BEHAVIOR OF LINEAR FORMS ON EXTREME POINTS[1]

BY
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Introduction

Suppose that L is a locally convex topological linear space, L^* is its conjugate space, K is a compact convex subset of L, and E_K is the set of all extreme points of K. By the Krein-Milman Theorem [9], [3, pp. 83–84], $K = \text{cl con } E_K$ and hence $\sup fE_K = \sup fK$ for each $f \in L^*$; moreover, the K-supremum of f is actually attained on E_K. If E_K is finite (that is, if K is a finite-dimensional convex polytope) then

(I) *for each* $f \in L^*$ *the number* $\sup fE_K$ *is an isolated point of the set* fE_K.

Here we are concerned with the restrictions which condition (I) imposes on the topological structure of E_K. The following results are established.

(a) *When L is finite-dimensional, condition* (I) *is equivalent to the finiteness of* E_K

(b) *When L is a Banach space,* (I) *implies the isolated points of* E_K *are dense in* E_K. *Conversely, for each infinite-dimensional Banach space L and each compact metric space Q in which the isolated points are dense, there is a compact convex set K in L such that* (I) *holds and* E_K *is homeomorphic with Q.*

(c) *In general,* (I) *does not imply E has isolated points. Indeed, for each totally disconnected compact Hausdorff space Q there is a locally convex space L and a compact convex set K in L such that* (I) *holds and* E_K *is homeomorphic with Q.*

These results are for compact convex sets, but some related results are obtained for locally compact closed convex sets containing no line.

Two lemmas

The lemmas of this section will not be used in proving (a), (c), or the first part of (b). They will be used for the second part of (b) and for a related finite-dimensional result.

For subsets X and Y of a metric space with distance function ρ, let

$$\delta(X, Y) = \inf \{\rho(x, y) : x \in X, y \in Y\}.$$

For a point x of the space let $\delta(x, Y) = \delta(\{x\}, Y)$. The following result may well be known, but lacking a specific reference we include a proof.

TOPOLOGICAL LEMMA. *For* $r = 1, 2$, *suppose that* A_r *is the set of all accumulation points of a compact metric space* Q_r *and that the set* $Q_r \sim A_r$ *of all*

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isolated points of Q_r is dense in Q_r. Then every homeomorphism of A_1 onto A_2 can be extended to a homeomorphism of Q_1 onto Q_2.

Proof. For the sake of notational simplicity, we assume Q_1 and Q_2 are disjoint and we use the symbol ρ for both distance functions. Since A_r is compact there is a function π_r on Q_r to A_r such that $\delta(q, A_r) = \rho(q, \pi_r(q))$ for all $q \in Q_r$. The hypotheses imply also that $Q_r \sim A_r$ is countably infinite and hence can be sequentially ordered. Consider an arbitrary homeomorphism ξ of A_1 onto A_2. The desired extension ψ of ξ will be constructed in the form $\psi = \xi \cup \eta \cup \zeta^{-1}$, where

(1) $\operatorname{dmn} \eta \cup \operatorname{rng} \zeta = Q_1 \sim A_1$ and $\operatorname{dmn} \zeta \cup \operatorname{rng} \eta = Q_2 \sim A_2$.

Let u be the first point of $Q_1 \sim A_1$ with respect to the sequential ordering. Assign u to $\operatorname{dmn} \eta$ and let $\eta(u)$ be the first point of $Q_2 \sim A_2$ such that

$$\rho(\eta(u), \xi\pi_1(u)) < \delta(u, A_1). \tag{2}$$

Let v be the first point of $Q_2 \sim A_2$ unless this point happens to be $\eta(u)$, in which case let v be the second point of $Q_2 \sim A_2$. Assign v to $\operatorname{dmn} \zeta$ and let $\zeta(v)$ be the first point of $Q_1 \sim A_1$ such that $\zeta(v) \neq u$ and

$$\rho(\zeta(v), \xi^{-1}\pi_2(v)) < \delta(v, A_2). \tag{3}$$

Proceed to define η and ζ, step by step, in the following manner: at each odd step, let u be the first point of $Q_1 \sim A_1$ not previously assigned to $\operatorname{dmn} \eta \cup \operatorname{rng} \zeta$, assign u to $\operatorname{dmn} \eta$ and let $\eta(u)$ be the first point of $Q_2 \sim A_2$ which satisfies (2) and has not been previously assigned to $\operatorname{dmn} \zeta \cup \operatorname{rng} \eta$; at each even step, let v be the first point of $Q_2 \sim A_2$ not previously assigned to $\operatorname{dmn} \zeta \cup \operatorname{rng} \eta$, assign v to $\operatorname{dmn} \zeta$ and let $\zeta(v)$ be the first point of $Q_1 \sim A_1$ which satisfies (3) and has not been previously assigned to $\operatorname{dmn} \eta \cup \operatorname{rng} \zeta$. Since $Q_r \sim A_r$ is dense in Q_r, the specified choices can be made. They lead to functions η and ζ for which (1) holds and

$$\operatorname{dmn} \eta \cap \operatorname{rng} \zeta = \emptyset = \operatorname{dmn} \zeta \cap \operatorname{rng} \eta.$$

With $\psi = \xi \cup \eta \cup \zeta^{-1}$, it is plain ψ is a biunique mapping of Q_1 onto Q_2 and we want to show ψ is a homeomorphism. Since ξ is a homeomorphism and Q_1 and Q_2 are compact, it suffices to show that if x_α and y_α are sequences in $Q_1 \sim A_1$ and $Q_2 \sim A_2$ respectively, with

$$x_\alpha \to x \in A_1, \quad y_\alpha \to y \in A_2, \quad \text{and} \quad \psi(x_i) = y_i,$$

then $\psi(x) = y$. By choosing a suitable subsequence if necessary, we may assume x_α is entirely in $\operatorname{dmn} \eta$ or entirely in $\operatorname{rng} \zeta$. For the first case note that $\pi_1(x_\alpha) \to x$, whence $\xi\pi_1(x_\alpha) \to \xi(x)$, and with

$$\rho(y_\alpha, \xi\pi_1(x_\alpha)) < \delta(x_\alpha, A_1) \to 0$$

it follows that

$$y = \lim y_\alpha = \lim \xi\pi_1(x_\alpha) = \xi(x) = \psi(x).$$

In the second case, y_α is entirely in dmn ζ and a similar argument applies.

GEOMETRICAL LEMMA. *If L is a normed linear space, K is a totally bounded subset of L whose closure is convex, and J is a complete convex proper subset of K, then there is a sequence B_α of open balls in $L \sim J$ satisfying the following five conditions*:

(i) lim (*radius of* B_α) $= 0$;
(ii) lim $\delta(B_\alpha, E_J) = 0$;
(iii) *each point of* $E_J \cap \mathrm{cl}\,(K \sim J)$ *is the limit of a subsequence of* B_α;
(iv) *any closed halfspace supporting* J *but not* K *misses some of* B_i*'s*;
(v) *each ball* B_i *is centered at a point of* $K \sim J$.

Proof. For each integer $n \geqq 2$ let

$$X_n = \{x \in K \sim J : 1/n < \delta(x, E_J) \leqq 1/(n-1)\}.$$

Being totally bounded, X_n contains a finite set Y_n such that $\delta(x, Y_n) < 1/n^2$ for each $x \in X_n$. Form the sequence B_α by listing first the balls of radius min $(1, \delta(y_1, J)/2)$ centered at the various points y_1 of Y_1, next the balls of radius min $(\frac{1}{4}, \delta(y_2, J)/2)$ centered at the various points y_2 of $Y_2, \cdots$, then the balls of radius min $(1/n^2, \delta(y_n, J)/2)$ centered at the various points y_n of $Y_n, \cdots$. Conditions (i), (ii), (iii) and (v) are readily verified. For (iv), let us consider an arbitrary closed halfspace W supporting J but not K. Since J is compact, E_J intersects the bounding hyperplane of W, whence there exist $w \in E_J \cap W$, $z \in K \sim W$ and $f \in L^*$ such that

$$\|f\| = 1 \quad \text{and} \quad W = \{v \in L : f(v) \leqq f(w)\}.$$

For each $\lambda \in [0, 1]$ let $z_\lambda = (1-\lambda)w + \lambda z$. Then $z_0 \in E_J$, $z_\lambda \in \mathrm{cl}\, K \sim J$ for all $\lambda \in]0, 1]$, and $\delta(z_\lambda, E_J)$ is a continuous function of λ. It follows that for all sufficiently large n there exists $\lambda(n) \in]0, 1]$ such that $z_{\lambda(n)} \in \mathrm{cl}\, X_n$. Further, there exists $y_n \in Y_n$ with $\| y_n - z_{\lambda(n)} \| \leq 1/n^2$. If b is any point of the open ball B_i centered at y_n and having radius $\leq 1/n^2$, then $\| b - z_{\lambda(n)} \| < 2/n^2$ and consequently, with $\|f\| = 1$,

$$f(b) - f(w) > f(z_{\lambda(n)}) - 2/n^2 - f(w) = \lambda(n) f(z - w) - 2/n^2.$$

But

$$1/n \leq \delta(z_{\lambda(n)}, E_J) \leqq \| z_{\lambda(n)} - w \| = \lambda(n) \| z - w \|,$$

so it follows that $\lambda(n) \geq 1/(n\| z - w \|)$ and

$$f(b) - f(w) > \frac{1}{n}\left(\frac{f(z-w)}{\|z - w\|} - \frac{2}{n}\right).$$

When n is sufficiently large the right-hand expression is positive and then B_i misses W.

COROLLARY. *If J is a compact convex subset of a separable normed linear space L, there is a sequence B_α of balls in $L \sim J$ satisfying the following four* *conditions*:

(i) $\lim$ (*radius of* B_α) $= 0$;

(ii) $\lim \delta(B_\alpha, E_J) = 0$;

(iii$'$) *each point of* E_J *is the limit of a subsequence of* B_α;

(iv$'$) *any closed halfspace supporting* J *misses some of the* B_i*'s.*

Proof. Let A denote the closed affine hull of J. A theorem in [4, p. 460] guarantees the existence of a nonsupport point p of J relative to A, or, equivalently, a point p such that the union of all rays from p through the various points of J is dense in A. We assume without loss of generality that p is the origin 0, whence A is a linear subspace of L and for each $f \in L^*$ it is true that

$$\sup fJ > 0 \quad \text{or} \quad f(a) = 0 \text{ for all } a \in A$$

Let p_α be a sequence of points dense in L and let λ_α be a sequence of positive numbers such that $\lambda_\alpha p_\alpha \to 0$. Finally, let

$$K = \operatorname{con}(2J \cup \{\lambda_1 p_1, \lambda_2 p_2, \cdots\}).$$

Then K is totally bounded by Mazur's Theorem [10], [3, p. 80] and hence by the Geometrical Lemma there is a sequence B_α of balls in $L \sim J$ satisfying conditions (i)–(v). Since $K \supset 2J$ it is then evident that (iii$'$) holds and it remains only to verify (iv$'$). Consider an arbitrary linear form $f \in L^* \sim \{0\}$. From the definition of K it is plain that 0 is a nonsupport point of K relative to L, whence $\sup fK > 0$. But $\sup fK \geq 2 \sup fJ$, so it follows that no closed halfspace supports both J and K and we then see from (iv) that any closed halfspace supporting J misses at least one of the B_i's.

The finite-dimensional case

We shall begin this section by proving assertion (a) from the introduction.

THEOREM. *A compact convex set* K *in* R^d *is a polytope if and only if, for each linear form* f *on* R^d, *the number* $\sup fE_K$ *is an isolated point of the set* fE_K.

Proof. The "only if" part is obvious. For the "if" part when $d = 2$, consider in R^2 a compact convex set K which has infinitely many extreme points. The boundary of K includes an accumulation point y of E_k, and R^2 admits a linear form f, not identically zero, such that $\sup fK = f(y)$. It is easily verified that $y \in E_K$ and $f(y)$ is an accumulation point of the set fE_K.

Now suppose the "if" part is known for $d = j$ (where $j \geq 2$) and consider a compact convex set K in R^{j+1} such that for every linear form f on R^{j+1}, the number $\sup fE_K$ is an isolated point of the set fE_K. We want to show K is a polytope and for this it suffices, in view of a characterization of polytopes in [8, p. 91], to show πK is a polytope whenever π is the orthogonal projection of R^{j+1} onto a j-dimensional linear subspace S of R^{j+1}. Of course πK is a compact convex set, and it is easily verified that $E_{\pi K} \subset \pi E_K$. For an arbitrary linear form g on S, $g\pi$ is a linear form on R^{j+1} and hence, by hypothesis,

$\sup g\pi E_K$ is an isolated point of the set $g\pi E_K$. But $gE_{\pi K} \subset g\pi E_K$ and

$$\sup gE_{\pi K} = \sup g\pi K = \sup g\pi E_K,$$

so $\sup gE_{\pi K}$ is an isolated point of the set $gE_{\pi K}$. Since g is arbitrary, we conclude from the inductive hypothesis that πK is a polytope. Since π is arbitrary, this implies K is a polytope and completes the proof.

R. R. Phelps has produced a second proof of the above theorem, based on the following fact rather than on the characterization of polytopes from [8]; If K is a compact convex set which satisfies condition (I) and H is a supporting hyperplane of K, then the intersection $K \cap H$ also satisfies condition (I).

COROLLARY. *Suppose that Q is a compact metric space and F is a finite-dimensional linear space of continuous real functions on Q. Suppose that for any finite $P \subset Q$ there exists $g \in F$ with* $\sup gQ > \sup gP$. *Then there exists $f_0 \in F$ such that the number* $\sup f_0 Q$ *is an accumulation point of the set $f_0 Q$.*

Proof. Let L denote the conjugate space of F and η the evaluation mapping of Q into L, so that $\eta_q(f) = f(q)$ for all $q \in Q$ and $f \in F$. Since all members of F are continuous, η is continuous and hence the set ηQ is compact. Let $K = \operatorname{con} \eta Q$, a compact convex subset of L. Note that $E_K \subset \eta Q$ and $K = \operatorname{con} E_K$, while the hypothesis about the existence of g implies K is not the convex hull of any finite subset of ηQ. It follows that E_K is infinite, and by the theorem just proved there is a member f_0 of F such that $\sup f_0 E_K$ is an accumulation point of $f_0 E_K$. Then $\sup f_0 Q$ is an accumulation point of $f_0 Q$ and the proof is complete.

In the corollary, F is not required to include the constant functions nor to separate the points of Q. Even if these requirements are added, the corollary may fail when F is infinite-dimensional. Indeed, let L be an infinite-dimensional separable Banach space, q_α a linearly independent sequence of points dense in L, and λ_α a sequence of positive numbers such that $\lambda_\alpha q_\alpha \to 0$. Let $Q = \{0, \lambda_1 q_1, \lambda_2 q_2, \cdots\}$, a compact subset of L, and let F denote the set of all restrictions to Q of the continuous affine forms on L. Then the corollary's conclusion fails, even though its hypotheses are satisfied except for the requirement that F should be finite-dimensional. The closed convex hull K of Q is compact and satisfies condition (I), though E_K is infinite. An elaboration of this example is used, in the second theorem of the next section, to prove the second part of (b).

Extreme points are important not only for compact convex sets, but also for those closed convex sets K which are locally compact and contain no line. Indeed, any such K (in a locally convex space L) is the closed convex hull of its set E_K of extreme points together with the union of its extreme rays [5, p. 237]. Some members of L^* may have infinite supremum on K, or have finite supremum and yet fail to attain a maximum; however, there are also members f of $L^* \sim \{0\}$ which do attain maxima on K, and each such f

attains its K-maximum on E_K [5, pp. 236–237]. This suggests the following weakened form of condition (I):

(II) *For each f in L^* which attains a maximum on K, the number* $\sup fE_K$ *is an isolated point of the set fE_K.*

Plainly (I) and (II) are equivalent when K is compact. When K is a closed convex subset of R^2 containing no line, (II) is equivalent to the finiteness of E_K. For $d \geq 3$, the situation in R^d is described by the following result in conjunction with the first theorem of the next section.

THEOREM. *If Q is a compact metric space in which the isolated points are dense, and if the set A of all accumulation points of Q is homeomorphic with a subset of a $(d-3)$-sphere, then there is a closed convex set K in R^d such that K contains no line,* (II) *holds, and E_K is homeomorphic with Q.*

Proof. Let τ denote the transformation of R^d onto itself carrying each point $x = (x^1, x^2, \cdots, x^d) \in R^d$ onto the point $\tau(x) = (-x^1, x^2, \cdots, x^d)$. Let $\| \;\; \|$ denote the Euclidean norm for R^d and let

$$R^{d-1} = \{x \in R^d : x^d = 0\}, \quad H = \{x \in R^{d-1} : \|x\| = 1,\ x^1 > 0\},$$
$$S^{d-3} = \{x \in R^{d-1} : \|x\| = 1,\ x^1 = 0\}.$$

Then H and τH are opposite hemispheres of the $(d-2)$-sphere

$$S^{d-2} = \{x \in R^{d-1} : \|x\| = 1\},$$

in which their common boundary is the $(d-3)$-sphere S^{d-3}. By hypothesis, there is a homeomorphism ξ of A into S^{d-3}. We will extend ξ to a homeomorphism ψ of Q into $H \cup S^{d-3}$ such that any halfspace in R^{d-1} supporting the set $\psi Q \cup \tau\psi Q$ at a point of ξA is in fact a supporting halfspace of S^{d-2}. Let us assume for the moment that ψ has been constructed, and let C denote the union of all rays in R^d which issue from the origin and intersect the $(d-1)$-ball

$$\{x \in R^d : x^d = -1, \|x - (-1, 0, \cdots, 0, -1)\| \leq 1\}.$$

Then C is a pointed closed convex cone which contains the ray T from the origin through the point $(-1, 0, \cdots, 0, 0)$; C is supported along this ray by a unique closed halfspace in R^d, namely by the halfspace $\{x \in R^d : x^d \leq 0\}$. Let

$$K = C + \operatorname{con} \psi Q$$

Plainly K is convex, and K is closed for C is closed and $\operatorname{con}\psi Q$ is compact. It is easily verified that $K \cap R^{d-1} \subset H + T$, whence it follows that K contains no line and $E_K = \psi Q$. It remains to show that (II) holds and of course to construct the homeomorphism ψ.

Consider an arbitrary linear form f on R^d, not identically zero, such that the K-supremum of f is attained on K and hence at a point of $E_K = \psi Q$. Let G

denote the hyperplane $\{x \in R^d : f(x) = \sup fK\}$. If G misses ξA then $\delta(G, \xi A) > 0$ and $G \cap E_K$ consists of a finite number of isolated points of E_K; then plainly $\sup fE_K$ is an isolated point of fE_K. If G intersects ξA then either $G = R^{d-1}$ or $G \cap R^{d-1}$ is a hyperplane in R^{d-1} which supports S^{d-2} at a point of S^{d-3} and therefore contains a translate of the ray T. This contradicts the special relationship, mentioned above, among C, T, and the halfspace $\{x \in R^d : x^d < 0\}$.

Only the construction of ψ remains. Applying the Geometrical Lemma with the roles of J and K played by the sets con ξA and con S^{d-2} respectively, and noting that $E_{\text{con}\,\xi A} = \xi A$, we see that there is a sequence b_α of points of con $S^{d-2} \sim$ con ξA such that $\delta(b_\alpha, \xi A) \to 0$, each point of ξA is the limit of a subsequence of b_α, and S^{d-2} is supported by every closed halfspace in R^{d-1} which supports con ξA and includes all the b_i's. Since all points of ξA are of unit norm, it is clear the these properties of the sequence b_α are possessed also by the sequence y_α, where $y_i = y_i/\| y_i \|$. Each point y_i is in H or τH, and we define $z_i = y_i$ when $y_i \in H$, $z_i = \tau(y_i)$ when $y_i \in \tau H$. Then every halfspace supporting the set

$$\xi A \cup \{z_1, z_2, \cdots\} \cup \tau\{z_1, z_2, \cdots\}$$

at a point of ξA is a supporting halfspace of S^{d-2}. By the Topological Lemma, ξ can be extended to a homeomorphism ψ of Q onto the set

$$\xi A \cup \{z_1, z_2, \cdots\} \subset H \cup S^{d-3},$$

and this completes the proof.

The normed case

The first part of assertion (b) above is extended by the following result, which involves condition (II) rather than condition (I).

THEOREM. *If L is a normed linear space, K is a locally compact closed convex subset of L containing no line, and* $\sup fE_K$ *is an isolated point of E_K whenever the linear from $f \in L^*$ attains a maximum on K, then the isolated points of E_K are dense in E_K.*

Proof. A point p of K is said to be *exposed* provided that there exists $f \in L^*$ such that the K-supremum of f is attained at p but nowhere else on K. Plainly (II) implies that each exposed point of K is an isolated point of E_K. But for K as described, it follows from [6, p. 91] that the exposed points of K are dense in E_K.

THEOREM. *For each infinite-dimensional Banach space L and each compact metric space Q in which the isolated points are dense, there is a compact convex subset K of L such that* (I) *holds and E_K is homeomorphic with Q.*

Proof. We assume without loss of generality that the Banach space L is separable. Let A_1 denote the set of all accumulation points of the compact metric space Q. Let B and S denote, respectively, the closed unit ball and the unit sphere of the Hilbert space l^2, taken in the weak topology. Thus B is a compact convex set and the set $S = E_B$ is dense in B. For each point $p = (p_2, p_3, \cdots)$ of the parallelotope $P = \times_{i=2}^{\infty} [2^{-i}, 2^{1-i}]$, let

$$\sigma(p) = ((1 - \textstyle\sum_2^{\infty} p_i^2)^{1/2}, p_2, p_3, \cdots) \in S.$$

As is easily verified, σ is a homeomorphism of P onto the set

$$\sigma P = \{(x_1, x_2, \cdots) \in S : x_1 > 0; i \geq 2 \Rightarrow 2^{-(i+1)} \leq x_i \leq 2^{-i}\}.$$

As is well known, every separable metric space can be topologically embedded in P. It follows that there is a homeomorphism τ of A_1 into σP.

We are going to construct a biunique linear transformation φ of l^2 into L such that the restriction of φ to B is continuous (hence a homeomorphism) and the compact convex set φB is not supported by a closed hyperplane at any point of the set $\varphi\sigma P$. Assuming for the moment that φ has been constructed, let us see how to complete the proof. Note that $E_{\varphi B} = \varphi S$, whence it follows (with the aid of Milman's theorem [11], [3, p. 84]) that $E_{\mathrm{cl\ con}\ Z} = Z$ for every compact set $Z \subset \varphi S$. Let $J = \mathrm{cl\ con}\ \varphi\tau A_1$, a compact convex set (by Mazur's theorem) with $E_J = \varphi\tau A_1$. The set $J \cup \varphi S$ is totally bounded and has as its closure the convex set φB. Further, every closed hyperplane supporting J includes a point of $\varphi\sigma P$ and hence does not support φS. Applying the Geometrical Lemma with the role of its K played by the set $J \cup \varphi S$, we obtain a countable subset I_2 of $\varphi S \sim J$ such that E_J is the set of all accumulation points of I_2 and I_2 is not contained in any closed halfspace supporting J. By the Topological Lemma, the homeomorphism $\varphi\tau$ of A_1 onto E_J can be extended to a homeomorphism ψ of Q onto $E_J \cup I_2$. Since ψQ is a compact subset of φS, ψQ is the set of all extreme points of the compact convex set $K = \mathrm{cl\ con}\ \psi Q$. Further, for every $f \in L^* \sim \{0\}$ there exists β_f such that

$$\sup fJ < \beta_f < \sup fI_2,$$

and since the set $\{y \in I_2 : f(y) > \beta_f\}$ is finite it is plain that $\sup fE_K$ is an isolated point of fE_K. Thus K satisfies condition (I) and the proof is complete except for the construction of φ.

By [7, p. 192], the separable Banach space L contains a sequence $u_1, u_2, \cdots$ of points of unit norm such that the linear hull of the u_i's is dense in L and such that $\xi_\alpha = \eta_\alpha$ whenever ξ_α and η_α are sequences of real numbers with $\sum_1^{\infty} \xi_i u_i = \sum_1^{\infty} \eta_i u_i$. In particular, the points u_i are linearly independent. For each point $x = (x_1, x_2, \cdots) \in l^2$ let

$$\varphi(x) = \textstyle\sum_1^{\infty} 4^{-i} x_i u_i \in L,$$

where the series converges because L is complete and $\| x_i u_i \| \leq \| x \|$. Then φ is a biunique linear transformation of l^2 into L and it is easily seen that the restriction of φ to l^2 is continuous. If the set φB is supported by a closed hyperplane at one of its points, that point is of the form $\varphi(y)$ for some $y \in S$

and L admits a continuous linear form f such that $\sup f\varphi B = f\varphi(y)$. But then $f\varphi$ is a continuous linear form on l^2 whose maximum on B is attained at y, whence (using known properties of l^2) there exists $\mu > 0$ such that

$$f\varphi(x) = \sum_1^\infty \mu y_i x_i, \qquad x = (x_1, x_2, \cdots) \in l^2.$$

On the other hand, it is clear that

$$f(\sum_1^\infty \xi_i u_i) = \sum_1^\infty \xi_i f(u_i)$$

whenever the series $\sum \xi_i u_i$ converges, and applying this to the series for $\varphi(x)$ we conclude that

$$f\varphi(x) = \sum_1^\infty 4^{-i} f(u_i) x_i, \qquad x = (x_1, x_2, \cdots) \in l^2.$$

It follows that $\mu y_i = 4^{-i} f(u_i)$. If $y \in \sigma P$ then

$$f(u_i) = \mu 4^i y_i \geq \mu 2^i,$$

contradicting the fact that f is continuous.

(By means of a slightly more careful construction, the theorem can be proved under the assumption that L is an infinite-dimensional F-space.

The general case

For a convex set K in an arbitrary topological linear space, condition (I) implies that every exposed point of K is an isolated point of E_K. Thus there are circumstances other than those of (b) under which (I) implies the existence of isolated extreme points. The most interesting of these is based on a recent theorem of Amir and Lindenstrauss [1], asserting that if L is a Banach space in its weak topology, then every compact convex subset of L is the closed convex hull of its set of exposed points. On the other hand, the following result shows that (I) does not imply the existence of isolated extreme points unless some supplementary condition is added.

THEOREM. *For each totally disconnected compact Hausdorff space Q there is a locally convex space L and a compact convex set K in L such that* (I) *holds and E_K is homeomorphic with Q.*

Proof. Let $\mathfrak{U}$ denote the class of all simultaneously open and closed subsets of Q. Since Q is compact and totally disconnected, every open subset of Q is a union of members of $\mathfrak{U}$. Let C denote the space of all continuous real functions on Q, S the subspace of C consisting of all linear combinations of members of $\mathfrak{U}$, M the space of all signed borel measures on Q, and K the subset of M consisting of all probability measures on Q. For each point q of Q, let μ_q denote the member of K which assigns unit measure to the set $\{q\}$. Define the bilinear form $\langle\ ,\ \rangle$ by setting

$$\langle \gamma, \mu \rangle = \int_{q \in Q} \gamma(q)\, d\mu(q)$$

for all $\gamma \in C$ and $\mu \in M$. Let L and L' denote the space M under its topologies $\sigma(M, S)$ and $\sigma(M, C)$ respectively, and let η denote the identity mapping of L' onto L.

Under the usual identification, L' is the conjugate space C^* in its weak* topology. It follows from a well-known theorem [2, pp. 501–502] that

$$E_K = \{\mu_q : q \in Q\},$$

and that, in the relative topology from L', K is a compact convex set for which E_K is homeomorphic with Q. But then K must have these same properties in the relative topology from L, for η is continuous and L is a Hausdorff space. For every linear form $f \in L^*$ there exist real numbers $\lambda_1, \cdots, \lambda_n$ and members $U_1, \cdots, U_n$ of $\mathcal{U}$ such that

$$f(\mu) = \langle \textstyle\sum_1^n \lambda_1 \chi_{U_i}, \mu \rangle$$

for all $\mu \in M$. Then clearly the set fE_K is finite and (I) holds.

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