NORMAL FIBRATIONS FOR COMPLEXES[1]

BY

NORMAN LEVITT

0. Introduction

Suppose a smooth manifold N is embedded in another M. Then there is the familiar notion of the normal bundle ξ of the embedding. However, one can regard this bundle as a spherical fibration. And it is easily seen that if T is a tubular neighborhood of N, then if we replace the map $\partial T \subseteq T$ by a fibration ξ', (which we view as a fibration over N), ξ' is fiber-homotopically equivalent to ξ. If one forgets about the smoothness (or PL) structure of N, and demands that it merely be an embedded Poincaré duality complex, then the same construction, according to Spivak, still yields a spherical fibration of the appropriate dimension, although it may no longer be fiber-homotopically equivalent to a bundle. (Here, we replace "tubular neighborhood" by "regular neighborhood"). In fact, we do not need to have an actual geometric embedding of N in M; it will suffice that a "thickening" $\bar{N}$ of N is a codimension 0 submanifold of M.

In this paper, we altogether abandon any conditions on N other than that it be of the homotopy type of a finite complex. One can then perform the same sort of construction, i.e. take a thickening $\bar{N}$ and look at the result of replacing $\partial\bar{N} \subseteq \bar{N}$ by a fibration ν.

What makes this interesting is that the stable homotopy type of the fiber of this fibration depends only on N and not at all on the thickening, thus generalizing the situation for Poincaré complexes. Moreover, if one suspends a thickening, then the fibration associated with the suspension is the suspension of the fibration associated with the original thickening. One can then ask questions about the fibrations to derive information about the thickening.

In particular, one shows that desuspending the fibration is roughly equivalent to desuspending the thickening. In the metastable range, in fact, one can restrict one's attention to the question of whether sections exist. This leads, among other things, to a generalization of Hirsch's compression theorem [l] in the metastable range. These results are exposited in the first three sections, along with some preliminary remarks on fibrations.

Sections 4, 5, and 6 utilize the notion of normal fibration of a thickening to get various results about Poincaré-duality spaces, among other things. It is shown, for instance, that one can talk about thickenings of a complex into Poincaré-duality pairs, and that the resulting theory greatly resembles that exposited by Wall [e] in the smooth and PL case. We also give a new proof

Received May 1, 1968.

[1] Results obtained at the Courant Institute of Mathematical Sciences, New York University, under a Ford Foundation grant.

of a theorem of Browder [n] on finite H-spaces. Finally, we give sufficient conditions, in terms of a "desuspended normal invariant" for embeddings up to homotopy type of complexes into spheres.

The tools chiefly used in this paper are the Stalling's embedding theorem [d] (both existence and uniqueness aspects) and the Browder codimension 1 theorem [c].

1. Preliminaries on fibrations

Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. Recall that the mapping cylinder M_f is defined as

$$(X \times I) \cup Y/(x, 1) = f(x)$$

with the identification topology. For our purposes however, we prefer to have a notion of mapping cylinder which involves the same point set, but with a slightly smaller topology. We define this by

$$\mathfrak{M}_f = (X \times I) \cup Y/(x, 1) = f(x)$$

with the topology defined by the following neighborhood basis: Let

$$p : (X \times I \cup Y) \to \mathfrak{M}_f$$

be projection. If $a \in \mathfrak{M}_f$ is in the image of $X \times [0, 1)$ under p, then a neighborhood of a is $p(U)$, where U is a neighborhood of $p^{-1}a \in X \times [0, 1)$ in $X \times [0, 1)$.

If $a \in p(Y)$ then a neighborhood of a is given by

$$p(U \cup f^{-1}(U) \times (1 - \varepsilon, 1])$$

where U is a neighborhood of the unique $y \in Y$ such that $py = a$, and $\varepsilon > 0$.

In this topology, there is a natural continuous map $i : M_f \to \mathfrak{M}_f$ which is given by the identity map on the underlying point set. If we regard X as the "top" of both $\mathfrak{M}_f$ and M_f, i.e. as $p(X \times \{0\})$, then we have

1.1. Lemma. $i : (M_f ; X) \to (\mathfrak{M}_f, X)$ is a homotopy equivalence of pairs.

Proof. The homotopy inverse is given by the map $j : \mathfrak{M}_f \to M_f$, which is

$$j(p(x, t)) = p(x, \min\ (1, 2t))$$

for $x \in X$, $t \in I$,

$$j(p(y)) = p(y)$$

for $y \in Y$.

It is easy to check that ji and ij are homotopic to the appropriate identity maps, Q.E.D.

The reason for introducing $\mathfrak{M}_f$ in place of M_f lies in the following easily verified fact.

1.2 Lemma. *Let* $\{(x_i, t_i)\}$, $x_i \in X$, $t_i \in I$ *be a set of points indexed by the di-*

rected set $\mathcal{J}$, so that $t_i \to 1, f(x_i) \to y, y \in Y$. Then

$$p(x_i, t_i) \to p(y) \in \mathfrak{M}_f.$$

We now identify Y with $p(Y) \subseteq \mathfrak{M}_f$. Now note that the obvious map $\pi_f : \mathfrak{M}_f \to Y$ given by

$$\pi_f(p(x, t)) = f(x), \quad (x, t) \in X \times I$$

$$\pi_f(y) = y, \quad y \in Y$$

is continuous.

Recall that if $\delta : V \to Y$ is a map of spaces and $W \subseteq V$, we say that (V, W, δ) is a pair fibration iff, given a space Z, a commutative diagram

$$\begin{tikzcd} & V \arrow[d, "\delta"] \\ Z \arrow[ur, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

and a homotopy $F : Z \times I \to Y, F_0 = f$, we can find a homotopy

$$G : Z \times I \to V, \quad G_0 = g \quad \text{and} \quad G(g^{-1}(W)) \subseteq W.$$

Thus $\sigma = \delta \mid W$ is a fiber map. 1.2 enables us to show the following, where it is understood that $X \times [0, 1)$ is identified with $p(X \times [0, 1))$, X with $X \times \{0\}$ $(= X)$, Y with $p(Y)$.

1.3 PROPOSITION. *Let $\sigma : X \to Y$ be a fiber map, and let $\pi_\sigma : \mathfrak{M}_\sigma \to Y$. Then $(\mathfrak{M}_\sigma, X, \pi_\sigma)$ is a pair fibration.*

Proof. Let

$$\begin{tikzcd} & \mathfrak{M}_\sigma \arrow[d, "\pi_\sigma"] \\ Z \arrow[ur, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

be commutative, and let $F : Z \times I \to Y, F_0 = f$. We will define the covering homotopy $G : Z \times I \to \mathfrak{M}_f$. First, define a homotopy $\tilde{G} : Z_1 \times I \to X$ where $Z_1 = g^{-1}(Z \times [0, 1)) \subseteq Z$, by noting that, if $\alpha : X \times [0, 1) \to X$ is the projection then $\sigma\alpha g \mid Z_1 = f \mid Z_1$; thus there is a covering homotopy

$$\tilde{G} : Z_1 \times I \to X$$

so that $\sigma\tilde{G} = F \mid Z_1$. We now define

$$G(z, t) = (\tilde{G}(z, t), s)$$

if $z \in Z_1$ and $g(z) = (x, s) \in X \times [0, 1)$,

$$G(z, t) = F(z, t)$$

if $g(z) \in Y$.

Obviously $\pi_\sigma G = F$, and we have to show that G is continuous. We show in fact that if $(z_i, t_i) \to (z, t)$, where $i \in \mathcal{I}$, $\mathcal{I}$ a directed set, then $G(z_i, t_i) \to G(z, t)$. First note that this is trivial if $G(z, t) \in X \times [0, 1)$. So let $G(z, t) = y \in Y$. It is then easy to find $(x_i, s_i) \in X \times I$ so that $p(x_i, s_i) = G(z_i, t_i)$. Clearly $\sigma(x_i) \to G(z, t) = F(z, t)$ because $\pi_\sigma G = F$. Moreover $s_i \to 1$. So by 1.2,

$$G(z_i, t_i) = p(x_i, s_i) \to G(z, t), \qquad \text{Q.E.D.}$$

Remark. If the spaces X, Y are "reasonable", e.g. X compact, Y Hausdorff, M_f is the same as $\mathfrak{M}_f$. However, in the subsequent chapters, we shall be looking at fibrations got by using the path-space construction to replace an inclusion $X \subseteq Y$ by a fibration $\sigma : X' \to Y$ where X, Y are finite complexes. Since X' may not be nice topologically we use $\mathfrak{M}_f$, rather than M_f. I would like to thank V. Sapojnikov for pointing out to me the need for such a technicality, which is sometimes ignored in the literature.

Now let $\sigma_1 : E_1 \to Y$, $\sigma_2 : E_2 \to Y$ be two fibrations over Y with fibers F_1, F_2. We use the notation $\delta_i = \pi_{\sigma_i} : \mathfrak{M}_{\sigma_i} \to Y$, $i = 1, 2$. Let

$$\mathfrak{M} = \{(v_1, v_2) \in \mathfrak{M}_{\sigma_1} \times \mathfrak{M}_{\sigma_2} \mid \delta_1 v_1 = \delta_2 v_2\}.$$

There is, of course, an obvious map $\delta : \mathfrak{M} \to Y$ given by

$$\delta(v_1, v_2) = \delta_1 v_1 = \delta_2 v_2, \quad (v_1, v_2) \in \mathfrak{M}.$$

Let $E = \{(v_1, v_2) \in \mathfrak{M} \mid v_1 \in E_1 \text{ or } v_2 \in E_2\}$.

1.4 DEFINITION. The *Whitney sum* $\sigma = \sigma_1 + \sigma_2$ is the map

$$\sigma : E \to Y,$$

where $\sigma = \delta \mid E$.

1.5 PROPOSITION. *σ is a fibration with fiber $F = F_1 * F_2$ (the join of F_1 and F_2).*

This is easily proved, using the fact that $(\mathfrak{M}_{\sigma_i}, E_i, \delta_i)$ are pair fibrations. It is also clear that, with E_i, σ_i as above, $\mathfrak{M} = \mathfrak{M}_\sigma$. Thus it becomes possible to show that, up to fiber homotopy equivalence,

$$\sigma_1 \oplus \sigma_2 = \sigma_2 \oplus \sigma_1$$

$$(\sigma_1 \oplus \sigma_2) \oplus \sigma_3 = \sigma_1 \oplus (\sigma_2 \oplus \sigma_3).$$

1.6 DEFINITION. Given a fibration $\tau : E \to Y$, the suspension $\Sigma\tau$ is defined as $\tau \oplus \varepsilon^1$, where $\varepsilon^1 : Y \times S^0 \to Y$ is the trivial S^0 fibration. Obviously, if $F =$ fiber of τ, the fiber of $\Sigma\tau$ is

$$S^0 * F = S^1 \wedge F = \Sigma F,$$

the suspension of F.

We also note that if $\sigma_i : E_i \to Y_i$, $i = 1, 2$ are fibrations with fiber F_i, and if $p_i : Y_1 \times Y_2 \to Y_i$ are projections, then we can define the fibration

$\sigma_1 \times \sigma_2$ over $Y_1 \times Y_2$ as $p_1^* \sigma_1 \oplus p_2^* \sigma_2$. An alternative description of $\sigma_1 \times \sigma_2$ is obtained by taking $\mathfrak{M} = \mathfrak{M}_{\sigma_1} \times \mathfrak{M}_{\sigma_2}$ and letting

$$E = \{ (v_1, v_2) \in \mathfrak{M} \mid \bar{p}_1(v_1) \in E_1 \text{ or } \bar{p}_2(v_2) \in E_2 \}.$$

Here $\bar{p}_i$ is the projection $\mathfrak{M}_{\sigma_1} \times \mathfrak{M}_{\sigma_2} \to \mathfrak{M}_{\sigma_i}$. As usual, we may note that if σ_i, $i = 1, 2$ are fibrations over Y, then

$$\sigma_1 \oplus \sigma_2 = \Delta^*(\sigma_1 \times \sigma_2)$$

where $\Delta : Y \to Y \times Y$ is the diagonal map.

2. The normal fibration of a complex

Let K^k be a finite C-W complex of homotopy dimension k. By a thickening of K^k we shall mean a smooth (or PL) compact manifold-with-boundary M^{n+k}, $n \geq 3$, such that $\partial M^{n+k} \subseteq M^{n+k}$ induces

$$\pi_1(\partial M^{n+k}) \cong \pi_1(M^{n+k}),$$

together with a homotopy equivalence

$$\phi : K^k \to_{\sim} M^{n+k}.$$

(Note that we do not insist that ϕ be a simple homotopy equivalence as Wall does in [e].) When there is no ambiguity, we shall use ϕ to denote the thickening. If ϕ_1, ϕ_2 are thickenings of K^k into M_1^{n+k}, M_2^{n+k} respectively, we say that ϕ_1 is equivalent to ϕ_2 if there is a smooth (PL) equivalence

$$\alpha : M_1^{n+k} \cong M_2^{n+k} \quad \text{with } \alpha\phi_1 \sim \phi_2 .$$

Let ϕ be a thickening of K^k into M^{n+k}. Consider the inclusion $\partial M^{n+k} \subseteq_i M^{n+k}$. By the standard path space construction (see [a]) we can replace this map by a fibration ν_ϕ, whose fiber we denote by F_ϕ. We call this the *normal fibration of the thickening*. It will be seen by standard arguments that F_ϕ will be $(n - 2)$-connected. ν_ϕ may be regarded as a fibration over K^k, and depends only on the equivalence class of ϕ. Our result is the following.

2.1 Theorem. *Let ϕ be a thickening of K^k. Then the stable homotopy type of F_ϕ is independent of ϕ.*

The proof of Theorem 2.1 is based on the following.

2.2 Lemma. *Let M^n be a compact smooth (PL) manifold-with-boundary and let ξ be an orthogonal (PL) k-plane bundle over M^n.*

Let E^{n+k} be the total space of the associated disk bundle, regarded as a smooth (PL) manifold with boundary.

Replacing inclusions by fibrations, let

$$F = \textit{fiber of } \partial M^n \subseteq M^n, \qquad G = \textit{fiber of } \partial E^{n+k} \subseteq E^{n+k}$$

Then

$$G = \Sigma^k F.$$

Proof. To compute G, we compute the fiber of the inclusion $\partial E^{n+k} \subseteq E^{n+k}$ where ∂E^{n+k} may be regarded as

$$\dot{E} \cup (E \mid \partial M^n)$$

where $\dot{E}$ is the total space of the sphere bundle associated to ξ and $E \mid \partial M^n$ is the total space of the disk bundle associated to $\xi \mid \partial M^n$. But if we replace $\partial M^n \subseteq M^n$ by the fibration π

$$\begin{array}{c} Y \\ F \downarrow \pi \\ M^n. \end{array}$$

We then can replace the pair $(M^n, \partial M^n)$ by $(\mathfrak{M}_\pi, Y)$.

Let ξ' be the corresponding bundle over $\mathfrak{M}_\pi$, and let $E', \dot{E}', \dot{E}' \mid Y$ correspond to $E, \dot{E}, E \mid \partial M^n$ in the obvious way.[2]

Then we have a homotopy equivalence of 4-tuples

$$(E, \partial E, \dot{E}, E \mid \partial M) \sim (E', \dot{E}' \cup (E' \mid Y), \dot{E}', E' \mid Y).$$

So therefore we need only study the fiber of the inclusion

$$\dot{E}' \cup (E' \mid Y) \subseteq E'.$$

But since $\pi \circ \xi' : E' \to_{\sim} M^n$, we are really studying the fiber of the natural map

$$\dot{E}' \cup (E' \mid Y) \to M^n. \tag{1}$$

However, if we set $\dot{\xi} = (k - 1)$-sphere bundle associated with ξ, it is obvious by the definitions of Section 1 that the map (1) is just the fibration which we have called $\dot{\xi} \oplus \pi$. By 1.4, therefore, the fiber of this map is

$$G = S^{k-1} * F = \Sigma^k F, \qquad \text{Q.E.D.}$$

The proof of 2.1 follows immediately, for if

$$\phi : K^k \to M^{m+k}, \qquad \psi : K^k \to N^{n+k}$$

are thickenings of K^k, then for suitable orthogonal (PL) bundles ξ, η over M^{m+k} and N^{n+k} respectively,

$$(E(\xi), \partial E(\xi)) \sim (E(\eta), \partial E(\eta))$$

where $E(\xi)$ is the total space of the associated disc bundle of ξ regarded as a manifold, and similarly for η.

Example. A thickening $\phi : K^k \to M^{n+k}$, $n \geq 3$, has $F_\phi \sim S^{n-1}$ if and only if K^k is a Poincaré duality space.

---

[2] Strictly speaking, if we wish to speak about Hurewicz fibrations rather than Serre fibrations, we may have to replace the fiber-bundle ξ' with a spherical fibration, i.e. a Hurewicz fibration with a homotopy S^{k-1} as fiber

Remark. If we allow "thickening" to mean a homotopy equivalence

$$\phi : K^k \to_\sim M^{n+k}$$

where $(M^{n+k}, \partial M^{n+k})$ is a Poincaré duality pair, then, defining ν_ϕ and F_ϕ in the same way, Theorem 2.1 still holds.

If $\phi : K^k \to M^{n+k}$ is a thickening, then the suspension of the thickening denoted by $\Sigma\phi$ is the map

$$\Sigma\phi : K^k \to M^{n+k} \times I$$

got by composing ϕ with the obvious section

$$M^{n+k} \to M^{n+k} \times \{0\} \subseteq M^{n+k} \times I.$$

2.3 COROLLARY. *If ψ is the suspension of the thickening ϕ, then $\nu_\psi = \Sigma\nu_\phi$.*

We now show that the converse to 2.3 holds, assuming 1-connectivity for technical reasons.

2.4 PROPOSITION. *Let K^k be 1-connected, $n - k \geq 4$ and let*

$$\psi : K^k \to M^n$$

be a thickening. Suppose $\nu_\psi = \Sigma\tau$.

Then there exists a thickening ϕ so that $\nu_\phi = \tau$ and $\Sigma\phi$ is equivalent to ψ.

Proof. The proof is based on the Browder co-dimension-1 theorem, together with a relative version of the Siebenmann splitting theorem [b], [c].

Let $\tau : E \to K^k$ be the fibration of the hypothesis. Then there is a homotopy equivalence

$$(2) \qquad (M^n, \partial M^n) \sim (\mathfrak{M}_\tau \times D^1, (E \times D^1) \cup (\mathfrak{M}_\tau \times S^0))$$

where

$$(E \times D^1) \cap (\mathfrak{M}_\tau \times S^0) = E \times S^0.$$

Easy algebraic topology then suffices to show that the pairs

$$(E \times D^1, E \times S^0) \quad \text{and} \quad (\mathfrak{M}_\tau \times S^0, E \times S^0)$$

satisfy Poincaré duality. Thus by Browder's result, $\partial M^n = V \cup W$, where V and W are co-dimension-0 submanifolds of ∂M^n, $\partial V = \partial W = V \cap W$, and where there is a homotopy equivalence of triads

$$(3) \quad (\partial M^n; V, W) \sim ((E \times D^1) \cup (\mathfrak{M}_\tau \times S^0); E \times D^1, \mathfrak{M}_\tau \times S^0)$$

so that (3) is consistent with (2). Now $(\mathfrak{M}_\tau, E)$ satisfies Poincaré duality in dimension $n - 1$, therefore by Siebenmann's result, we have

$$(M^n - W, E - W) \cong (N^{n-1}, \partial N^{n-1}) \times \mathbf{R},$$

(where $\cong$ denotes either diffeomorphism or PL-homeomorphism, whichever is appropriate.) Thus

$$(M^n, \partial M^n) \cong (N^{n-1} \times D^1, \partial N^{n-1} \times D^1 \cup N^{n-1} \times S^0).$$

All the assertions of 2.4 follow easily, Q.E.D.

In fact, in the metastable range we can state an even stronger result: We claim that one need only assume the existence of a section for the fibration of the thickening.

2.5. THEOREM. *Let ψ be a thickening $K \to M^{n+k}$, $n + k \geq 6$, $3k/2 + 2$. Then if ν_ψ admits a section, the thickening ψ desuspends up to equivalence to a thickening $\phi : K^k \to N^{n+k-1}$ (with $\nu_\psi = \Sigma\nu_\phi$).*

Proof. By adding 2-cells and 3-cells to K^k, we may as well assume that ψ is a simple homotopy equivalence. If ν_ϕ admits a section then ϕ factors through ∂M^{n+k} up to homotopy, i.e. there is a map $\phi_1 : K^k \to \partial M^{n+k}$ so that

$$\begin{tikzcd} K^k \arrow[r, "\phi_1"] \arrow[dr, "\phi"'] & \partial M^{n+k} \arrow[d, phantom, "\cap"] \\ & M^{n+k} \end{tikzcd}$$

homotopy commutes.

Now the map ϕ is infinitely connected and the inclusion $\partial M^{n+k} \subseteq M^{n+k}$ is $(n - 1)$-connected. It follows that ϕ_1 is $(n - 2)$-connected. But $(n - 1) \geq 2k - (n + k - 1)$.

Hence, by the Stallings embedding theorem [d], there exists a codimension—1 submanifold $N^{n+k+1} \subseteq \partial M^{n+k}$ and a simple homotopy equivalence

$$\phi_2 : K^k \to N^{n+k-1}$$

so that

$$\begin{tikzcd} K^k \arrow[r, "\phi_2"] \arrow[dr, "\phi_1"'] & N^{n+k-1} \arrow[d, phantom, "\cap"] \\ & \partial M^{n+k} \end{tikzcd}$$

homotopy commutes.

Easy geometry and the s-cobordism theorem then easily suffice to show that ϕ_2 is the required desuspension, Q.E.D.

Now let $\phi : J^j \to M^m$, $\psi : K^k \to N^n$ be thickenings of J^j and K^k respectively. Then there is a thickening

$$\phi \times \psi : J^j \times K^k \to M^m \times N^n.$$

Our result then becomes

2.6. PROPOSITION. $\nu_{\phi\times\psi} = \nu_\phi \times \nu_\psi$.

Proof. The proof is essentially like that of 2.1. Let E_ϕ, E_ψ be the total spaces of ν_ϕ and ν_ψ respectively, and let $\mathfrak{M}_\phi$ and $\mathfrak{M}_\psi$ be the mapping cylinders. Then we claim that there is a homotopy equivalence of 5-tuples:
$(M^m \times N^n, \partial(M^m \times N^n), M^m \times \partial N^n, \partial M^m \times N^n, \partial M^m \times \partial N^n)$
$\sim (\mathfrak{M}_\phi \times \mathfrak{M}_\psi, (\mathfrak{M}_\phi \times E_\psi) \cup (E_\phi \times \mathfrak{M}_\psi), \mathfrak{M}_\phi \times E_\psi, E_\phi \times \mathfrak{M}_\psi, E_\phi \times E_\psi)$
But then the obvious map

$$\delta : \mathfrak{M}_\phi \times \mathfrak{M}_\psi \to M^m \times N^n$$

is such that

$$\sigma = \delta \mid (\mathfrak{M}_\phi \times E_\psi) \cup (E_\phi \times \mathfrak{M}_\psi)$$

is a fibration. It is, in fact, the product $\nu_\phi \times \nu_\psi$ as described in Section 1.

3. Sections and compressions in the metastable range

Let K^k be a finite CW complex of homotopy dimension k and let V^{n+k} be a (smooth or PL) manifold with or without boundary By an embedding of K^k in V^{n+k} up to homotopy type we mean a commutative diagram

$$\begin{tikzcd} K^k \arrow[r, "\phi"] \arrow[dr, "\phi_1"'] & M^{n+k} \arrow[d, phantom, "\cap"] \\ & V^{n+k} \end{tikzcd}$$

where M^{n+k} is a codimension-0 submanifold of V^{n+k} and ϕ is a homotopy equivalence, i.e. a thickening of K^k into M^{n+k}. The normal fibration of the embedding is, by definition, the normal fibration ν_ϕ of the thickening.

Our main result, then, is

3.1 THEOREM. *Let $n + k \geq 3(k + 1)/2$, 6 and let*

$$\begin{tikzcd} K^k \arrow[r, "\phi"] \arrow[dr, "\phi_1"'] & M^{n+k} \arrow[d, phantom, "\cap"] \\ & V^{n+k} \end{tikzcd}$$

be an embedding up to homotopy type. Let ϕ_1 be homotopic to the constant map into V^{n+k}, and let ν_ϕ be fiberhomotopically trivial over the j-skeleton of M^{n+k}.

Then if $n \geq k - j + 1$, ν_ϕ admits a section.

Proof. Let $M = M^{n+k}$, $V = V^{n+k}$, $\nu = \nu_\phi$. Define the "Thom space" $T(\nu)$ as $M/\partial M = V/V\text{-int } M \sim \mathfrak{M}_\nu/E$ where $E \to M$ is the fibration corresponding to $\partial M \subseteq M$. We claim that the obvious map $M \to T(\nu)$ is homotopic to the constant map, because it factors through $M \subseteq V$, which is homotopically trivial since ϕ_1 is homotopically trivial.

Now let F be the fiber of ν, and let

$$\begin{tikzcd} E(F) \arrow[d, "u(F)"] \\ B(F) \end{tikzcd}$$

be the universal F-fibration, so that there is a map of fibrations

$$\begin{tikzcd} E \arrow[r, "A"] \arrow[d] & E(F) \arrow[d, "u(F)"] \\ M \arrow[r, "\alpha"] & B(F) \end{tikzcd}$$

By our assumption, α restricted to the j-skeleton of M is trivial.

Therefore, we consider $B^j(F)$ = the j-connective covering of $B(F)$. We let $c : B^j(F) \to B(F)$ be the covering and let

$$u^j(F) : E^j(F) \to B^j(F)$$

be $c*(u(F))$. It follows that there exists a β so that $\alpha = c\beta$ and we then have a map of fibrations

$$\begin{tikzcd} E \arrow[r, "B"] \arrow[d] & E^j(F) \arrow[d, "u^j(F)"] \\ M \arrow[r, "\beta"] & B^j(F) \end{tikzcd}$$

But note that F will be $(n - 2)$-connected. Consequently $E^j(F)$ has connectivity at least min $(j, n - 2)$. So let $u^j = u^j(F)$, $E^j = E^j(F)$, $T(u^j) =\cdot \mathfrak{M}_{u^j}/E^j$. It will then follow by Toda's version of the triad theorem [d] that the natural homomorphism of homotopy groups

$$\pi_i(\mathfrak{M}_{u^j}, E^j) \to \pi_i(T(u^j))$$

is at least an isomorphism for $i \leq k$ and an epimorphism for $i = k + 1$. Since $M \to T(\nu)$ is homotopically trivial, so is $M \to T(u^j)$, thus, since M is of the homotopy type of a k-complex, the map

$$c^j : M \to B^j \sim \mathfrak{M}_{u^j}$$

lifts to a map $M \to E^j$. Since ν is the fibration $c^{j*}u^j$, it follows that ν admits a section, Q.E.D.

Remark. The proof of Theorem 3.1 is a generalization of the proof of a theorem of Handel [f].

To translate the conclusion of 3.1 into geometric terms, we note that if ν_ϕ

admits a section, then we may alter ϕ by a homotopy so that im $\phi \subseteq \partial M^{n+k}$.

3.2 Definition. Suppose

$$\begin{tikzcd} K^k \arrow[r, "\phi"] \arrow[dr, "\phi_1"] & M^{n+k} \arrow[d, phantom, "\cap"] \\ & V^{n+k} \end{tikzcd}$$

is an embedding up to homotopy type, and suppose ν_ϕ admits a section (so that im ϕ may be assumed to lie in ∂M^{n+k}). Then the section is called non-linking iff ϕ is homotopic to 0 in $V^{n+k} - \operatorname{int} M^{n+k}$.

We then obtain the following generalization following generalization of Hirsch's result [f].

3.3 Theorem. *Let $n + k \geq (3k + 4)/2$, 6 and let*

$$\begin{tikzcd} K^k \arrow[r, "\phi"] \arrow[dr, "\phi_1"] & M^{n+k} \arrow[d, phantom, "\cap"] \\ & V^{n+k} \end{tikzcd}$$

be an embedding up to homotopy type so that ν_ϕ admits a non-linking section.

Then if $V^{n+k} - \operatorname{int} M^{n+k}$ is at least $(k - n + 3)$-connected, there exists an embedding up to homotopy type

$$\begin{tikzcd} K^k \arrow[r, "\psi"] \arrow[dr, "\psi_1"] & U^{n+k-1} \arrow[d, phantom, "\cap"] \\ & S^{n+k-1} \end{tikzcd}$$

and an embedding (smooth or PL)

$$(S^{n+k-1}, U^{n+k-1}) \subseteq_i (V^{n+k} - \operatorname{int} M^{n+k}, \partial M^{n+k})$$

so that

$$\begin{tikzcd} K^k \arrow[r, "\psi"] \arrow[d, "\phi"'] & U^{n+k-1} \arrow[d, "i"] \\ M^{n+k} & \partial M^{n+k} \arrow[l, phantom, "\supseteq"] \end{tikzcd}$$

homotopy commutes.

Moreover, $i(S^{n+k-1})$ bounds a (smooth or PL) $(n + k)$-disc in $V^{n+k} - \operatorname{int} M^{n+k}$.

Proof. By 2.5, we may assume that there is a co-dimension—0 submanifold $U^{n+k-1} \subseteq \partial M^{n+k}$, so that $\operatorname{im} \phi \subseteq U^{n+k-1}$, and so that the map of K into U^{n+k-1} (which we call ψ) is a homotopy equivalence.

Since the section is non-linking, the inclusion

$$U^{n+k-1} \subseteq V^{n+k} - \operatorname{int} M^{n+k}$$

is homotopic to the constant map into the base point. (We may as well assume that the base point of $V^{n+k} - \operatorname{int} M^{n+k}$ is in its interior.) Now, by the Stallings embedding theorem, U^{n+k-1} has a k-dimensional spine $\bar{K}^k$. It follows by the Stallings-Hirsch-Zeeman engulfing theorem [g] that there exists a disc

$$D^{n+k} \subseteq V^{n+k} - \operatorname{int} M^{n+k}$$

with $\bar{K}^k \subseteq D^{n+k}$, and, since $\bar{K}^k \subseteq \partial(V^{n+k} - \operatorname{int} M^{n+k})$, we may assume that

$$\bar{K}^k \subseteq S^{n+k-1} \subseteq D^{n+k}.$$

Moreover, since U^{n+k-1} is a regular neighborhood of $\bar{K}^k$ in

$$\partial(V^{n+k} - \operatorname{int} M^{n+k}),$$

one can assume without loss of generality that $U^{n+k-1} \subseteq S^{n+k-1}$, Q.E.D.

3.4 COROLLARY. *Let $n + k \geq (3k + 5)/2$, 6 and let*

$$\begin{tikzcd} K^k \arrow[r, "\phi"] \arrow[dr, "\phi_1"'] & M^{n+k} \arrow[d, phantom, "\cap"] \\ & S^{n+k} \end{tikzcd}$$

be an embedding up to homotopy type which admits a non-linking section. Then K^k embeds up to homotopy type in S^{n+k-1}.

Proof. It is easily seen that $S^{n+k} - \operatorname{int} M^{n+k}$ is $(n - 2)$-connected. But $(n - 2) \geq k - n + 3$ therefore 3.4 applies and the assertion follows.

Remark. If, in the above theorems, we eliminate the assumption that $n + k \geq 6$, the results still hold if we assume $n \geq 3$, and that $(M^{n+k}, \partial M^{n+k})$ is $(n - 1)$-connected.

4. Normal fibration for a pair; applications

Let (K, J) be a finite C-W pair. We say that the homotopy dimension of (K, J) is (k, j) iff there is a homotopically equivalent pair (X, Y) with $\dim X = k$, $\dim Y = j$, and if no other homotopically equivalent pair (X', Y') has $\dim X' < k$ or $\dim Y' < j$. We write (K^k, J^j) for a pair of homotopy dimension (k, j).

4.1 DEFINITION. Let (K^k, J^j) be a finite C-W pair. By a *thickening* of

(K^k, J^j) we mean the following:

(i) a smooth (PL) compact manifold-with-boundary M^{n+k}, $n \geq 3$, $n + k - 4 \geq j$;

(ii) a codimension-0 submanifold N^{n+k-1} of ∂M^{n+k} such that the inclusions $\partial N^{n+k-1} \subseteq N^{n+k-1}$, $\partial M^{n+k} - \operatorname{int} N^{n+k-1} \subseteq N^{n+k-1}$ induce isomorphism of the fundamental group;

(iii) a homotopy equivalence of pairs

$$\phi : (K^k, J^j) \to_{\sim} (M^{n+k}, N^{n+k-1}).$$

Note that $\phi \mid J^j$ is a thickening of J^j into N^{n+k-1}.

It follows easily by the techniques of Wall [e; §7] that if K^k is homotopic to J^j plus cells of dimension $\leq s$ (s will be k except, possibly, when $j = k$), then the inclusion

$$\partial M^{n+k} - N^{n+k-1} \subseteq M^{n+k}$$

will be $(n + k - s - 1)$-connected. When there is no ambiguity, we write ϕ to denote the thickening. For notational convenience, we write $E(\phi)$ for the space $\partial M^{n+k} - \operatorname{int} N^{n+k-1}$.

4.2 Definition. Let $\phi : (K^k, J^j) \to (M^{n+k}, N^{n+k-1})$ be a thickening, and let ν_ϕ be the fibration corresponding to the inclusion $E(\phi) \to M^{n+k}$.

ν_ϕ is called the normal fibration of the thickening. The fiber is denoted by F_ϕ and is called the fiber of the thickening.

We can regard ν_ϕ as a fibration over K^k.

4.3 Lemma. *The stable homotopy type of F_ϕ is independent of the thickening.*

The proof is similar to that of 2.1.

We also may define the *suspension* of a thickening in the obvious way, i.e. if $\phi : (K^k, J^j) \to (M^{n+k}, N^{n+k-1})$ is a thickening then the suspension is the map

$$\psi : (K^k, J^j) \to (M^{n+k}, N^{n+k-1}) \times I$$

got by composing ϕ with

$$(M^{n+k}, N^{n+k-1}) \to (M^{n+k}, N^{n+k-1}) \times \{0\}.$$

Again, we denote the suspension by $\Sigma\phi$. Equivalence of thickenings is defined by the obvious generalization of the absolute case. We have, in analogy to 2.3,

4.4 Lemma. *If ϕ is a thickening and $\psi = \Sigma\phi$, then $\nu_\psi = \Sigma\nu_\phi$.*

Let ϕ be a thickening of (K^k, J^j) into (M^{m+k}, N^{n+k-1}) where n is large and where $(M^{n+k}, N^{n+k-1}) \subseteq (D^{n+k}, S^{n+k-1})$. Then ν_ϕ is called the $(n + k)$-dimensional representative of the stable normal fibration of the pair (K^k, J^j). Since n is large, this is well defined, and the $(n + k)$-dimensional representative suspends to the $(n + k + 1)$-dimensional representative. We leave it to the reader to verify these details.

Given a pair (K^k, J^j), we use the following notation:

(1) k_1 = homotopy dimension of K, (which may be less than k).
(2) k_2 = the integer $\leq k$ such that

$$H^{k_2}(K^k, J^j) \neq 0, \qquad H^i(K^k, J^j) = 0, \qquad i > k_2.$$

(3) k_3 = connectivity of (K^k, J^j).

We note that $k_2 > k_1$ implies $k_2 = k$. Moreover, given a thickening

$$\phi : (K^k, J^j) \to (M^{n+k}, N^{n+k-1})$$

we have obvious associated thickenings,

$$\phi_1 : K^{k_1} \to M^{n+k}, \qquad \phi_2 : J^j \to N^{n+k-1}.$$

We also note that pair $(M^{n+k}, E(\phi))$ is $(n + k - k_2 - 1)$-connected, while the pair $(M^{n+k}, \partial M^{n+k})$ is $(n + k - k_1 - 1)$-connected.

4.5 DEFINITION. Let (K^k, J^j), (A^a, B^b) be finite CW pairs. Let ν_ϕ, ν_ψ represent the stable normal fibrations of (K^k, J^j), (A^a, B^b) (with respect to appropriate thickenings ϕ, ψ into manifold pairs of the *same* dimension).

Then we say that (K^k, J^j), (A^a, B^b) are *normally equivalent* iff there exists a homotopy equivalence $\alpha : K \to_{\sim} A$ such that $\alpha^* \nu_\psi = \nu_\phi$.

We then have

4.6 THEOREM. *Let (K^k, J^j), (A^a, B^b) be normally equivalent pairs.*

Then if $k_3 \geq 2$ and
(i) $2k_3 - k_2 \geq 0$,
(ii) $k_3 - k_1 \geq 0$,
(K^k, J^j) is homotopically equivalent to (A^a, B^b).

Proof. We ignore the case where k_3 is infinite, as this is trivial.

In choosing finite representatives for the stable normal fibrations of (K^k, J^j) and (A^a, B^b) we can take thickenings ϕ into (M_1^{n+k}, N_1^{n+k-1}), ψ into (M_2^{n+k}, N_2^{n+k-1}) with n as large as we like and, in fact, we can take $M_1^{n+k} = M_2^{n+k} = M$, so assume

$$\begin{tikzcd} K \arrow[rr, "\alpha", "\sim"'] \arrow[dr, "\phi_1", "\sim"'] & & A \arrow[dl, "\psi_1"', "\sim"] \\ & M & \end{tikzcd}$$

homotopy commutes.

Since n can be taken arbitrarily large, we assume $n + k - k_3$ is conveniently larger than k_1. Look now at the diagram

$$\begin{tikzcd} \cdots H^{i-1}(M) \arrow[r] & H^{i-1}(E(\phi)) \arrow[r] & H^i(M, E(\phi)) \arrow[r] \arrow[d, "\cong"] & H^i(M) \\ & & H_{n+k-i}(K^k, J^j) & \end{tikzcd}$$

where the vertical isomorphism follows from Poincaré duality. $H^i(M)$ vanishes for $i > k_1$, and, since (K^k, J^j) is k_3-connected, $H^i(E(\phi))$ vanishes for $i \geq n + k - k_3 - 1$. Since $E(\phi)$ is of the homotopy type of a finite complex, it is of the homotopy type of an $(n + k - k_3 - 2)$-complex. We now look at $E(\psi) \subseteq \partial M$. Since $\alpha^* \nu_\psi = \nu_\phi$, there is a homotopy commutative diagram

$$\begin{tikzcd} E(\phi) \arrow[rr, "\bar{\alpha}", "\simeq"'] \arrow[dr, phantom, "\cap" description] & & E(\psi) \arrow[dl, phantom, "\cap" description] \\ & M & \end{tikzcd}$$

Since $E(\phi)$ and $E(\psi)$ are of homotopy dimension $n + k - k_3 - 2$, and since M^{n+k} has a k_1-dimensional spine, it follows that we must have

$$(n + k - k_3 - 2) + k_1 + 1 < n + k$$

in order to have a homotopy-commutative diagram

$$\begin{tikzcd} E(\phi) \arrow[rr, "\bar{\alpha}", "\simeq"'] \arrow[dr, phantom, "\cap" description] & & E(\psi) \arrow[dl, phantom, "\cap" description] \\ & \partial M & \end{tikzcd}$$

But this is guaranteed by inequality (ii) of the hypothesis.

Now let r be the connectivity of the inclusion $E(\phi) \subseteq \partial M$. Since $\partial M \subseteq M$ is $(n + k - k_1 - 1)$-connected, and since $E(\phi) \subseteq M^{n+k}$ is $(n + k - k_2 - 1)$-connected, it follows that

$$r \geq n + k - 1 - \max\,[k_1 + 1, k_2].$$

Therefore, inequalities (1) and (ii) together imply that

$$r \geq 2(\text{homotopy dim}\, E(\phi)) - (n + k - 1) + 2.$$

It then follows by the uniqueness aspect of the Stallings embedding theorem (see [e] or [f] that the embeddings $E(\phi) \subseteq \partial M$, $E(\psi) \subseteq \partial M$ are concordant, i.e. there is an h-cobordism W between $E(\phi)$ and $E(\psi)$ and an embedding $\beta : W \subseteq \partial M \times I$ so that

$$\beta : E(\phi) \subseteq \partial M \times \{0\} \quad \text{and} \quad \beta : E(\psi) \subseteq \partial M \times \{1\}$$

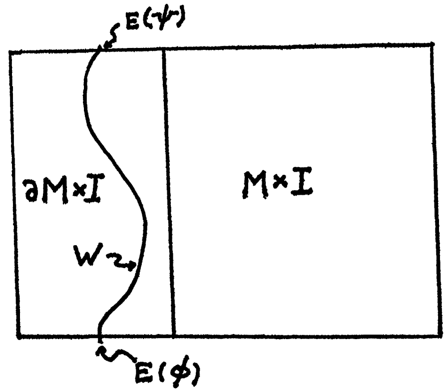

FIGURE 1

in the obvious way. We can therefore extend this concordance to a concordance in M. We illustrate the situation schematically in the figure.

It is easy to see that (M, N_1) is homotopically equivalent to (M, N_2) so that

$$\begin{array}{ccc}(K^k, J^j) \xrightarrow{\phi} (M, N_1) \sim (M, N_2) & \xrightarrow{\psi} & (A^a, B^b) \\ \cup\!| & & |\!\cup \\ K & \xrightarrow[\sim]{\alpha} & A\end{array}$$

homotopy commutes. This completes the proof.

We now apply this theorem to help obtain a theory of "thickenings" into Poincaré-duality pairs which, in many ways resembles that discussed by Wall [e] in the smooth and PL case.

For brevity, we adopt the following definition of Poincaré-duality pair:

4.7 DEFINITION. Let $(P, \partial P)$ be a finite CW pair of homotopy dimension $(p, p - 1)$. We say that $(P, \partial P)$ is a *Poincaré duality pair* iff given a thickening,

$$\phi : (P, \partial P) \to (M^{n+p}, N^{n+p-1}).$$

We have ν_ϕ an $(n - 1)$-spherical fibration over P, and, for

$$\phi' : \partial P \to N^{n+p-1},$$

$\nu_{\phi'}$ is the restriction of ν_ϕ to ∂P.

If P is a finite CW complex, we say that it is a *Poincaré-duality space* iff the pair $(P, 0)$ is a Poincaré-duality pair.

We say that p is the *dimension* of the Poincaré-duality pair.

We will abbreviate "Poincaré-duality pair (space)" by writing "P-pair (-space)." We recall that if ϕ is a thickening of a p-dimensional P-pair $(P, \partial P)$ into an orientable manifold pair, then ν_ϕ is an orientable spherical fibration if and only if $(P, \partial P)$ satisfies the Poincaré-duality formula in dimension p, i.e.

$$H^i(P) \cong H_{p-i}(P, \partial P), \qquad H^i(P, \partial P) \cong H_{p-i}(P)$$

with the isomorphism being induced by cap product with a generator of $H_p(P, \partial P) \cong \mathbf{Z}$. If, on the other hand, ν_ϕ is not an orientable fibration when the range of ϕ is orientable, then $(P, \partial P)$ satisfies the more general form of Poincaré-duality (with co-efficients in an orientation bundle). We also note that if $(P, \partial P)$ is a p-dimensional P-pair, then ∂P is a $(p-1)$-dimensional P-space. Our definition is to a certain extent redundant. We need only have defined P-pair as a pair (K^k, J^j) so that for some thickening

$$\phi : (K^k, J^j) \to (M^{n+k}, N^{n+k-1}),$$

ν_ϕ is an $(n-1)$-spherical fibration, and $\nu_{\phi'} = \nu_\phi \mid J$. It then follows without too much work that this holds for any thickening and that $j = k - 1$. For details we refer the reader to Spivak [g] where the necessary techniques are developed. Recall also that P-pairs are said to be equivalent if they are homotopically equivalent.

Now let K^k be a finite CW complex. By a Poincaré thickening (which we abbreviate to P-thickening) of K^k, we mean a P-pair $(P, \partial P)$ of dimension $n + k$, $n \geq 3$ together with a homotopy equivalence $\phi : K^k \to P$. We also require that $(P, \partial P)$ be $(n-1)$-connected. If $(P_1, \partial P_1)$ and $(P_2, \partial P_2)$ are $(n+k)$-dimensional P-pairs and $\phi_1 : K^k \to P_1$, $\phi_2 : K^k \to P_2$ are P-thickenings we call the thickenings *equivalent* iff there is a homotopy equivalence $f : (P_1, \partial P_1) \sim (P_2, \partial P_2)$ so that

$$\begin{tikzcd} & P_1 \arrow[r] & (P_1, \partial P_1) \arrow[dd, "f", "\sim"'] \\ K^k \arrow[ur] \arrow[dr] & & \\ & P_2 \arrow[r] & (P_2, \partial P_2) \end{tikzcd}$$

homotopy commutes. We use $PTh^{n+k}(K^k)$ to denote the set of equivalence classes of P-thickenings of K^k in dimension $n + k$. It is easy to see that if $(P, \partial P)$ is a P-pair of dimension $n + k$, then

$$(P \times I, (\partial P \times I) \cup P \times I)$$

is a P-pair of dimension $n + k + 1$; thus we have a natural suspension operation

$$\Sigma_n : PTh^{n+k}(K^k) \to PTh^{n+k+1}(K^k)$$

just as in the smooth or PL case.

It is also amusing to note that if $\phi : K^k \to (P, \partial P)$ is a P-thickening of K^k, one may define a normal fibration ν_ϕ by replacing $\partial P \subseteq P$ by a fibration (and regarding the result as a fibration over K^k, if we wish). Clearly, if P is a compact smooth (PL) manifold and ∂P its boundary (up to homotopy type) we get the same object ν_ϕ which we defined in Chapter 2. We can also assert, as in Chapter 2, that if we set F_ϕ = fiber of ν_ϕ, then the stable homotopy type of F_ϕ is independent of the thickening and, in fact coincides with the stable homotopy type defined for the case of smooth or PL thickenings. To prove this, one makes an easy generalization of the proof of Theorem 2.1.

4.8 THEOREM. *In the sequence*

$$\cdots PTh^{n+k}(K^k) \xrightarrow{\Sigma^n} PTh^{n+k+1}(K^k) \cdots$$

Σ^n is epimorphic for $n \geq k + 1$ and monomorphic for $n \geq k + 2$; moreover, for $n \geq k + 2$, $PTH^{n+k}(K^k)$ is naturally isomorphic to $\mathcal{K}^{\sim}F(K^k)$, where $\mathcal{K}^{\sim}F$ denotes the group of stable spherical fibrations over K^k.

Before we proceed to the proof of 4.8, we remark that this result is very similar to that of Wall [e] or Mazur [h] in the smooth and PL cases. The chief difference is that in these cases one gets epimorphism for $n = k$ and monomorphism for $n = k + 1$. Thus in changing from the manifold categories to the Poincaré-duality category one "loses" a dimension, at least insofar as these results indicate. This is curious in view of the results of an earlier paper of the author [i] in which it is shown that, for an appropriate notion of embedding K^k embeds in a Poincaré-duality space of dimension $2k + 2$, thus, in effect "losing" one dimension as compared to the usual results for manifolds. It is not known whether these results are best possible, and it would be interesting to learn whether this "loss" of one dimension is a real phenomenon or whether it can be eliminated by introducing a more perspicacious technique.

Proof of 4.8. First note that if $\phi : K^k \to P$, where $(P, \partial P)$ is an $(n + k)$-dimensional P-pair, then the stable normal fibration ν of the pair may be regarded as an element of $\mathcal{K}^{\sim}F(K^k)$ and depends only on the equivalence class of the thickening. This gives a map

$$PTh^{n+k}(K^k) \xrightarrow{T^{n+k}} \mathcal{K}^{\sim}F(K^k)$$

which, moreover, commutes with the suspension map, i.e.

$$\begin{tikzcd} PTh^{n+k}(K^k) \arrow[rr, "\Sigma^{n+k}"] \arrow[dr, "T^{n+k}"'] & & PTh^{n+k+1}(K^k) \arrow[dl, "T^{n+k+1}"] \\ & \mathcal{K}^{\sim}F(K^k) & \end{tikzcd}$$

commutes. This is checked by a straightforward application of the definitions.

First we show that for $n \geq k + 2$, T^{n+k} is a monomorphism. Let

$$\phi_1 : K^k \to P_1, \qquad \phi_2 : K^k \to P_2$$

be thickenings which have the same image under T^{n+k}. By definition there will be a homotopy equivalence $\alpha : P_1 \sim P_2$ which exhibits a normal equivalence between $(P_1, \partial P_1)$ and $(P_2, \partial P_2)$. Now note that the cohomology dimension of $(P_1, \partial P_1)$ is $n + k$; the homotopy dimension of P_1 is, of course, k, and the connectivity of $(P_1, \partial P_1)$ is $n - 1$. By plugging in these observations, one easily ascertains that the inequalities (i) and (ii) hold, along with the rest of the hypothesis of 4.6. Therefore, α extends to a homotopy equivalence of pairs $(P_1, \partial P_1) \sim (P_2, \partial P_2)$. Since α is homotopy consistent with ϕ_1, ϕ_2, the thickenings are therefore equivalent.

To show that

$$T^{n+k} : PTh^{n+k}(K^k) \to \mathcal{K}^{\sim}F(K^k)$$

is onto for $n \geq k + 1$, let $\psi : K^k \to_{\sim} M^{n+k+r}$ be a thickening of K^k into a regular neighborhood in a high dimensional euclidean space, i.e. M^{n+k+r} is a submanifold of $\mathbf{R}^{n+k+r}$, r being large. Let u be an element in $\mathcal{K}^{\sim}F(K^k)$; then u is represented by an $(r - 1)$-spherical fibration $\xi : \mathcal{E} \to K^k$. Up to homotopy we may assume that $\mathcal{E}$ is a $(k + r - 1)$-complex and that ξ is a map into $M^{n+k+r} = M$. By the fact that M may be taken to have a k-dimensional spine, and by general position it follows that up to homotopy ξ factors through a map $\xi_1 : \mathcal{E} \to \partial M$. (Here we use the fact that $n \geq k + 1$.)

Now the pair $(M, \partial M)$ is $(n + r - 1)$-connected; ξ is $(r - 1)$-connected, hence ξ_1 is $(r - 1)$-connected. But $n \geq k + 1$ assures that the hypotheses of Stallings embedding theorem [d], [e] are satisfied, and thus one may assume that there is a codimension-0 submanifold $E = E^{n+k+r-1} \subseteq \partial M$ and a homotopy equivalence $f : \mathcal{E} \sim E$ so that

$$\begin{tikzcd} \mathcal{E} \arrow[r, "f"] \arrow[dr, "\xi_1"'] & E \arrow[d, phantom, "\cap|"] \\ & \partial M \end{tikzcd}$$

homotopy commutes.

Let $N = \partial M - \operatorname{int} E$. One can assume that $\partial E \subseteq E$ is $(n - 1)$-connected; thus $N \subseteq \partial M$ is $(n - 1)$-connected and therefore $N \subseteq M$ is $(n - 1)$-connected. Moreover, $\partial N \subseteq N$ induces isomorphism $\pi_1(\partial N) \cong \pi_1(N)$.

It is easily checked by Poincaré-duality that the space N has cohomology dimension $n + k - 1$, and therefore has homotopy dimension $n + k - 1$. Thus the pair (M, N) has homotopy dimension $(n + k, N + k - 1)$. The normal fibration of the pair thickening is $(n - 1)$-spherical. But since $\pi_1(N) \cong \pi_1(M)$, it is easily checked that the fibration corresponding to $\partial N \subseteq N$ is the induced $(n - 1)$-spherical fibration. (See [g] for the relevant techniques.) Thus (M, N) is a P-pair of dimension $n + k$ and ψ induces an obvious $(n + k)$-dimensional P-thickening ψ' of K^k. Since r is large, it is easily verified that we may assume $(M, N) \subseteq D^{n+k+r}, S^{n+k+r-1}$. It then follows from definitions that $T^{n+k}(\psi') = u$, Q.E.D.

5. Normal fibrations for products; applications

Let K^k, J^j be finite complexes, and let

$$\phi : K^k \to M^{m+k}, \qquad \psi : J^j \to N^{n+j}$$

be thickenings (into smooth or PL manifolds). Then there is an obvious product thickening

$$\phi \times \psi : K^k \times J^j \to M^{m+k} \times N^{n+j}.$$

5.1 Lemma. *$\nu_{\phi\times\psi}$ is the product fibration $\nu_\phi \times \nu_\psi$.*

Lemma 5.1 was proved as Proposition 2.6.

5.2 Corollary. *If ν_ϕ represents the stable normal fibration for K^k and ν_ψ represents the stable normal fibration for J^j, then $\nu_\phi \times \nu_\psi$ represents the stable normal fibration of $K^k \times J^j$.*

5.3 Corollary. *If F_ϕ, F_ψ are the fibers of thickenings of K^k, J^j respectively, then a thickening of $K^k \times J^j$ has fiber of the stable homotopy type of the join $F_\phi * F_\psi$.*

We now apply these facts to the study of finite H-spaces. We use M^n to denote an H-space homotopically equivalent to a finite complex and of homotopy dimension n. We will assume M^n is 1-connected, n odd, ≥ 5. M^n has a homotopy inverse. e, u, and j denote respectively the base point, multiplication, and inverse of M^n.

First note that by a theorem of Browder [j], M^n satisfies Poincaré-duality in dimension n, and thus by Spivak [g] is a P-space.

Thus, $M^n \times M^n$ is a P-space as well. We recall a definition of [f] and [i].

5.4 Definition. If N is a P-space of dimension n, K' a finite complex of dimension $\leq n$, we say that a map $f : K \to N$ is *homotopic to an embedding* iff there exists a pair of n-dimensional P-pairs $(N_1, \partial N_1)$ $(N_2, \partial N_2)$ with $\partial N_1 = \partial N_2$, a P-thickening $\phi : K \to_\sim N_1$ and a homotopy equivalence

$$\alpha : N \to_\sim N_1 \cup_{\partial N_1 = \partial N_2} N_2$$

so that

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d, "\phi"] & N \arrow[d, "\alpha"] \\
N_1 \arrow[r, phantom, "\subseteq"] & N_1 \underset{\partial N_1 = \partial N_2}{\cup} N_2
\end{tikzcd}$$

homotopy commutes.

The triple $\{(N_1, \partial N_1), (N_2, \partial N_2), \alpha\}$ is called a splitting of N. It is easily shown that if N is a P-space, then the map

$$\mathrm{id} \times * : N \to N \times N$$

is homotopic to an embedding. This follows from the fact that a point embeds in a P-space. (See Wall [k].) Moreover, if N is an n-dimensional P-space and if Q is a q-dimensional P-space, of $\geq n + 3$, then given an embedding of N into Q, with $\{(Q_1, \partial Q_1), (Q_2, \partial Q_2), \alpha\}$ the corresponding splitting, the map $\partial Q_1 \subseteq Q_1$ is, up to homotopy, a $(q - n - 1)$-spherical fibration, called the *normal fibration* of the embedding. In the above case, we may assume that

$$\mathrm{id} \times * : N \to N \times N$$

is homotopic to an embedding so as to have trivial normal fibration.

Now if $M = M^n$ is a finite H-space with homotopy inverse, we have:

5.5 LEMMA. *Let Δ be the diagonal map $\Delta : M \to M \times M$ and let*

$$\alpha : M \times M \to M \times M \quad by \quad \alpha(x, y) = (x, \mu(x, y)).$$

Then α is a homotopy equivalence so that

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id} \times e"'] & M \arrow[d, "\Delta"] \\ M \times M \arrow[r, "\alpha"] & M \times M \end{tikzcd}$$

homotopy commutes.

Consequently

5.6. LEMMA. *Δ is homotopic to an embedding with trivial normal fibration.*

However, we have the following theorem from [f]:

5.7 THEOREM. *Let P, Q, R be 1-connected P-spaces, and let*

$$f : P \to Q, \qquad g : Q \to R$$

be homotopic to embeddings with normal fibrations ν_f, ν_g over P and Q respectively. Then $g \circ f$ is homotopic to an embedding with normal fibration $\nu_f \oplus f^* \nu_g$.

5.8 COROLLARY. *If M^n is a 1-connected finite H-space with homotopy inverse, then the stable normal fibration of M is a trivial spherical fibration.*

Proof. Consider $M \times M$: It is easily seen by 5.1 that if ν_ϕ is a representative for the stable normal fibration of M, then, $\nu_\phi \times \nu_\phi$ is a representative of the stable normal fibration of $N \times N$. By 5.6 and 5.7, it follows that

$$\Delta^* (\nu_\phi \times \nu_\phi) \oplus \varepsilon^n$$

is a representative for the stable normal fibration of N, where ε^n is the trivial S^{n-1}-fibration, n. But then it follows that

$$\Delta^* (\nu_\phi \times \nu_\phi) \oplus \varepsilon^n = \nu_\phi \oplus \nu_\phi \oplus \varepsilon^n$$

is stably equivalent to ν_ϕ, since both are representatives of the stable normal fibration of N. It follows that ν_ϕ is trivial, since stable spherical fibrations form a group.

5.9 THEOREM. *If M^n is as above and n is odd, ≥ 5, then M^n is of the homotopy type of an n-dimensional π-manifold.* (This was first proved by Browder [n].)

Proof. Let $\bar{M} = \bar{M}^{n+k}$, (with k large) be a codimension-0 submanifold of S^{n+k} of the homotopy type of M and with $\pi_1(\partial\bar{M}) \to \pi_1(\bar{M})$ an isomorphism. Then by definitions and 5.8, there is a homotopy equivalence

$$h : (\bar{M}, \partial\bar{M}) \to_{\sim} (M \times D^k, M \times S^{k-1}).$$

Let $\gamma \in H_{n+k}(\bar{M}, \partial\bar{M})$ be an orientation of the orientable manifold $\bar{M}$. Then the element $\gamma' \in H_{n+k}(\bar{M}/\partial\bar{M})$ is the image of γ under the projection map. But look at

$$h' : \bar{M}/\partial\bar{M} \to M \times D^k/M \times S^{k-1} = T(\varepsilon^k)$$

where $T(\varepsilon^k)$ is the Thom complex of the trivial bundle. Clearly, $h'_*\gamma'$ generates the top-dimensional homology of $T(\varepsilon^k)$. But the usual collapsing map construction shows that γ' is spherical, hence $h'_*\gamma'$ is spherical and $T(\varepsilon)$ is reducible. The Browder-Novikov theorem [n] then implies that M^n is of the homotopy type of a smooth manifold with trivial stable normal bundle, Q.E.D.

Remark. The condition that n be odd comes from the existence of the Index obstruction for $n = 4k$ and the Arf-Kervaire invariant for $n = 4k + 2$. Browder has shown that the theorem holds for $n = 4k$ as well [n], because the index obstruction vanishes.

6. Embeddings up to homotopy type

Let K^k be a 1-connected finite complex and let ν_ϕ be a representative of the stable normal fibration of K^k, for some thickening

$$\phi : K^k \to \bar{M}^{k+r}, \qquad \bar{M}^{k+r} \subseteq S^{k+r}.$$

Then if E_ν is the total space of $\nu = \nu_\phi$, we have

$$(\mathfrak{M}_\nu, E_\nu) \sim (M, \partial M).$$

Moreover, the "Thom complex" $T(\nu) = \mathfrak{M}_\nu/E_\nu$ has homology $\mathbf{Z}$ in dimension $k + r$ and this homology is spherically generated; thus $T(\nu)$ is reducible.

Now suppose the fibration ν desuspends j times, i.e. $\nu = \sum^j \mu$. We then claim

6.1 LEMMA. $T(\nu) = \Sigma^j T(\mu)$.

The proof proceeds as in the case of bundles or spherical fibrations. It is also not hard to show

6.2 LEMMA. *$(\mathfrak{M}_\mu, E_\mu)$ is of the homotopy type of a P-pair of dimension $k + r - j$.*

Now suppose $T(\mu)$ is reducible. Let $\mathfrak{D}$ be the "double" of $(\mathfrak{M}_\mu, E_\mu)$, i.e. $\mathfrak{D}$ is the space $\mathfrak{M}_\mu \cup \mathfrak{M}_\mu$ with the two copies of E_μ identified. $\mathfrak{D}$ is then a P-space of dimension $k + r - j$. To avoid ambiguity, let ε^1 denote the trivial S^0 fibration over K^k and $\varepsilon^1(\mathfrak{D})$ the trivial S^0-fibration over $\mathfrak{D}$. Set $\mu' = \mu \oplus \varepsilon^1$. Then we easily see that $T(\mu')$ is reducible. Moreover, we claim

6.3 LEMMA. *There is a degree-1 map from $T(\mu')$ to $T(\varepsilon^1(\mathfrak{D}))$.*

The proof of 6.3 is easily arrived at, but the explicit details are tedious to write down and will be left to the reader. In consequence of 6.3, we see that $T(\varepsilon^1(\mathfrak{D}))$ is reducible. It follows from Browder [c] or Levitt [f] that $\mathfrak{D}$ embeds up to homotopy type in $S^{k+r-j+2}$, i.e. there is a $W^{k+r-j+2} \subseteq S^{k+r-j+2}$ with $\mathfrak{D} \sim W$, and with $\partial W \subseteq W$ a trivial S^1-fibration. But then let

$$\mathcal{W} = \mathfrak{D} \times D^2, \qquad \partial\mathcal{W} = \mathfrak{D} \times S^1.$$

Then $\mathcal{W} = \mathcal{K} \cup \mathcal{K}'$ where $\mathcal{K}$, $\mathcal{K}'$ are the two copies of $\mathfrak{M}_\mu \times D^2$, and where $\mathcal{K} \cap \mathcal{K}' = E_\mu \times D^2$. By the Browder codimension-1 theorem [c] it can be shown that there are codimension-0 submanifolds U, V of W, $W = U \cup V$, and a homotopy equivalence

$$(\mathcal{W}, \mathcal{K}, \mathcal{K}', \mathcal{K} \cap \mathcal{K}') \sim (W, U, V, U \cap V).$$

Thus since $K^k \sim \mathcal{K} \sim U$, $\mathcal{K}^k$ embeds up to homotopy type in $S^{k+r-j+2}$. We have thus proved

6.4 THEOREM. *Let μ be a fibration over K^k so that $\Sigma^j\mu = \nu_\phi$, where ν_ϕ is a $k + r$-dimensional representative of the stable normal fibration of K^k.*
Then if $T(\mu)$ is reducible, K^k embeds in $S^{k+r-j+2}$ up to homotopy type.

Another way of stating this is that, if ϕ is an embedding up to homotopy type of K^k in S^{k+r}, and ν_ϕ is the normal fibration of the embedding, then we can reduce the ambient dimension of the embedding by $j - 2$ if we can desuspend ν_ϕ j times and desuspend along with it the element in $\pi_{k+r}(T(\nu_\phi))$ which generates $H_{k+r}(T(\nu_\phi))$ under the Hurewicz map.

BIBLIOGRAPHY

a. E. SPANIER, *Algebraic topology*, McGraw-Hill, New York, 1966.
b. L. SIEBENMANN, Ph.D. Dissertation, Princeton University, 1965.
c. W. BROWDER, *Embedding 1-connected manifolds*, Bull. Amer. Math. Soc., vol 72 (1966), pp. 225–231.
d. J. STALLINGS, *The embedding of homotopy types into manifolds*, mimeographed notes, Princeton University.
e. C. T. C. WALL, *Classification problems in differential topology IV, Thickenings*, Topology, vol. 5 (1966), pp. 73–94.
f. N. LEVITT, *Applications of engulfing*, Ph.D. Dissertation, Princeton University, 1967.
g. M. SPIVAK, *Spaces satisfying Poincaré duality*, Topology, vol. 6 (1967), pp. 77-102.
h. B. MAZUR, *Differential topology from the point of view of simple homotopy theory*, Publ. Math., l'I.H.E.S., France, no. 15, 1963.

i. N. LEVITT, *On the structure of Poincaré duality spaces*, Topology,
j. W. BROWDER, *Torsion in H-spaces*, Ann. of Math., vol. 14 (1961), pp. 24–51.
k. C. T. C. WALL, *Poincaré complexes I*, mimeographed notes.
l. M. HIRSCH, *Embeddings and compressions of smooth manifolds and polyhedra*, Topology, vol. 4 (1966), pp. 361–369.
m. D. HANDEL, *On the normal bundle of an embedding*, Topology, vol. 5 (1967), pp. 65–68.
n. W. BROWDER, *Homotopy type of differentiable manifolds*, Proc. Aarhus Symposium, 1962.

NEW YORK UNIVERSITY
NEW YORK, NEW YORK