A REPLACEMENT THEOREM FOR NILPOTENT GROUPS WITH MAXIMUM CONDITION

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The object of this note is to extend some recent work of J. Thompson and of G. Glauberman to nilpotent groups with maximum condition on subgroups. Our results derive from a slight simplification of the proofs as given in [1]. We are indebted to D. Hertzig and M. Suzuki for some corrections and improvements and to G. Glauberman for a preprint of his work.

We shall first consider a finitely generated nilpotent group S. Such a group has a finite normal series with cyclic factors of either prime or infinite order. We shall let $\mathfrak{A}_0(S)$ be the family of self-centralizing abelian subgroups of maximum torsion-free rank of S and will let $\mathfrak{A}(S)$ be the subfamily consisting of those members whose torsion subgroups have maximal orders. We let $J(S)$ denote the subgroup of S generated by the members of $\mathfrak{A}(S)$.

In general $Z(H)$, $C(H)$, and $N(H)$ will denote center, centralizer, and normalizer, respectively of H. $C_K(H)$ will mean $C(H) \cap K$ and $N_K(H)$ will mean $N(H) \cap K$. The commutator $[a, b]$ will mean $a^{-1}b^{-1}ab = a^{-1}a^b$ and $[A, B]$ will denote the subgroup generated by all $[a, b]$ with $a \in A$, $b \in B$; $[A, B, C]$ will mean $[[A, B], C]$, A^2 will denote $[A, A]$ and A^n will denote $[A^{n-1}, A]$. It will be convenient to let $[A, 1B] = [A, B]$ and then to let $[A, nB]$ mean $[A, (n-1)B, B]$.

Our first theorem includes the replacement Theorems 3.1 and 4.1 of [1].

THEOREM 1. *Let S be a finitely generated nilpotent group and let B be a normal subgroup of S with B^2 central in $BJ(S)$. Let A be in $\mathfrak{A}(S)$ with $[B, A, A] \neq 1$, and further so that if B has an involution then either B is abelian or $[B, A, A, A] \neq 1$. Then there is an $A^* \in \mathfrak{A}(S)$ so that*

$$A \cap B < A^* \cap B \quad \textit{and} \quad [A^*, A, A] = 1.$$

Proof. Without loss of generality we may assume that $S = AB$. Then since B^2 is central in $BJ(S)$ (and consequently in S) and A is self-centralizing, $B^2 < A$ and $A \cap B$ is normal in S (for any subgroup of B containing B^2 is normal in B).

If we use bars to denote elements and subgroups mod $A \cap B$, then $\bar{S}$ is the semi-direct product $[\bar{B}]\bar{A}$. Now we let $B_1 = C_B(A)$ (therefore $B_1 = A \cap B$) and inductively let B_n be the set of $b \in B$ so that $[b, A] \leqq B_{n-1}$. Since $[B, A, A] \neq 1$, $B_3 > B_2$. Now $\bar{B}_3\bar{A}$ is nilpotent of class at most 2 and therefore for any fixed $x \in B_3$, the map ϕ defined by $a\phi = [\bar{x}, \bar{a}]$ for $a \in A$ is a

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homomorphism from A onto $[\bar{x}, \bar{A}]$ whose kernel C is the complete inverse image of the centralizer $C_{\bar{A}}(\bar{x})$.

We first show that we may always pick an $x \in B_3$, $x \notin B_2$ so that $[x, A]$ is abelian. If B is abelian, this is the case for all $x \in B_3$, $x \notin B_2$ since $[x, A] \leqq B$. If $[B, 3A] \neq 1$, we may without loss of generality take S (which is BA) to be $B_4 A$ so that $[B, 4A] = 1$. Then $[S^3, S^3] \leqq S^6 = (BA)^6$. But $[BA, BA] \leqq B^2[B, A]$, and inductively we get that

$$(BA)^5 \leqq B^2[B, 4A] = B^2;$$

thus $(BA)^6 = 1$ since B^2 is central. It follows that S^3 and hence $[S, 2A]$ is abelian. Since $[B, 3A] \neq 1$, $[S, 3A] \neq 1$ and hence $[S, 2A] \nleqq B_1$, $[S, A] \nleqq B_2$. Thus when $[B, 3A] \neq 1$, those x in $[B_4, A]$ not in $[B_3, A] = B_2$ have the property that $[x, A]$ is abelian.

The last case to consider is when $[B, 3A] = 1$ and B $(= B_3)$ has no involution. We proceed as follows. In general we have

$$[x, ac] = [x, c][x, a]^c = [x, c][x, a][x, a, c]$$

and similarly $[x, ca] = [x, a][x, c][x, c, a]$. Then if a and c commute and $[x, a]$ and $[x, c]$ commute,

$$(1) \qquad [x, a, c] = [x, c, a].$$

We use (1) in S modulo B^2 with $x \in B$, $a, c \in A$, since for $x \in B$, $[x, A]$ is abelian modulo B^2; furthermore, since B^2 is central in S we get for x, y in B, $a, c \in A$ that

$$(2) \qquad [x, a, c, y] = [x, c, a, y].$$

Then since B^2 is central in B we see that

$$[x, a, c, y]^{-1} = [x, a, c, y^{-1}] = [[x, a, c]^{-1}, y];$$

and since $x \in B = B_3$, we see from $[x, a, cc^{-1}] = 1$ that

$$[x, a, c]^{-1} = [x, a, c^{-1}].$$

Together we have

$$(3) \qquad [x, a, c, y] = [x, a, c^{-1}, y^{-1}].$$

By the Hall identity (Lemma 4.1 (b) of [1]),

$$[x, a, c^{-1}, x^{-1}]^c[[c, x], [x, a]]^{x^{-1}}[x^{-1}, [x, a]^{-1}, c]^{[x,a]} = 1,$$

which simplifies (since B^2 is central) to

$$(4) \qquad [x, a, c^{-1}, x^{-1}][[c, x], [x, a]] = 1.$$

Then from (3) and (4) and the fact that B^2 is central,

$$[x, a, c, x] = [[x, c], [x, a]];$$

and by symmetry, $[x, c, a, x] = [[x, a], [x, c]]$. Thus from (2) it follows that $[[x, a], [x, c]]$ is its own inverse and is therefore 1 since B has no involution. Thus in all cases there is an $x \in B_3$, $x \notin B_2$, so that $[x, A]$ is abelian as we wished to show.

From (1) it now follows that $[x, A, C] = 1$ and then, if A^* denotes $[x, A]C$, that A^* is abelian. Since B has class at most 2, $A \cap B = C \cap B$ and consequently $A/C \cong A^*/C$; for

$$A/C \cong \bar{A}/\bar{C} \cong [x, A](A \cap B)/(A \cap B) \cong [x, A](B \cap C)/(B \cap C)$$
$$\cong [x, A]/([x, A] \cap C) \cong [x, A]C/C \cong A^*/C.$$

It follows that A^* has the same torsion-free rank as A. Furthermore, since a nilpotent group has a normal torsion subgroup, by restricting consideration to the torsion subgroup we deduce (sence $[x, a]$ is periodic when a is) that the torsion subgroup of A^* has the same order as that of A and hence that $A^* \in \mathfrak{A}(S)$. Since $x \notin B_2$; $A^* \cap B > A \cap B$; and since $[x, A] \leqq B_2$ and $[B_2, A, A] = 1$, it follows that $[A^*, A, A] = 1$ and the theorem is proved.

COROLLERY 1. *Suppose B is abelian or has no involution with B^2 central in $BJ(S)$. If A is chosen in $\mathfrak{A}(S)$ so that for no $A_1 \in \mathfrak{A}(S)$ is $A \cap B < A_1 \cap B$, then $[B, A, A] = 1$.*

We now introduce a notion of stability in terms of which we can formulate our next theorem. This "stability" includes, as can readily be checked, the notion of p-stability as given in [1]. Suppose that a group G has a finitely generated nilpotent subgroup S such that for each normal subgroup K of G, $S \cap K$ is intravariant in K and let T be any characteristic subgroup of G maximal in that $S \cap T = 1$. We shall say that G is *S-stable* if when S and T are as above and if for arbitrary $P \leqq S$ such that $PT \lhd G$, $x \in N(P)$ with $[P, x, x] = 1$ implies that $x^n \in SC(P)$ for all $n \in N(P)$. This means in particular that for P a normal subgroup of an S-stable group G with $P \leqq S$, $[P, x, x] = 1$ implies that $x \in \bigcap_{g \in G} S^g C(P)$.

Our next results include Theorem 4.3 and Theorem A as well as Corollaries 3.2 and 3.5 of [1].

THEOREM 2. *Let G be an S-stable group and let B be a normal subgroup of G contained in S; suppose further that B is abelian if S contains an involution. Then $Z(J(S)) \cap B$ is normal in G.*

Proof. We first assume that B^2 is central in $BJ(S)$ (and of course that $B \neq 1$).

Let C denote $Z(J(S)) \cap B$ and let V be the normal closure of C in G. We must show that $C = V$. First we pick an $A \in \mathfrak{A}(S)$ so that for no $A_1 \in \mathfrak{A}(S)$ is $V \cap A$ contained properly in $V \cap A_1$. By Corollary 1, this implies that $[V, A, A] = 1$. If L denotes $\bigcap_{g \in G} S^g C(V)$, then $L \lhd G$ and since G is S-stable, $A \leqq L$. Hence $Z(J(S)) \leqq X$ with X denoting $Z(J(S \cap L))$. By the Frattini argument, $G = L(N(S \cap L))$. Since X is characteristic in $S \cap L$, $G = LN(X)$.

If $Z(J(S)) \neq X$, there is an $A \in \mathfrak{A}(S) A \nleqq L$ so that for no $A_1 \in \mathfrak{A}(S)$ with $A_1 \nleqq L$, is $V \cap A$ properly contained in $V \cap A_1$. Since $A \nleqq L$, S-stability implies that $[V, A, A] \neq 1$, and Theorem 1 implies that there is an A^* with $V \cap A^* > V \cap A$ and $[A^*, A, A] = 1$. By the maximality of $V \cap A$ in the choice of A, $A^* \leqq L$ and hence $A^* \geqq X$. Since $C \lhd L$ and $C \leqq X$, it follows that $V \leqq X$. We then have (since $V \leqq X \leqq A^*$) the contradiction

$$1 \neq [V, A, A] \leqq [X, A, A] \leqq [A^*, A, A] = 1.$$

We conclude that $Z(J(S)) = X$ and hence that C (which is then $X \cap B$) is normal in $G = LN(X)$. This proves the theorem for the case that B^2 is central in $BJ(S)$.

If B^2 is not central in $BJ(S)$ we can assume inductively (on the class of B) that $Z(J(S)) \cap B^2 \lhd G$. But

$$C = Z(J(S)) \cap B = Z(J(S)) \cap V$$

and hence (since V is the normal closure of C and $[C, V] \leqq C \cap B^2$ a normal subgroup of G) $[V, V] \leqq C \cap B^2 \leqq C$. Thus V^2 is central in $V(J(S))$, and by the first part of the proof with V in place of B it follows that

$$C = Z(J(S)) \cap V \lhd G$$

and hence $C = V$ as was to be shown.

COROLLARY 2. *Let G be an S-stable group, let $B = \bigcap_{g \in G} S^g$, and suppose that $B \geqq C(B)$; suppose further that B is abelian if S contains an involution. Then*

1. *the center Z of $J(S)$ is a characteristic subgroup of G;*
2. *if B is abelian then B is the only element of $\mathfrak{A}(S)$;*
3. *$G = C(Z(S))N(J(S))$.*

Proof of 1. Since Z is an abelian normal subgroup of S, $[B, Z, Z] = 1$. The S-stability then implies that $Z^g \leqq SC(B)$ for all $g \in G$. Consequently $Z \leqq B$. Since S is intravariant, B is normal in the holomorph H of G, and consequently by the theorem, $Z \lhd H$ or Z is characteristic in G as was to be shown.

Proof of 2. If $\mathfrak{A}(S)$ has an element other than B, choose an $A \in \mathfrak{A}(S)$, $A \neq B$, so that $A \cap B$ is maximal. If $[B, A, A] \neq 1$, then by Theorem 1 there is an A^* with

$$A \cap B < A^* \cap B \quad \text{and} \quad [A^*, A, A] = 1.$$

By the maximality of $A \cap B$, $A^* = B$. Then $[B, A, A] = 1$ and S-stability implies that $A \leqq B$ as in the proof of 1. Thus B is the only element of $\mathfrak{A}(S)$ as was to be shown.

Proof of 3. Since $B \leqq S$, $Z(S) \leqq C(B) \leqq B$. Then $Z(S) \leqq Z(B)$. Since $B \lhd G$, $Z(B) \lhd G$, and by Theorem 2, if Z denotes $Z(B) \cap Z(J(S))$, then $Z \lhd G$ (and consequently $C(Z) \lhd G$). By the Frattini argument,

$$G = C(Z)N(S \cap C(Z)).$$

Since $J(S)$ centralizes Z, $S \cap C(Z) \geqq J(S)$ so that $J(S \cap C(Z)) = J(S)$, and $N(S \cap C(Z)) \leqq N(J(S))$. Thus $G = C(Z)N(J(S))$. Since $Z(S) \leqq Z$, $C(Z(S)) \geqq C(Z)$, and $G = C(Z(S))N(J(S))$ as was to be shown.

Reference

1. George Glauberman, *A characteristic subgroup of a p-stable group*, Canadian J. Math., 1968, to appear.

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