ON LIMIT-PRESERVING FUNCTORS

BY

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Following Lambek [2] we shall use the suggestive term "infimum" for the generalized inverse limit of Kan. "Supremum" is defined dually. In [1], the infimum (supremum) is known as a "left root" ("right root"). The terms "inf-complete" and "inf-preserving" are used in the obvious way.

If $\mathfrak{A}$ is a small category then $[\mathfrak{A}, \text{Ens}]$ shall denote the category of all (covariant) functors from $\mathfrak{A}$ to the category Ens of sets. $[\mathfrak{A}, \text{Ens}]_{\text{inf}}$ shall be the full subcategory of inf-preserving functors.

The theorem below answers an open question raised in the introduction to [2]. As Lambek points out this result implies that $[\mathfrak{A}, \text{Ens}]_{\text{inf}}$ is sup-complete and can be regarded as a nicely behaved completion of $\mathfrak{A}^{\circ}$, the dual or opposite category of $\mathfrak{A}$.

Theorem. *Let $\mathfrak{A}$ be a small category. Then $[\mathfrak{A}, \text{Ens}]_{\text{inf}}$ is a reflective subcategory of $[\mathfrak{A}, \text{Ens}]$.*

Notation. In what follows, "Γ" shall always be used to denote a functor whose domain is a small category, I. We shall also always use $A_i = \Gamma(i)$ for $i \in I$.

If $\Gamma : I \to \mathfrak{A}$ has an inf we shall denote it by $(A, u) = \inf \Gamma$ where $u = \{u_i : A \to A_i \mid i \in I\}$ is the required natural transformation from the constant functor to Γ.

If $\Gamma: I \to \text{Ens}$ then $\inf \Gamma = (A, u)$ always exists and we may assume that $A \subseteq \prod A_i$ and that each u_i is the restriction of the projection function $p_i : \prod A_i \to A_i$. It then follows that $x \in A$ iff $x \in \prod A_i$ and $h(p_i(x)) = p_j(x)$ whenever $h \in \Gamma(\text{Hom}\,(i, j))$.

Lemma 1. *Let $G : \mathfrak{A} \to \text{Ens}$ be an inf-preserving functor whose action on morphisms is denoted by $G(f) = \bar{f}$. Let F be a function from the class of objects of $\mathfrak{A}$ to the class of sets. Assume $F(A) \subseteq G(A)$ for all $A \in \mathfrak{A}$. Then F can be regarded, in the natural way, as an inf-preserving functor iff*

(1) *for each morphism $f : B \to A$ it is true that*

$$\bar{f}(F(B)) \subseteq F(A);$$

(2) *whenever $(A, u) = \inf \Gamma$, for $\Gamma : I \to \mathfrak{A}$, then*

$$F(A) \supseteq \bigcap \bar{u}_i^{-1}(F(A_i)).$$

Proof. Clearly (1) is equivalent to the statement that F is functorial in the natural way. Notice that (1) and (2) imply $F(A) = \bigcap \bar{u}_i^{-1}(F(A_i))$. It suffices to show that $\inf\,(F\Gamma) = \bigcap \bar{u}_i^{-1}(F(A_i))$.

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Since G is inf-preserving, one can regard $G(A) = \inf (G\Gamma) \subseteq \prod G(A_i)$. The functions $\{\bar{u}_i\}$ can be regarded as the restrictions of the projection maps $\{p_i\}$. It then follows that $G(A)$ is the set of all $x \in \prod G(A_i)$ for which $h(p_i(x)) = p_j(x)$ for all $h \in \Gamma(\mathrm{Hom}\,(i, j))$.

Similarly $x \in \inf (F\Gamma)$ if $x \in \prod F(A_i)$ and $h(p_i(x)) = p_j(x)$ for all suitable h. It follows that

$$\inf (F\Gamma) = G(A) \cap \prod F(A_i) = \bigcap \bar{u}_i^{-1}(F(A_i)).$$

Important Remark. We shall say that $\Gamma : I \to \mathfrak{A}$ and $\Gamma' : I' \to \mathfrak{A}$ are *similar* if $\inf \Gamma = (A, u)$ and $\inf \Gamma' = (A, u')$ both exist and the *unindexed sets* of morphisms $\{u_i\}$ and $\{u_i'\}$ are the same. Observe that if condition (2) of the above lemma is satisfied for Γ then the condition is also satisfied for all Γ' which are similar to Γ. Moreover, since $\mathfrak{A}$ is a small category, there clearly exists a representative *set* of functors such that whenever $\inf \Gamma$ exists, Γ is similar to a functor in the representative set. From here on, we shall assume that a fixed representative set of this type has been chosen.

Definition. Let G and F be as in the above lemma. In what follows we let Γ vary over the fixed representative set of functors mentioned above. We then define functions $F^{\#}$ and F^* (mapping the objects of $\mathfrak{A}$ into sets) by

$$F^{\#}(A) = \bigcup\{\bar{f}(F(B)) \mid f : B \to A\}$$

$$F^*(A) = \bigcup\{\bigcap \bar{u}_i^{-1}(F(A_i)) \mid (A, u) = \inf \Gamma\}.$$

Moreover, for each ordinal, α, we shall define the function F_α by $F_0 = F$ and

$$F_\alpha = (F_{\alpha-1})^{\#*} \quad \text{if} \quad \alpha - 1 \quad \text{exists}$$

and

$$F_\alpha(A) = \bigcup\{F_\beta(A) \mid \beta < \alpha\} \quad \text{if} \quad \alpha \neq 0 \quad \text{and} \quad \alpha - 1 \quad \text{does not exist.}$$

Lemma 2. *Let F and G be as above. Let m be an infinite cardinal for which*

(1) card $(F(A)) \leq m$ *for all* $A \in \mathfrak{A}$,

(2) *the set of all morphisms of $\mathfrak{A}$ has cardinal less than m,*

(3) *m exceeds the cardinal of the fixed representative set of functors,* $\{\Gamma : I \to \mathfrak{A}\}$,

(4) *whenever $\Gamma : I \to \mathfrak{A}$ is in the fixed representative set then* card $I \leq m$.

It follows that card $(F^{\#}(A)) \leq m$ *and* card $(F^*(A)) \leq m^m$ *for all* $A \in \mathfrak{A}$.

Proof. Straightforward. Notice that $F^*(A) \subseteq \bigcup\{\prod F(A_i)\}$.

Lemma 3. *Let γ be the smallest ordinal whose cardinal exceeds the cardinal of the set of all morphisms of $\mathfrak{A}$. Let G and F be as in Lemma 1. Then F_γ is the smallest inf-preserving subfunctor of G for which $F(A) \subseteq F_\gamma(A) \subseteq G(A)$ for all $A \in \mathfrak{A}$.*

Proof. It clearly suffices to show that F_γ satisfies the conditions of Lemma 1. To verify (1), let $f : B \to A$ be given and let $x \in F_\gamma(B)$. Then $x \in F_\beta(B)$

for some $\beta > \gamma$ and so

$$\bar{f}(x) \in F_{\beta+1}(A) \subseteq F_\gamma(A).$$

As for (2), let $(A, u) = \inf \Gamma$ and let $x \in \bigcap \bar{u}_i^{-1}(F_\gamma(A_i))$. Then for each i, there exists $\beta_i < \gamma$ such that $\bar{u}_i(x) \in F_{\beta_i}(A_i)$. Moreover, we can choose $\beta_i = \beta_j$ if $u_i = u_j$. Hence the set of distinct β_i's has no more elements than the set of morphisms of $\mathfrak{A}$. Clearly there exists $\beta < \gamma$ such that $\beta_i < \beta$ for all i. It follows that

$$x \in \bigcap_i \bar{u}_i^{-1}(F_\beta(A_i)) \subseteq F_{\beta+1}(A) \subseteq F_\gamma(A).$$

DEFINITION. Let F and G be as in Lemma 1. For convenience we shall use "$\bar{F}$" to denote the smallest inf-preserving functor "between F and G" (i.e. $\bar{F} = F_\gamma$).

More generally, let $\eta : E \to G$ be a natural transformation for which $G \in [\mathfrak{A}, \text{Ens}]_{\text{inf}}$. We shall then use "$\bar{E}$" to denote the smallest inf-preserving subfunctor of G through which η factors. Clearly $\bar{E} = F_\gamma$ where $F(A)$ is the set-theoretic range of $\eta(A)$.

We define $\eta : E \to G$ to be *dense* if $G \in [\mathfrak{A}, \text{Ens}]_{\text{inf}}$ and $\bar{E} = G$. Observe that every $\eta : E \to G$ factors through a dense transformation (*viz.* $E \to \bar{E} \to G$), if $G \in [\mathfrak{A}, \text{Ens}]_{\text{inf}}$.

LEMMA 4. *Let $\eta : E \to G$ and $\lambda, \mu : G \to H$ be natural transformations where G and H are inf-preserving. If η is dense then $\lambda\eta = \mu\eta$ implies $\lambda = \mu$.*

Proof. Let $\sigma : F \to G$ be the difference kernel (or equalizer) of λ and μ in the category $[\mathfrak{A}, \text{Ens}]$ (see [2, p. 8] for the existence of σ). It follows from the construction of difference kernels that F may be regarded as a subfunctor of G and that η factors through F. Moreover F is inf-preserving in view of [2, pp. 19–21]. But η is dense, hence $F = G$ and so $\lambda = \mu$.

Proof of the theorem. Let $E \in [\mathfrak{A}, \text{Ens}]$ be given. Let $\{\eta_i : E \to G_i\}$ be a representative class of dense transformations such that every other dense transformation from E is equivalent to exactly one η_i. By applying Lemma 2, one can obtain an upper bound for card $G_i(A)$ which is independent of i and A. This implies that the class $\{\eta_i : E \to G_i\}$ is a set.

Let $\eta : E \to \prod G_i$ be determined by $p_i\,\eta = \eta_i$ for all i, where $p_i : \prod G_i \to G_i$ is a projection transformation. In view of [2, pp. 19–21], we see that $\prod G_i \in [\mathfrak{A}, \text{Ens}]_{\text{inf}}$. We shall factor η through a dense transformation, $\bar{\eta} : E \to \bar{E}$ composed with $\mu : \bar{E} \to \prod G_i$ which injects $\bar{E}$ as a subfunctor of $\prod G_i$.

We claim that $\bar{\eta} : E \to \bar{E}$ reflects E into $[\mathfrak{A}, \text{Ens}]_{\text{inf}}$. For if $\lambda : E \to H$ is given with $H \in [\mathfrak{A}, \text{Ens}]_{\text{inf}}$, we can factor λ through a dense transformation. Since $\{\eta_i : E \to G_i\}$ is representative we can assume $\lambda = \theta\eta_i$ for suitable i and θ. This implies $\lambda = (\theta p_i\,\mu)\bar{\eta}$. Moreover, $(\theta p_i\,\mu)$ is uniquely determined in view of Lemma 4.

References

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