RESTRICTED GORENSTEIN RINGS

BY

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Recently, Faith [4] and Levy [7] studied restricted quasi-Frobenius rings. In this note we discuss some generalizations.

We start by observing that R is a restricted quasi-Frobenius ring iff R is a restricted Gorenstein ring. Furthermore, R is restricted Gorenstein ring, and R is a Gorenstein ring iff R/M^2 is a Gorenstein ring for every maximal ideal M in R.

We define a sequence of classes of rings $G_0, G_1 \cdots$ by $G_i = \{R \mid R/M^2$ is a Gorenstein ring whenever M is a prime ideal and ht $M \geq i\}$. It turns out that R is a G_1-ring iff R is a direct sum of ideals $R_1 \cdots R_t$, and R_i is an Artinian ring, or, a Dedekind domain for every i, $i = 1, \cdots, t$. Also, R is a G_i-ring for $i \geq 2$ iff Krull-dim $R < i$.

We also study the classes $G^{j+1} = \{R \mid R/I$ is a Gorenstein ring whenever I contains a prime ideal M such that ht $M \geq j\}$. It turns out that G^i contains all rings of Krull dimension less than i. Rings of Krull dimension i that have finite global dimension are G^i-rings. In G^1 there are rings of Krull dimension one, the global dimension of which is not finite.

0. Preliminaries

All rings are presumed to be commutative rings with an identity.

For a prime ideal M in R we denote by R_M the local ring of R at M. We set ht M = Krull-dim R_M.

A ring R is a Gorenstein (quasi-Frobenius) ring if R is a Noetherian ring, and inj $\dim_R R < \infty$ (inj $\dim_R R = 0$).

A ring R is a restricted Gorenstein (quasi-Frobenius) ring if R/I is a Gorenstein (quasi-Frobenius) ring whenever I is a non-zero ideal.

Let F be a field and A an F vector space. By $\dim_F A$ we denote the (vector space) dimension of A over F.

By Spec R we denote the variety associated to R by taking the Zariski topology on the set of prime ideals of R.

We quote some useful facts.

A. Let R be an Artinian local ring, with radical M, and set $k = R/M$. Then R is a Gorenstein ring iff R is a quasi-Frobenius ring, and this is so iff $\dim_k \mathrm{Hom}_R(k, R) = 1$ (cf. [1]).

B. Let R be a Noetherian ring, and M a prime ideal. The kernel of the canonical map $R \to R_M$ is the intersection of the primary components of (0) which are disjoint from $R - M$, e.g. [9, Theorem 18, p. 225].

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C. Let M be a prime ideal in a Noetherian ring R, that is generated by r elements, then $\operatorname{ht} M \leq r$ [8, p. 26].

D. Let R be a Noetherian ring of finite Krull dimension. If R is a Gorenstein ring, then

$$\text{inj dim}_R R = \text{Krull-dim } R = \text{Sup}_M \text{ inj dim}_{R_M} R_M ,$$

and

$$\text{gl dim } R = \text{Sup}_M \text{ gl dim } R_M ,$$

where M ranges over all maximal ideals of R.

E. Let R be a Noetherian domain. If $\dim_{R/M} M/M^2 = 1$, for every maximal ideal M in R, then R is a Dedekind domain [2].

A ring R is indecomposable if for every two non-zero ideals, M, N such that $R = M + N$, the intersection $M \cap N$ is non-zero. Notice that R is indecomposable iff 0 and 1 are the unique idempotents in R.

F. Let R be a Noetherian ring without nilpotent elements, then R_M contains no nilpotent elements for every prime ideal M in R.

It might be helpfull to think of $\dim_{R/M} M/M^2$ for a maximal ideal M in R, as the "dimension of the tangent space to Spec R at M". The condition $M \supsetneq N$ for a prime ideal N in R can then be viewed as "M is not an isolated point on Spec R" or "M lies on a subvariety of Spec R, of dimension at least one". Finally, R_M being a regular local ring should be understood as "M is a non-singular point of Spec R".

1. G_0-rings

A ring S is a G_0-ring if for every prime ideal M, S/M^2 is a Gorenstein ring.

If S is an Artinian G_0-ring, then it readily follows that S is a direct sum of ideals $S_1 \cdots S_t$, where S_i is a local Artinian, uniserial ring for $i = 1, \cdots, t$ [cf. 7]. Furthermore, one easily verifies that such a ring S is necessarily a G_0-ring. This completes the study of Artinian G_0-rings.

Let S be a Noetherian ring, and let N be its nilpotent radical. Set $R = S/N$. Since idempotents from R can be lifted to S, it follows that R is a direct sum of ideals $R_1 \cdots R_t$ iff S is a direct sum of ideals $S_1 \cdots S_t$, and $R_i = S_i/N \cap S_i$ for $i = 1, \cdots, t$. In particular S is indecomposable iff R is indecomposable.

Let M' be a maximal ideal in S, and $M = M'/N$ the corresponding maximal ideal in R.

Lemma 1. *If* $\dim_{S/M'} M'/M'^2 = 1$ *then* R_M *is a regular local ring of dimension less than or equal to one.*

Proof. If $M'^2 \not\supset N$, then from $\dim_{S/M'} M'/M'^2 = 1$ it follows that $M' = M'^2 + N$. Since $M^2 = M'^2 + N/N = M'/N$, it follows that in R_M, $MR_M = (MR_M)^2$. Since MR_M is the Jacobson radical of R_M,

$$\bigcap_{n=1}^{\infty} (MR_M)^n = (0),$$

therefore $MR_M = (0)$ and R_M is a field.

Otherwise $M'^2 \supset N$ and thus $\dim_{R/M} M/M^2 = \dim_{S/M'} M'/M'^2 = 1$. Hence there exists an element m such that $M = Rm + M^2$. Therefore in R_M we have $MR_M = mR_M + M^2R_M = mR_M + (MR_M)^2$, and by the lemma of Nakayama we conclude that $MR_M = mR_M$. Therefore, Krull-dim $R_M \leq 1$. If R_M is Artinian, then since R contains no nilpotent elements it follows that R_M is a field. Otherwise Krull-dim $R_M = 1$, and since $MR_M = mR_M$, it results that R_M is a regular local ring, of dimension one. This completes the proof of the lemma.

PROPOSITION 2. *Let S be a Noetherian indecomposable ring. If*

$$\dim_{S/M'} M'/M'^2 = 1$$

for every maximal ideal M' in S, and if S is not an Artinian ring, then S is a Dedekind domain.

Proof. By Lemma 1, R_M is a regular local ring of dimension at most 1 for every maximal ideal M in R. Thus gl dim $R \leq 1$. Since R is not an Artinian ring it follows that gl dim $R = 1$. But an indecomposable ring of global dimension one is a domain, since for every element $r \neq 0$ in R, Rr is a projective ideal, hence the exact sequence $0 \to \text{ann}(r) \to R \to Rr \to 0$ splits, thus $\text{ann}(r) = (0)$. Therefore R is a Dedekind domain. This implies that N is a prime ideal in S. Since S is not an Artinian ring, then for every maximal ideal M' of S, it follows that M' contains N. The condition $\dim_{S/M'} M'/M'^2 = 1$ implies the existence of an element m in M' such that $M' = Sm + M'^2$. Localizing at M' it now readily follows that $M'S_{M'} = mS_{M'}$, i.e. $S_{M'}$ is a principal ideal ring. Since $M' \supsetneq N$, it results that Krull-dim $S_{M'} \geq 1$. Hence $S_{M'}$ is a regular local ring. In particular $S_{M'}$ is a domain. Since (0) in S is a primary ideal, the canonical map $S \to S_{M'}$ is an embedding. In particular this implies $N = 0$. Therefore $S = R$ is a Dedekind domain.

The assumption $\dim_{S/M'} M'/M'^2 = 1$ holds whenever S/M'^2 is a quasi-Frobenius ring. Since we consider only maximal ideals, S/M'^2 is a quasi-Frobenius ring iff S/M'^2 is a Gorenstein ring.

Remark that if S is a Dedekind domain then all its proper residue rings are quasi-Frobenius rings [7].

Finally, notice that if Krull-dim $R = 1$, then for every maximal ideal M, $M^2 \neq 0$.

We therefore proved

THEOREM 3. *Let R be an indecomposable Noetherian ring of Krull dimension one. Then the following are equivalent:*

(i) *R is a Dedekind domain.*

(ii) *All proper residue rings of R are quasi-Frobenius (Gorenstein).*

(iii) *R/M^2 is a quasi-Frobenius (Gorenstein) ring for every maximal ideal M in R.*

LEMMA 4. *Let R be a Noetherian ring that is a direct sum of ideals $R_1 \neq 0$ and $R_2 \neq 0$. Then R is a G_0-ring iff R_1 and R_2 are both G_0-rings.*

Proof. The proof is an immediate consequence of the fact, that for each prime ideal P in R either $P \cap R_1 = R_1$ or else $P \cap R_2 = R_2$.

We thus obtain the following combining Theorem 3, Lemma 4, and the Artinian case.

THEOREM 5. *Let R be a Noetherian ring, then the following are equivalent:*

(i) *R is a direct sum of ideals $R_1 \cdots R_t$. Each R_i is either an Artinian, uniserial, local ring, or a Dedekind domain.*

(ii) *R and all its proper residue rings are Gorenstein rings.*

(iii) *R/M^2 is a quasi-Frobenius (Gorenstein) ring for every maximal ideal M in R.*

Under each of these equivalent conditions, inj $\dim_{R/I} R/I \leq 1$ for every ideal I in R, and gl dim $R/M \leq 1$ for every prime ideal M in R.

If R is not an indecomposable ring, then one easily verifies that condition (ii) is equivalent with

(ii)* All proper residue rings of R are Gorenstein rings.

Theorem 5 can be viewed as the characterization of G_0-rings. Another characterization may be obtained from

PROPOSITION 6. *Let R be a Noetherian ring and M a maximal ideal; then R/M^2 is a quasi-Frobenius ring* iff *R_M is a principal ideal ring.*

Proof. If R/M^2 is a quasi-Frobenius ring then $\dim_{R/M} M/M^2 = 1$. This implies the existence of an element m in M for which $M = Rm + M^2$. Then localizing at M this implies $MR_M = mR_M$, therefore R_M is a principal ideal ring.

Conversely, R_M being a principal ideal ring is either a uniserial Artinian ring or else a Dedekind domain. In any event R_M/M^2R_M is a quasi-Frobenius ring. Since $R/M^2 = (R/M^2)_{(M/M^2)} = R_M/M^2R_M$ it follows that R/M^2 is a quasi-Frobenius ring.

2. G_1-rings

Let S be a Noetherian ring, N its nilpotent radical and set $R = S/N$. Throughout this section we assume that R is an indecomposable, non-Antinian ring, unless otherwise specified.

We recall that R is a G_1-ring if R/M^2 is a Gorenstein ring whenever M is a non-minimal prime. Then if R is an Artinian ring then obviously R is a G_1-ring.

Observe that if R is a domain then R is a G_1-ring iff R is a G_0-ring; therefore we have

PROPOSITION 7. *Let R be a G_1-domain; then R is a Dedekind domain.*

Our first aim is to study the ring R instead of studying the ring S; we need

LEMMA 8. *If S is a G_1-ring then so is every residue ring of S.*

Proof. Let I be any ideal in S, and consider the ring S/I. If S/I is an Artinian ring, then obviously we are done. Otherwise, there exists prime ideals M_1, M_2 in S such that $I \subset M_1 \subset_{\neq} M_2$. Since S is a G_1-ring we have that S/M_2^2 is a Gorenstein ring. Assume furthermore that M_2 is a maximal ideal in S. This implies by Proposition 6 that S_{M_2} is a principal ideal ring. Since $M_1 \subset_{\neq} M_2$ it turns out that Krull-dim $S_{M_2} \geq 1$, therefore S_{M_2} is a regular local ring. In particular $M_1 S_{M_2} = 0$. It therefore follows that Krull-dim $S \leq 1$. In particular Krull-dim $S/I \leq 1$, therefore it suffices to prove that for every non-minimal prime M' in S/I, we have that $(S/I)/M'^2$ is a quasi-Froebenius ring. But from $I \subset M_1$ we now obtain

$$IS_{M_2} \subset M_1 S_{M_2} = 0 \quad \text{and} \quad (S/I)_{(M_2/I)} = S_{M_2}/IS_{M_2},$$

i.e. $(S/I)_{(M_2/I)}$ is a regular local ring. Therefore by proposition 6 we have that $(S/I)/(M_2/I)^2$ is a quasi-Frobenius ring. This proves that S/I is a G_1-ring

In particular if S is a G_1-ring then R is a G_1-ring. We quote the following as a corollary.

COROLLARY 9. *If S is a G_1-ring then* Krull-dim $S \leq 1$.

This was proved while proving Lemma 8.

Since R is presumed to be not an Artinian ring we will restrict ourselves to the case Krull-dim $R = 1$. In R there are no nilpotent elements, thus there are no nilpotent elements in R_M. Hence if Krull-dim $R_M = 0$, R_M is a field. If Krull-dim $R_M \neq 0$ then necessarily Krull-dim $R_M = 1$. This implies that M is not a minimal prime. Therefore, if R is a G_1-ring, R/M^2 is a quasi-Frobenius ring hence R_M is a regular local ring.

We therefore proved that R_M is a regular local ring for every maximal ideal in R if R is a G_1-ring. Since R is assumed to be indecomposable, from gl dim $R = \operatorname{Sup}_M$ gl dim $R_M = 1$—where M ranges over all maximal ideals M in R—it follows that R is a Dedekind domain.

Thus, if S is a G_1-ring, we have by Lemma 8 that R is a G_1-ring, hence R is a Dedekind domain. It follows that N is a prime ideal in S. Since Krull-dim $S = 1$, we have that for every maximal ideal M in S, $M \supset_{\neq} N$. Hence S/M^2 is a quasi-Frobenius ring. By Proposition 6, S_M is a principal ideal ring. $M \supset_{\neq} N$ implies that Krull-dim $S_M \geq 1$, therefore S_M is a regular local ring. The canonical map $S \to S_M$ is thus an embedding of S in a domain, therefore $N = 0$, i.e. $R = S/N = S$ is a Dedekind domain.

Finally, notice that if R is the direct sum of ideals $R_1 \neq 0$ and $R_2 \neq 0$ then R is a G_1-ring iff R_1 and R_2 are G_1-rings.

We therefore established the following:

THEOREM 10. *Let R be a Noetherian ring. Then the following are equivalent*:

(i) *R is a G_1-ring.*

(ii) *R is a direct sum of ideals $R_1 \cdots R_t$. For every i, $1 \leq i \leq t$, R_i is either an Artinian ring or else a Dedekind domain.*

(iii) *For every proper ideal I of R that properly contains a prime ideal N of R, R/I^2 is a Gorenstein ring.*

(iv) *For every maximal ideal M of R, if M is not a minimal prime then R/M^2 is a quasi-Frobenius ring.*

Under each of these equivalent conditions R/I is a quasi-Frobenius ring whenever I is a proper ideal that properly contains a prime ideal N. Furthermore, for any prime ideal N in R, $\operatorname{gl\,dim} R/N \leq 1$.

3. G_i-rings

A similar treatment to the one used above will lead to the following conclusion: If M is a maximal ideal and R/M^2 is a quasi-Frobenius ring then $\operatorname{ht} M \leq 1$. Therefore we have

THEOREM 11. *Let R be a Noetherian ring. Then R is a G_i-ring $(i > 2)$ iff* Krull-dim $R < i$.

Some further properties that can be easily derived are that if R is a G_i-ring then so are all its residue rings. Conversely, if all residue rings of R are G_i-rings then R is a G_i-ring if R is not a domain. If R is a domain then R is a G_{i+1}-ring.

If I is an ideal that contains a prime ideal M in a G_i-ring R, then R/I is a $G_{i-\operatorname{ht} M}$-ring.

4. G^i-rings

We start with G^1-rings. Recall that R is a G^1-ring if R/I is a Gorenstein ring whenever I contains a prime ideal. This readily implies that if R is a G^1-domain, then R is a Dedekind domain. Futhermore, if N is a prime ideal then R/N is a Dedekind domain. This results since if N is non-maximal, and M is any maximal ideal containing N, then $(M/N)^2 = M^2 + N/N \neq M/N$; thus $R/M^2 + N$ is a quasi-Frobenius ring. In particular this implies that Krull-dim $R \leq 1$.

Since every Artinian ring is obviously a G^1-ring, and since if R is a direct sum of ideals $R_1 \cdots R_t$ then R is a G^1-ring iff R_i is a G^1-ring for $i = 1, \cdots, t$, we will restrict ourselves to indecomposable rings of Krull dimension 1. Furthermore, if S is a G^1-ring, and N its nilpotent radical, then $R = S/N$ is again a G^1-ring. Then R is an indecomposable G^1-ring without nilpotent elements.

If M is a maximal ideal in R and M is a minimal prime, then R_M is necessarily a field. If M is a maximal ideal that properly contains a prime ideal M_1, then R_M/M_1R_M is a regular local ring of dimension one and R_{M_1} is a field. This implies the existence of an element m in M such that $MR_M = mR_M$

$+ M_1 R_M$. Since $R_{M_1} = (R_M)_{M_1}$ is a field it follows that the kernel of the canonical map $R_M \to R_{M_1}$ is $M_1 R_M$. Unless $M_1 R_M = (0)$, this implies that $(0) = M_1 R_M \cap \cdots \cap M_t R_M$, where $M_i R_M$ are prime ideals for every $i, i = 1, \cdots, t$ and $M_i R_M \neq MR_M$.

Set $N_1 = M_2 R_M \cap \cdots \cap M_t R_M$, and set $M' = M_1 R_M + N_1$. Since $R_M/_{M_1 R_M}$ is a regular local ring, and since M' is a direct sum, it follows that for some integer $j, j > 0, m^j \in N_1$. Since N_1 is the intersection of prime ideals, this immediately implies that $m \in N_1$. Hence $M' = MR_M$, and $N_1 = mR_M$. One may now proceed by induction to prove that MR_M is the direct sum of the cyclic ideals N_k for $k = 1, \cdots, t$ where N_k is the intersection of $M_i R_M$ for $i \neq k$.

Conversely, if for every maximal ideal M in R, MR_M is a direct sum of cyclic ideals then R is a G^1-ring. We will be done if we can prove that for any prime ideal K in R, R_M/KR_M is a uniserial, Artinian ring, or a Dedekind domain, whenever $K \subset M$. But if K is a prime ideal then $KR_M \subset MR_M$ is a prime ideal. Therefore, if $MR_M = m_1 R_M + \cdots + m_2 R_M$ (direct sum), then KR_M contains $m_1 \cdots m_r$ except, maybe, for one i, say $m_i \notin KR_M$. Thus R_M/KR_M is a residue ring of $R_M/_{(m_1 \cdots m_{i-1}, m_{i+1} \cdots m_r)}$ that is a regular local ring of dimension one, since m_i is not nilpotent modulo $(m_1 \cdots m_{i-1}, m_{i+1} \cdots m_r)$.

Since in S, every prime ideal contains N, the above result means that S is a G^1-ring iff for every maximal ideal M in S, MS_M/NS_M is a direct sum of cyclic ideals.

The Artinian rings and the Dedekind domains are trivial examples of G^1-rings.

Another example results by taking a ring S with non-zero nilpotent radical N, so that S/N is a Dedekind domain. Such a ring may be obtained as a residue ring of a domain A of global dimension 2, by a primary ideal.

As for G^i-ring for $i \geq 2$, one easily verifies that if R is a G^i-ring then so are all its residue rings. Conversely, if all proper residue rings of R are G^i-rings then R is a G^{i+1}-ring. It will be a G^i-ring if R is not a domain.

Also if R is a direct sum of ideals $R_1 \cdots R_t$, then R is a G^i-ring iff R_j are G^i-ring for $j = 1, \cdots, t$.

Finally if R is a G^i-ring then, by arguments similar to the one used above it will result that Krull-dim $R \leq i - 1$ or else Krull-dim $R = i$ and R/M is a Dedekind domain for every prime ideal M in R for which ht $M = i - 1$. The converse may be verified easily. Remark that these conditions are satisfied by rings of finite global dimension, and of Krull-dim $R = i$. At least for $i = 1$ we had G^1-rings of Krull dimension one having infinite global dimension.

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