $\dot{R}^2_{\sim}$ correspond to a syllable $g_{\vee}^{-1}\, g_{\vee}$ of Type b in $\mathfrak{e}_{\sim}$, $\mathfrak{e}^{\#}_{\sim}$, or $\mathfrak{e}^{\#\#}_{\sim}$ then these edges are not separated by the $\dot{Q}_f^1$'s in $\dot{R}^2_{\sim}$.

(10.4) Another essential property of the relator-diagrams $\mathfrak{R}_{\sim}$ is the following: if i is a fixed integer, $1 \leqq i \leqq b$, then the decompositions Θ_{jk}, Θ_l ($j = 1, \cdots, s; k = 1, \cdots, t_j + 1; l = u_A + 1, \cdots, u$) contain all together just one 2-dimensional element that is associated with[9] the relator $\mathfrak{r}_i$. However, the decompositions $\Theta_{u+1}, \cdots, \Theta_v$ may contain some more 2-dimensional elements associated with $\mathfrak{r}_i$; but in this case, if $\Theta_{\approx}$ (where $\approx$ stands for two fixed indices j_0k_0 or for one fixed index l_0 with $u_A < l_0 \leqq u$) and $\Theta_m (u < m \leqq v)$ each contain an element associated with $\mathfrak{r}_i$, then Θ_m is isomorphic to a "part" of $\Theta_{\approx}$, i.e., we have $K_m^2 \subset K_{\approx}^2$ and there exists a homeomorphism

$$\alpha_m : R_m^2 \rightarrow R_{\approx}^2$$

of R_m^2 into $R_{\approx}^2$ such that $\{\varkappa_m = \alpha_m^{-1} \cdot (\varkappa_{\approx} \,|\, K_m^2)$, where α_m carries elements of Θ_m onto elements of $\Theta_{\approx}$, preserving the association of these elements to the g's and $\mathfrak{r}$'s. Moreover, $\alpha_m(\dot{R}_m^2)$ intersects $\mathring{R}^2_{\approx}$ in just one of the open arcs Q_f^1.

11. Conclusion. It remains to show that the presentation $\mathfrak{P}(\Psi)$ can be transformed into $(\{g_{\vee}\}, \{g_{\vee}, \overset{b-a}{*}\})$ by cancellation operations as asserted in the theorem. Guided by the relator-diagrams we first transform $\mathfrak{P}(\Psi)$ into a presentation whose relators are derived from the $\mathfrak{r}^{\#}_{\sim}$'s by deleting all the cancellation words $\mathfrak{e}$, $\mathfrak{e}^{\#}$, $\mathfrak{e}^{\#\#}$. The rest is obvious.

Step i. *Removing the cancellation syllables of Type* a *from the* $\mathfrak{r}^{\#}_{\sim}$'s (see Fig.

[9] Of course, there may be other 2-elements associated with relators, say $\mathfrak{r}_{i_1}, \mathfrak{r}_{i_2}, \cdots$, such that $\mathfrak{r}_{i_1} = \mathfrak{r}_{i_2} = \cdots = \mathfrak{r}_i$ (letter by letter), but then $i_1, i_2, \cdots, i$ are pairwise distinct.

14). We transform the relator-diagrams $\mathfrak{R}_{\sim}$ in the following way:

Let P_d^1 be an open arc in ${}^{\circ}K_{\approx}^2 \cap V_*^2$ [$\approx$ stands, as in (10.4), for j_0k_0 or l_0; $j_0 = 1, \cdots, s$; $k_0 = 1, \cdots, t_{j_0} + 1$; $l_0 = u_A + 1, \cdots, u$] that joins a point, say p, in ${}^{\cdot}V_{X''j}^2$ to a point in ${}^{\cdot}V_{Ze}^2$. Then $\varkappa_{\approx}(P_d^1)$ is an arc $\bar{Q}_d^1$ in $R_{\approx}^2$, corresponding to a word $\mathfrak{q}_d = \mathfrak{e}_{Qd}\, g_{Ze}^{-1}$ of Type 3 as considered in (10.2). From $\varkappa_{\approx}(p)$ there originate two arcs, say $N_{\rightarrow}^1$, $N_{\leftarrow}^1$, in ${}^{\cdot}R_{\approx}^2$, that correspond (if oriented towards $\varkappa_{\approx}(p)$) to the same word

$$\mathfrak{n} = g_{Xj}^{-1}\, g_{Yj1}^{-1}\, g_{Yj1} \cdots g_{Yjt_j}^{-1}\, g_{Yjt_j}$$

where $\mathfrak{n}\mathfrak{n}^{-1}$ is a syllable of Type a in $\mathfrak{e}_{\approx}$ as discussed in (10.1)). Now we replace $R_{\approx}^2$ by a disk $R_{\approx}^{*2}$ corresponding to an identification of $N_{\rightarrow}^1$ to $N_{\leftarrow}^1$, i.e., so that there is a map $\beta_{\approx}$ of $R_{\approx}^2$ onto $R_{\approx}^{*2}$ which is one-to-one on $R_{\approx}^2 - (N_{\rightarrow}^1 + N_{\leftarrow}^1)$, on $N_{\rightarrow}^1$, and on $N_{\leftarrow}^1$, which maps $N_{\rightarrow}^1$ and $N_{\leftarrow}^1$ onto the same arc N^1, and which maps the elements of $\Theta_{\approx}$ onto elements of a cell-decomposition $\Theta_{\approx}^*$ of $R_{\approx}^{*2}$ (where $R_{\approx}^{*2}$ is disjoint from M^3 and from the R^2's). We replace $\mathfrak{R}_{\approx}$ by the relator-diagram $\mathfrak{R}_{\approx}^*$ (consisting of $\Theta_{\approx}^*$ and the association of its 1- and 2-elements to the g's and $\mathfrak{r}$'s as carried over by $\beta_{\approx}$). We denote the open arc ${}^{\circ}[\beta_{\approx}(Q_d^1) + N^1]$ by Q_d^{*1} and $\beta_{\approx}(Q_f^1)$ by Q_f^{*1} for all $Q_f^1 \subset R_{\approx}^2$ ($f = 1 \cdots, w; f \neq d$); then to Q_d^{*1} there corresponds the word

$$\mathfrak{q}_d^* = \mathfrak{n}\mathfrak{q}_d = g_{Xj}^{-1}\, g_{Yj1}^{-1}\, g_{Yj1} \cdots g_{Yjt_j}^{-1}\, g_{Yjt_j}\, \mathfrak{e}_{Qd}\, g_{Ze}^{-1},$$

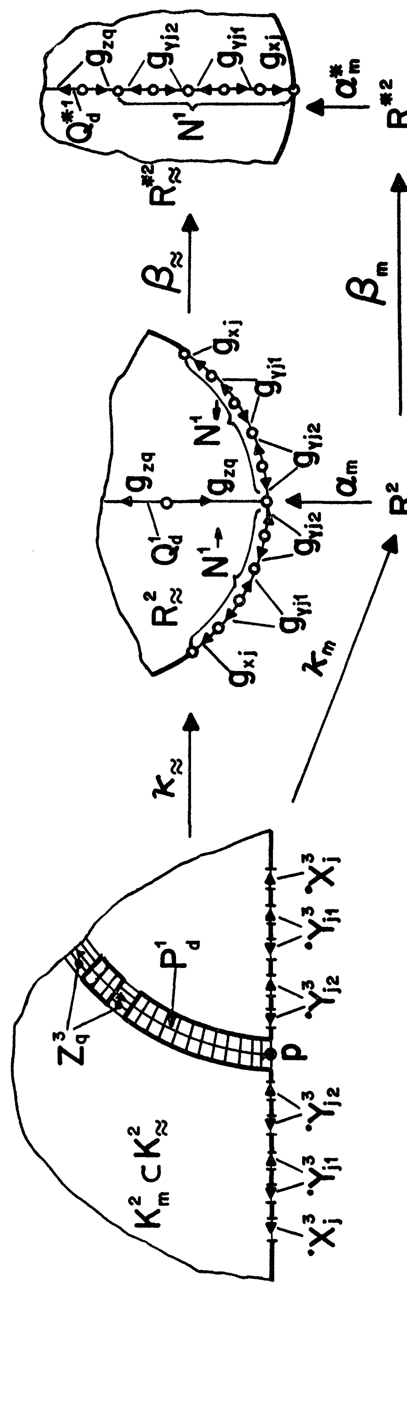
let us call this "of *Type* 3*". To ${}^{\cdot}R_{\approx}^{*2}$ there corresponds a word $\mathfrak{r}_{\approx}^{*\#}$ that is obtained from $\mathfrak{r}_{\approx}^*$ by deleting a syllable $\mathfrak{n}\mathfrak{n}^{-1}$. We remark that $\mathfrak{R}_{\approx}^*$ is a relator-diagram corresponding to $\mathfrak{P}(\Psi)$ and that $\mathfrak{R}_{\approx}^*$ has also the properties stated for $\mathfrak{R}_{\sim}$ in (10.3).

We carry out the above procedure for all those disks K_m^2 that contain the open arc P_d^1; (these disks K_m^2 lie in $K_{\approx}^2$, and $m > u$). We denote the corresponding maps by $\beta_m : R_m^2 \rightarrow R_m^{*2}$, and the relator-diagrams and decompositions so obtained by $\mathfrak{R}_m^*$ and Θ_m^*, respectively. If $K_{\sim}^2$ [$\sim$ stands for jk or m as in Sec. 10] does not contain P_d^1 then we simply denote the identity map on $R_{\sim}^2$ by $\beta_{\sim}$, and $\mathfrak{R}_{\sim}$, $R_{\sim}^2$, $\Theta_{\sim}$ by $\mathfrak{R}_{\sim}^*$, $R_{\sim}^{*2}$, $\Theta_{\sim}^*$, respectively; etc. Now the relator-diagrams $\mathfrak{R}^*$ have again the property stated in (10.4): We obtain the required homeomorphisms α_m^* ($m = u + 1, \cdots, v$) by taking

$$\begin{aligned} \alpha_m^* &= \alpha_m \quad \text{if} \quad K_m^2 \not\subset K_{\approx}^2 \\ &= \beta_{\approx}\, \alpha_m\, \beta_m^{-1} \quad \text{if} \quad K_m^2 \subset K_{\approx}^2 \qquad \text{(see Fig. 14)}, \end{aligned}$$

where we assume that β_m has been chosen in such a way that if two points p_1, p_2 of ${}^{\cdot}R_{\approx}^2$ have the same image point under $\beta_{\approx}$ then $\alpha_m^{-1}(p_1)$ and $\alpha_m^{-1}(p_2)$ have the same image point under β_m.

We carry out the procedure described in the above two paragraphs for all P_d^1's of the type considered and we obtain in this way relator-diagrams $\mathfrak{R}^{\mathrm{I}}$ corresponding to $\mathfrak{P}(\Psi)$. (We use the notation $R_{\sim}^{\mathrm{I}2}$, $\Theta_{\sim}^{\mathrm{I}}$, $Q_f^{\mathrm{I}1}$, $\mathfrak{r}_{\sim}^{\mathrm{I}\#}$, α_m^{I}, etc., for the disks, decompositions, etc., of the $\mathfrak{R}_{\sim}^{\mathrm{I}}$'s.) The words $\mathfrak{r}_{\sim}^{\mathrm{I}\#}$ are still of



$K_m^2 \subset K_\approx^2$
Z_q^3
P_d^1
p
X_j^3
Y_{j1}^3
Y_{j2}^3
$\kappa_\approx$
κ_m
$R_\approx^2$
Q_d^1
N^1
g_{zq}
g_{yj2}
g_{yj1}
g_{xj}
α_m
R_m^2
$\beta_\approx$
β_m
$R_\approx^{*2}$
Q_d^{*1}
α_m^*
R_m^{*2}

FIG. 14

the form ($\#$), but we have the following essential simplification: all the cancellation words $\mathfrak{e}^{\mathrm{I}}_{\sim}$, $\mathfrak{e}^{\mathrm{I}\#}_{\sim}$, $\mathfrak{e}^{\mathrm{I}\#\#}_{\sim}$ are products of syllables $g_{\vee}^{-1} g_{\vee}$ of Type b in such a way that both edges in $\dot{}R^{\mathrm{I}2}_{\sim}$ that correspond to such a syllable lie in the boundary of the same 2-dimensional element of $\Theta^{\mathrm{I}}_{\sim}$ (since all syllables of Type a have been deleted, but (10.3) has been preserved). The $\mathfrak{R}^{\mathrm{I}}_{\sim}$'s have also the property (10.2), modified by admitting $\mathfrak{q}$'s of Type 3^*, and the properties (10.3) and (10.4).

Step ii. *"Outer" cancellations* (see Fig. 15). We consider an arc, say L^1, in $\dot{}R^{\mathrm{I}2}_{\approx}$ ($_\approx$ stands again for j_0k_0 or l_0) that corresponds to a syllable $g_{\vee}^{-1} g_{\vee}$ of Type b in $\mathfrak{e}^{\mathrm{I}}_{\approx}$, $\mathfrak{e}^{\mathrm{I}\#}_{\approx}$, or $\mathfrak{e}^{\mathrm{I}\#\#}_{\approx}$. Because of (10.3) all of L^1 lies in the boundary of just one 2-dimensional element of $\Theta^{\mathrm{I}}_{\approx}$ corresponding to a relator, say $\mathfrak{r}_{i_0}$ ($i_0 = 1, \cdots, b$). So we may cancel the corresponding syllable $g_{\vee}^{-1} g_{\vee}$ in $\mathfrak{r}_{i_0}$ (cancellation operation of Type 1) which yields a new relator, say $\mathfrak{r}^{\mathrm{I}*}_{i_0}$ and a new presentation

$$\mathfrak{P}^{\mathrm{I}*} = (\{g_{\mathrm{X}j}, g_{\mathrm{Y}jk}, g_{\mathrm{Z}m}\}, \{\mathfrak{r}^{\mathrm{I}*}_i\})$$

where $\mathfrak{r}^{\mathrm{I}*}_i = \mathfrak{r}_i$ if $i \neq i_0$.

Now we replace $R^{\mathrm{I}2}_{\approx}$ by a disk $R^{\mathrm{I}*2}_{\approx}$ corresponding to shrinking L^1 to one point, i.e., so that there is a map $\beta^{\mathrm{I}*}_{\approx}$ of $R^{\mathrm{I}2}_{\approx}$ onto $R^{\mathrm{I}*2}_{\approx}$ which[10] is one-to-one on $R^{\mathrm{I}2}_{\approx} - L^1$, which maps L^1 into one point, and which maps the elements of Θ^{I} onto elements of a cell-decomposition $\Theta^{\mathrm{I}*}_{\approx}$ of $R^{\mathrm{I}*2}_{\approx}$. Now $\mathfrak{R}^{\mathrm{I}*}_{\approx}$ (consisting of $\Theta^{\mathrm{I}*}_{\approx}$ and the association of its 1- and 2-elements to the g's and $\mathfrak{r}^{\mathrm{I}*}$'s as induced by $\beta^{\mathrm{I}*}_{\approx}$) is a relator-diagram corresponding to $\mathfrak{P}^{\mathrm{I}*}$ (since by (10.4) $\mathfrak{r}_{i_0}$ occurs just once in $\Theta^{\mathrm{I}}_{\approx}$). In the same way we replace all those relator-diagrams $\mathfrak{R}^{\mathrm{I}}_m$ whose decompositions Θ^{I}_m contain a 2-element associated with $\mathfrak{r}_{i_0}$ by relator-diagrams $\mathfrak{R}^{\mathrm{I}*}_m$ (defined by maps $\beta^{\mathrm{I}*}_m : R^{\mathrm{I}2}_m \to R^{\mathrm{I}*2}_m$ that map the arcs $(\alpha^{\mathrm{I}}_m)^{-1}(L^1)$ into single points). For the remaining $\mathfrak{R}^{\mathrm{I}}_{\sim}$'s we take $\beta^{\mathrm{I}*}_{\sim}$ to be the identity on $R^{\mathrm{I}2}_{\sim}$, and we take $\mathfrak{R}^{\mathrm{I}*}_{\sim} = \mathfrak{R}^{\mathrm{I}}_{\sim}$, etc. Then the $\mathfrak{R}^{\mathrm{I}*}$'s are relator-diagrams corresponding to $\mathfrak{P}^{\mathrm{I}*}$ and possess the properties (10.1), (10.2, modified by admitting $\mathfrak{q}^{\mathrm{I}*}$'s of Type 3^*), (10.3), and (10.4).

We carry out the above procedure for all arcs of the considered type, and we obtain in this way (by cancellation operations of Type 1) a presentation

$$\mathfrak{P}^{\mathrm{II}} = (\{g_{\mathrm{X}j}, g_{\mathrm{Y}jk}, g_{\mathrm{Z}m}\}, \{\mathfrak{r}^{\mathrm{II}}_i\})$$

and corresponding relator-diagrams $\mathfrak{R}^{\mathrm{II}}$. (We use the notation $Q^{\mathrm{II}1}_f$, $\mathfrak{r}^{\mathrm{II}\#}_{\sim}$, α^{II}_m, etc., in the obvious way.) Now the cancellation words $\mathfrak{e}^{\mathrm{II}}_{\sim}$, $\mathfrak{e}^{\mathrm{II}\#}_{\sim}$, $\mathfrak{e}^{\mathrm{II}\#\#}_{\sim}$ are empty, except, may be, if $\sim$ stands for m with $m > u$. Furthermore, the boundaries of all those open arcs $Q^{\mathrm{II}1}_f$ that lie in some $R^{\mathrm{II}2}_{\sim}$ are equal to just one point in $\dot{}R^{\mathrm{II}2}_{\sim}$ (compare Fig. 16).

Step iii. *"Inner" cancellations.* Now we consider those arcs L^1 in the $Q^{\mathrm{II}1}_f$'s that lie in disks $R^{\mathrm{II}2}_{\approx}$ and that correspond to syllables $g_{\vee}^{-1} g_{\vee}$. As in Step ii we cancel, step by step, all the corresponding syllables in the $\mathfrak{r}^{\mathrm{II}}_i$'s (cancellation operations of Type 1), and we obtain in this way a presentation

$$\mathfrak{P}^{\mathrm{III}} = (\{g_{\mathrm{X}j}, g_{\mathrm{Y}jk}, g_{\mathrm{Z}m}\}, \{\mathfrak{r}^{\mathrm{III}}_i\}).$$

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[10] These maps are not semilinear, but can be taken piecewise algebraic.

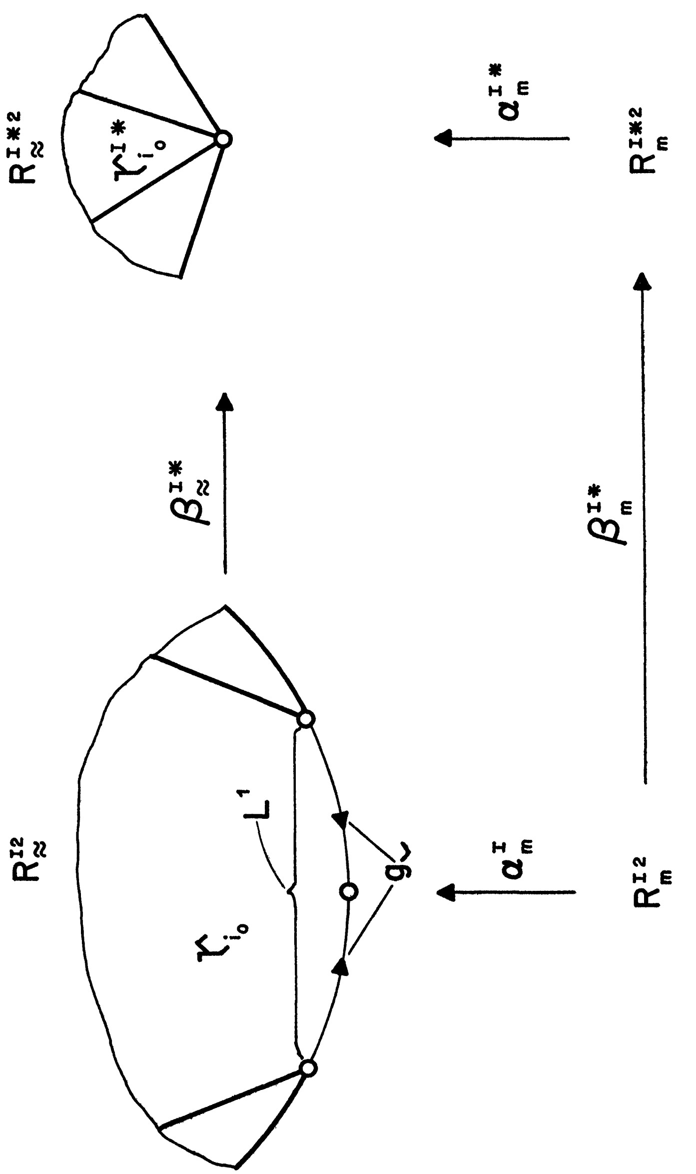

FIG. 15

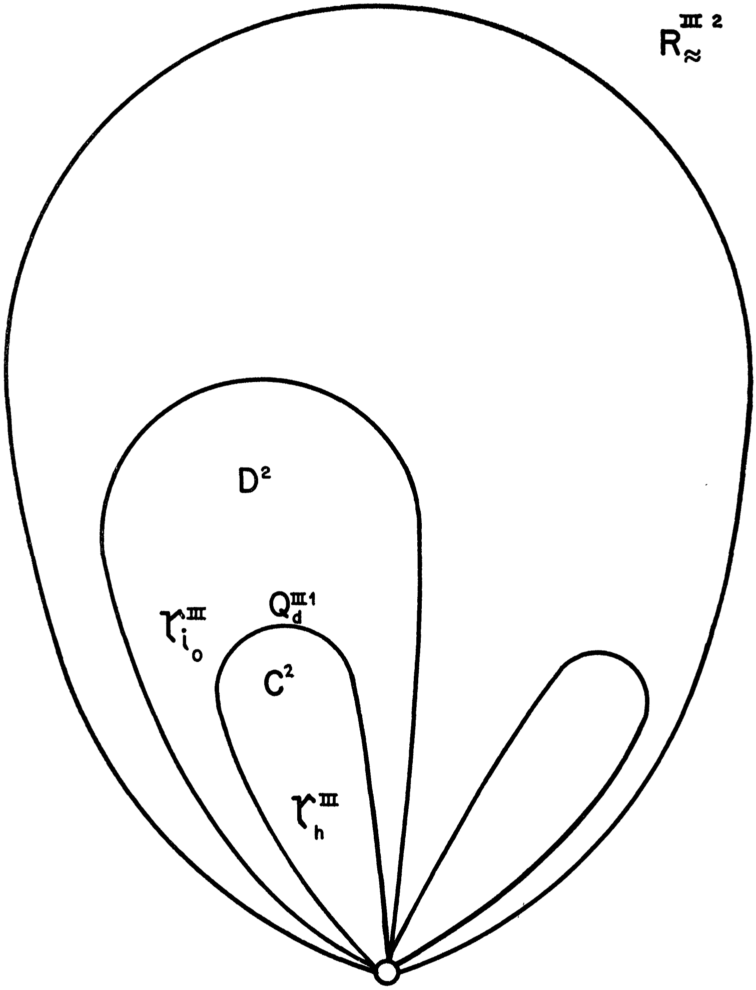

FIG. 16

Then, again as in Step ii, we construct relator-diagrams $\mathfrak{R}_{\approx}^{III}$ and $\mathfrak{R}_m^{III}$ that correspond to $\mathfrak{P}^{III}$ by shrinking arcs to points. (Note that the arcs $(\alpha_m^{II})^{-1}(L^1)$ may lie in $^{\cdot}R_m^{II2}$ as well as in $^{\circ}R_m^{II2}(m > u)$.) The $\mathfrak{R}^{III}$'s possess again the properties (10.1), (10.2, modified), (10.3), (10.4).

The words $\mathfrak{r}_{\sim}^{III\#}$ read from the $^{\cdot}R^{III2}$'s are of the form

$$\mathfrak{r}_{j1}^{III\#} = g_{Yj1}\ ; \quad \mathfrak{r}_{jk}^{III\#} = g_{Yjk}\, g_{Yjk-1}^{-1}\ (k = 2, \cdots, t_j);$$

$$\mathfrak{r}_{Yjt_j+1}^{III\#} = g_{Xj}\, g_{Yjt_j}\ ;$$

$$(\mathrm{III}\,\#)\qquad \mathfrak{r}_m^{\mathrm{III}\#} = \text{either } g_{\mathrm{Z}m}\,,\ \text{or } g_{\mathrm{Z}m}\, g_{\mathrm{X}j_m}^{\pm 1}\,,\ \text{or } g_{\mathrm{Z}m}\, g_{\mathrm{X}j_m}^{-1}\, g_{\mathrm{Y}j_m k_m}^{-1}\,,$$
$$\text{or } g_{\mathrm{Z}m}\, g_{\mathrm{Y}j_m k_m}^{-1}\,,\ \text{or } g_{\mathrm{Z}m}\, g_{\mathrm{Z}\lambda(m)}^{-1}\,,\quad \lambda(m) < m,$$

and the words $\mathfrak{q}_f^{\mathrm{III}}$ $(f = 1, \cdots, w)$ read from the $Q_f^{\mathrm{III}1}$'s are of the form

$$\mathfrak{q}_f^{\mathrm{III}} = \text{either } g_{\mathrm{Z}l}\, g_{\mathrm{Z}m}^{-1}\,,\ \text{or } g_{\mathrm{X}j}^{\pm 1}\, g_{\mathrm{Z}m}^{-1}\,,\ \text{or } g_{\mathrm{Z}m}^{-1}\,,\ \text{or } g_{\mathrm{Y}jk}\, g_{\mathrm{Z}m}^{-1}\,,\ \text{or } g_{\mathrm{Y}jk}\, g_{\mathrm{X}j}\, g_{\mathrm{Z}m}^{-1}\,.$$

Step iv. *Deleting Q^1's* (see Fig. 16). Provided that $w \neq 0$ there exists a disk $R_{\approx}^{\mathrm{III}2}$ that contains at least one of the open arcs $Q_f^{\mathrm{III}1}$. Then there is at least one open arc, say $Q_d^{\mathrm{III}1}$ that lies in $R_{\approx}^{\mathrm{III}2}$ in such a way that $\bar{Q}_d^{\mathrm{III}1}$ is the boundary of a 2-dimensional element, say C^2, of $\Theta_{\approx}^{\mathrm{III}}$. There is just one other element, say D^2, of $\Theta_{\approx}^{\mathrm{III}}$ whose boundary contains $\bar{Q}_d^{\mathrm{III}1}$. Let $\mathfrak{r}_h^{\mathrm{III}}$ and $\mathfrak{r}_{i_0}^{\mathrm{III}}$ $(h \neq i_0)$ be the relators associated with C^2 and D^2, respectively; then $\mathfrak{q}_d^{\mathrm{III}}$ is equal to a member of $\langle \mathfrak{r}_h^{\mathrm{III}} \rangle$, and some member of $\langle \mathfrak{r}_{i_0}^{\mathrm{III}} \rangle$ can be written as $(\mathfrak{q}_d^{\mathrm{III}})^{\pm 1} \mathfrak{r}_{i_0}^{\mathrm{IV}}$ (where $\mathfrak{r}_{i_0}^{\mathrm{IV}}$ is some word in the g's). Now we replace $\mathfrak{r}_{i_0}^{\mathrm{III}}$ by $\mathfrak{r}_{i_0}^{\mathrm{IV}}$ (cancellation operation of Type 2 and of length 1, 2, or 3) and we obtain in this way a presentation $\mathfrak{P}^{\mathrm{IV}}$ from $\mathfrak{P}^{\mathrm{III}}$.

Then we construct relator-diagrams $\mathfrak{R}_{\sim}^{\mathrm{IV}}$ that correspond to $\mathfrak{P}^{\mathrm{IV}}$ as follows: First we delete from $\Theta_{\approx}^{\mathrm{III}}$ those elements that lie in $Q_d^{\mathrm{III}1}$, and we replace the elements C^2, D^2 by the open disk $C^2 + Q_d^{\mathrm{III}1} + D^2$; this yields $\Theta_{\approx}^{\mathrm{IV}}$ (where $R_{\approx}^{\mathrm{IV}2} = R_{\approx}^{\mathrm{III}2}$ and the new 2-dimensional element of $\Theta_{\approx}^{\mathrm{IV}}$ is associated with $\mathfrak{r}_{i_0}^{\mathrm{IV}}$). If the relator $\mathfrak{r}_{i_0}^{\mathrm{III}}$ is associated with a 2-dimensional element, say D_0^2, of a decomposition Θ_m^{III}, different from $\Theta_{\approx}^{\mathrm{III}}$, then [by (10.4)] $m > u$, and $\alpha_m^{\mathrm{III}}(R_m^{\mathrm{III}2})$ contains $Q_d^{\mathrm{III}1}$ in its interior (since otherwise the closed curve $\bar{Q}_d^{\mathrm{III}1}$ would lie in $\alpha_m^{\mathrm{III}}({}^{\cdot}R_m^{\mathrm{III}2})$, but would not be equal to $\alpha_m^{\mathrm{III}}({}^{\cdot}R_m^{\mathrm{III}2})$ in contradiction to the fact that $\alpha_m^{\mathrm{III}}(R_m^{\mathrm{III}2})$ is a disk). Hence ${}^{\circ}R_m^{\mathrm{III}2}$ contains $(\alpha_m^{\mathrm{III}})^{-1}(Q_d^{\mathrm{III}1})$ which is one of the $Q_f^{\mathrm{III}1}$'s, say $Q_c^{\mathrm{III}1}$, and Θ_m^{III} possesses an element $C_0^2 = (\alpha_m^{\mathrm{III}})^{-1}(C^2)$ that is associated with $\mathfrak{r}_h^{\mathrm{III}}$. Then we delete from Θ_m^{III} those elements that lie in $Q_c^{\mathrm{III}1}$, and we replace the elements C_0^2 and D_0^2 by $C_0^2 + Q_c^{\mathrm{III}1} + D_0^2$ (which we associate with $\mathfrak{r}_{i_0}^{\mathrm{IV}}$). This yields Θ_m^{IV}. For the remaining $\mathfrak{R}_{\sim}^{\mathrm{III}}$'s we take $\mathfrak{R}_{\sim}^{\mathrm{IV}} = \mathfrak{R}_{\sim}^{\mathrm{III}}$. We write $\mathfrak{r}_i^{\mathrm{IV}} = \mathfrak{r}_i^{\mathrm{III}}$ if $i \neq i_0$.

Now the $\mathfrak{R}_{\sim}^{\mathrm{IV}}$'s have again the properties (10.1), (10.2, modified), (10.3), (10.4), and $\mathfrak{r}_{\sim}^{\mathrm{IV}\#} = \mathfrak{r}_{\sim}^{\mathrm{III}\#}$, but the number w^{IV} of open arcs $Q_f^{\mathrm{IV}1}$ $(f = 1, \cdots, w^{\mathrm{IV}})$ is smaller than w.

Step v. *Deleting all Q^1's.* We repeat the procedure of Step iv as often as possible and finally obtain in this way (after at most w steps) a presentation $\mathfrak{P}^{\mathrm{V}} = (\{g_{\vee}\}, \{\mathfrak{r}_i^{\mathrm{V}}\})$ and corresponding relator-diagrams $\mathfrak{R}_{\sim}^{\mathrm{V}}$ such that each decomposition $\Theta_{\sim}^{\mathrm{V}}$ possesses just one 2-dimensional element. That means that each of the $v + s$ words $\mathfrak{r}_{\sim}^{\mathrm{V}\#} = \mathfrak{r}_{\sim}^{\mathrm{III}\#}$ [see (III $\#$)] is at the same time a member of a class, say $\langle \mathfrak{r}_{i_\sim}^{\mathrm{V}} \rangle$, where we may assume that the notation is so arranged that $1 \leqq i_\sim \leqq v + s$; (we do not specify the remaining relators $\mathfrak{r}_{v+s+1}^{\mathrm{V}}, \cdots, \mathfrak{r}_b^{\mathrm{V}}$). Further we may assume that $\mathfrak{r}_{\sim}^{\mathrm{V}\#}$ is a cyclic permutation of $\mathfrak{r}_{i_\sim}^{\mathrm{V}}$ and not one of $(\mathfrak{r}_{i_\sim}^{\mathrm{V}})^{-1}$; (this can be arranged by proper choice of the direction in which $\mathfrak{r}_{i_\sim}^{\mathrm{V}}$ is read when the last open arc Q^1 is removed in ${}^{\circ}R_{\sim}^{\mathrm{III}2}$).

Step vi. Obviously we can transform $\mathfrak{P}^{\mathrm{V}}$ by a sequence of cancellation operations of Type 2 and length 1 (a relator out of $\langle g_{\mathrm{Y}jk}\, g_{\mathrm{Y}jk-1}^{-1} \rangle$ is replaced by a relator $g_{\mathrm{Y}jk}$ where another relator is equal to $g_{\mathrm{Y}jk-1}$, $k = 2, \cdots, t_j$; etc.) into a presentation

$$\mathfrak{P}^{\mathrm{VI}} = (\{g_{\vee}\}, \{g_{\vee}, \mathfrak{r}_{v+s+1}^{\mathrm{VI}}, \cdots, \mathfrak{r}_b^{\mathrm{VI}}\})$$

where $\mathfrak{r}_i^{\mathrm{VI}} = \mathfrak{r}_i^{\mathrm{V}}$ for $i = v + s + 1, \cdots, b$. Now we can transform $\mathfrak{P}^{\mathrm{VI}}$ by a sequence of cancellation operations of Type 2 and length 1 (a relator $g_{\vee}^{\pm 1}\mathfrak{r}''$ is replaced by $\mathfrak{r}''$ where another relator is equal to $g_{\vee}$) into the presentation $(\{g_{\vee}\}, \{g_{\vee}, *^{b-(v+s)}\})$. This finishes the proof of the theorem.

Bibliography

1. S. Adjan, *The algorithmic unsolvability of problems concerning certain properties of groups*, Doklady Akad. Nauk SSSR, vol. 103 (1955), pp. 533–535.
2. J. W. Alexander, *On the subdivision of polyhedra in 3-space*, Proc. Nat. Acad. Sci. U.S.A., vol. 10 (1924), pp. 6–8.
3. W. Haken, *On homotopy 3-spheres*, Illinois J. Math., vol. 10 (1966), pp. 159–178.
4. ———, *Some results on surfaces in 3-manifolds*, Studies in modern topology, Math Assoc. Amer. studies in mathematics, vol. 5(1968), pp. 39–98.
5. ———, *Trivial loops in homotopy 3-spheres*, Illinois J. Math., vol. 11 (1967), pp. 547–554.
6. M. Rabin, *Recursive unsolvability of group theoretic problems*, Ann. of Math. (2), vol. 67 (1958), pp. 172–194.
7. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, B. G. Teubner, Leipzig, 1934.
8. E. A. Van Kampen, *On some lemmas in the theory of groups*, Amer. J. Math., vol. 55 (1933), pp. 268–273.

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