FREE INVOLUTIONS OF HOMOTOPY $S^l \times S^l$'S

BY

R. WELLS

Introduction

A *homotopy* $S^l \times S^l$ will be a smoothing of the piecewise linear $S^l \times S^l$. If $l \geq 3$, it follows from de Sapio 13 that a homotopy $S \times S$ is stably parallelizable. We will be interested only in the case l even, $l \geq 8$, and $l \neq 2^j - 2$ for all j. Then by a standard argument a homotopy $S^l \times S^l$, since it is stably parallelizable, is of the form $S^l \times S^l \# \Sigma$ where Σ is a suitable homotopy l-sphere.

An *involution of* $S^l \times S^l \# \Sigma$ will be a fixed point free, orientation preserving, diffeomorphism $\rho : S^l \times S^l \# \Sigma \rightarrow S^l \times S^l \# \Sigma$ of order two. An involution ρ is *weakly equivalent* to ρ' if there is an orientation preserving diffeomorphism ψ carrying the domain of ρ onto that of ρ' such that $\rho' \circ \psi = \psi \circ \rho$. If is clear that weak equivalence classes of involutions are in bujective correspondence with the oriented diffeomorphism classes of the manifolds $M = S^l \times S^l \# \Sigma/\rho$. To classify the involutions up to weak equivalence, we attempt to classify the manifolds M up to oriented diffeomorphism.

It will turn out that, given M, there is a unique even integer k mod $2^{\varphi(l)}$ such that $f^*(v(M))$ is stably equivalent to $k\xi_l$ for any map $f : P_l \rightarrow M$ such that $\pi_1(f)$ is an isomorphism, where ξ_l is the canonical line bundle over P_l. This integer will be called the *type* of M.

Let γ be the unique l-plane bundle over P_l stably equivalent to $(2^{\varphi(l)} - l - 1 - k)\xi_l$, with Euler class a generator or zero, depending on which is possible. (Exactly one of these cases will be possible.)

Suppose now that M is of type k. Then its normal bundle is stably equivalent to $k\xi + \beta$, where β pulls back from a unique element $\alpha(M) \in \widetilde{KO}(T(\gamma))$ by means of a canonical map $M \rightarrow T(\gamma)$. The oriented diffeomorphism classes of manifolds of type k form a group $\Gamma(\gamma)/G$, and

$$\Gamma(\gamma)/G \xrightarrow{\alpha} \widetilde{KO}(T(\gamma))$$

turns out to be a homomorphism in Section 4.

The next problem is to describe the kernel K/G of α. For this we need a J-homomorphism

$$\widetilde{KO}^{-1}(S(\gamma)) \xrightarrow{J} \pi^s_{2l+k}\, T(k\xi_\infty)$$

where $S(\gamma)$ is the sphere bundle of γ above and ξ_∞ is the canonical line bundle over RP_∞. The homomorphism J is defined using the Thom construction, exactly as the standard J homomorphism is defined. Then there is a homomorphism $\varphi : K/G \rightarrow \Lambda$ where Λ is the cokernel of J. It follows from the

Received November 1, 1968.

theorem of Section 2 that φ is an epimorphism, and from the theorem of Section 5 that the kernel of φ is an image of Z_2. And it follows from Section 6 that there is a fixed map $Z_2 \to K\tilde{O}(T(\gamma))$ such that α factors uniquely through this map. Thus we may take $\alpha : \Gamma(\gamma)/G \to Z_2$.

Thus $\Gamma(\gamma)/G$ is described by the exact sequences:

$$\begin{tikzcd}
 & & & Z_2 \arrow[d] & \\
 & 1 \arrow[r] & K/G \arrow[r] & \Gamma(\gamma)/G \arrow[r, "\alpha"] \arrow[d, "\varphi"] & Z_2 \\
K\tilde{O}^{-1}(S(\gamma)) \arrow[rr, "J"] & & \pi^s_{2l+k} T(k\xi_\infty) \arrow[r] & \Lambda \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}$$

Section 1 contains preliminaries. In Section 2 we study a special case of the problem of killing middle homotopy groups of manifolds, and arrive at the theorem that will make φ an epimorphism. In Section 3 we study mappings and embeddings $P_l \to M$, to obtain (1) the type of M is well-defined, (2) a useful decomposition of M. In Section 4, we use that decomposition to prove that if type $(M) = k$, then $v(M)$ differs from $k\xi$ by a stable bundle of index 0. This fact enables us to show that Im $(\alpha) = 0$ or Z_2. In Section 5, we define a group $\Gamma(\gamma)$ of which $\Gamma(\gamma)/G$ is a quotient. Finally in Section 5, we define J and φ. That φ is an epimorphism follows already from Theorem 2, and that φ has kernel at most of order 2 follows from Theorem 4 of that section.

As a by-product, in Section 6, we obtain the following theorem.

THEOREM 6. *If $l \equiv 4, 6$ (8) and M is the quotient of $S^l \times S^l$ by an involution, then $v(M)$ is stably an even multiple of the canonical line bundle.*

For a counterexample in the case $l \equiv 0$ (8) see [10].

Wall's theorems on non-simply connected surgery [8] are crucial to the argument, and some theorems, especially Theorem 2, resemble special cases of Theorem 6.5 of [8]. To derive Theorem 2 from Wall's theorem, one would have to factor the natural map $M \to RP_\infty$ of Section 2 through $S(\gamma + \varepsilon) \to P_l \to RP_\infty$. If this could be done, a much stronger theorem than Theorem 2 would result. A special case of this problem, factoring the natural map $M \to RP_\infty$ for certain M through $P_l \to RP_\infty$ occurs in Section 5. In that case there is a solution, and Wall's theorem applies to conclude $M = S(\gamma + \varepsilon)$.

I. Preliminaries

In this section we fix notation.

P will always denote infinite-dimensional projective space, and P_j will always denote j-dimensional projective space. The canonical line bundle over P will be ξ_∞, except in Section 2, where it will be ξ. The canonical line bundle over

P_j will be ξ_j. The order of the reduced stable class of ξ_j in $K\tilde{O}(P_j)$ will we $2^{\varphi(j)}$. If γ is any vector bundle, $E(\gamma)$ will be its associated cell bundle and $S(\gamma)$ its associated sphere bundle. The Stiefel-Whitney class of γ will be $\omega(\gamma)$ and the Pontryagin class of γ will be $P(\gamma)$. Two bundles γ and γ' will be *isomorphic* if there is a bundle map $\gamma \to \gamma'$ covering a homeomorphism of the base spaces. If A is a submanifold of B, then $v(A:B)$ will be the normal bundle of A in B, and $\tau(A)$ will be the tangent bundle of A; $v(A)^m$ is the (stable) normal bundle of A in Euclidean space of codimension m. The trivial bundle of dimension i is denoted by ϵ^i.

Modules over the group ring of Z_2 will be called Z_2-molecules. Special ones will be $\bar{Z}$, on which Z_2 operates by changing signs; $\overline{Z+Z}$ and $\overline{Z_2+Z_2}$, on which Z_2 operates by changing signs; $Z+Z$ and Z_2+Z_2, on which Z_2 operates by changing components. If X is a space with $\pi_1(X) = Z_2$, then $\bar{Z}$, $\overline{Z+Z}$, $\overline{Z_2+Z_2}$ will also denote the bundles of coefficients over X associated with these modules. Then $H_*(X;A)$ and $H^*(X;A)$ will denote as usual the homology and cohomology of X with coefficients in the bundle of coefficients associated with the Z_2-module A.

Suppose $A \subset X$ and $B \subset Y$ are subspaces such that

$$A \subset X \subset X \cup CA \quad \text{and} \quad B \subset Y \subset Y \cup CB$$

are confibrations (this assumption holds for all inclusions throughout). Then if $f: A \to B$ is a map, $X \times 0 \cup_f Y \times 1$ will denote, by abuse of language, the space $X \times 0 \cup Y \times 1$ modulo the identification $(x, 0) \sim (f(x), 1)$ for $x \in A$. If CA is the cone over A, then $X \times 0 \cup_1 CA \times 1$ will be written $X \cup CA$, by abuse of notation. Then the suspension of X will be

$$SX = CX \cup CX = CX \times 0 \cup_1 CX \times 1.$$

If f is a homeomorphism of A onto B, we have the transposition homeomorphism

$$T : X \times 0 \cup_f Y \times 1 \to Y \times 0 \cup_{f^{-1}} X \times 1$$

defined by $T(x, 0) = (x, 1)$ and $T(y, 1) = (y, 0)$ on the representative level. Denote the ith stable homotopy group of X by $\pi_i^s(X)$. Then

$$T_* : \pi_i^s(SX) \to \pi_i^s(SX)$$

is sign reversal.

II. $k\xi$-cobordism

Let ξ be the canonical line bundle on infinite real projective space. Let $k\xi$ be the k-fold Whitney sum of ξ with itself, and let $T(k\xi)$ be the Thom space of $k\xi$. Then the elements of $\tilde{\pi}^s_{n+k}(T(k\xi))$ may be interpreted as $k\xi$-cobordism classes, where a $k\xi$-manifold is a pair $(M, \mathfrak{F})$ with

$$v(M)^m \xrightarrow{\mathfrak{F}} k\xi + \varepsilon^{m-k}$$

an isotopy class of bundle maps, and m is large.

Consider $\alpha \in \tilde{\pi}^s_{2l+k}(T(k\xi))$ where l and k are even. We seek a 'canonical' representative of α. To begin with, let $\eta \to P_l$ be the $(l+1)$-dimensional reduction of $(2^{\varphi(l)} - l - 1 - k)\xi_l$, where ξ_l is the canonical line bundle over P_l. Let $E(\eta)$ be its associated cell bundle and $S(\eta)$ its associated sphere bundle. Then there is an isotopy class $\mathfrak{F}_0$ of bundle maps $v(E(\eta))^m \to k\xi + \varepsilon^{m-k}$. We denote its restriction to $v(S(\eta))^m$ also by $\mathfrak{F}_0$. Then let $(M, \mathfrak{F})$ be a representative of α. Since P is connected, we may carry out 0-modifications of $(M, \mathfrak{F})$ in order to assume M is connected. If the maps $M \to P$ covered by $\mathfrak{F}$ does not pull back non-trivially the generator of $H^1(P : Z_2)$, we may replace $(M, \mathfrak{F})$ by $(M, \mathfrak{F}) + (S(\eta), \mathfrak{F}_0)$ before the 0-modifications, without changing α. Now a series of 1-modifications kill off the kernel of $\pi_1(M) \to \pi_1(P) = Z_2$, so we may assume that map to be an isomorphism. Then since $\pi_p(P) = 0$ for $p > 1$, we may perform p-modifications to insure that $\pi_i(M) \approx \pi_i(P)$ for all $i < l$.

Finally, we arrive at a representative $(M, \mathfrak{F})$ of α such that $\pi_i(M) \approx \pi_i(P)$ for $i < l$. If

$$\hat{M} \xrightarrow{\pi} M$$

is the double cover of M, we have $H_0(\hat{M}) = H_{2l}(\hat{M}) = Z$ and $H_l(\hat{M})$ free and $H_i(\hat{M}) = 0$ otherwise. Let $\rho : \hat{M} \to \hat{M}$ be the transposition. Then ρ_* turns $H_*(\hat{M})$ into a graded Z_2-module, and the intersection pairing $H_l(\hat{M}) \times H_l(\hat{M}) \to Z$ is a totally orthogonal, symmetric pairing invariant under ρ_*.

LEMMA 1 (Wall). *If $x \in H_l(\hat{M})$ is such that $x \cdot x = 0$ and $x \cdot \rho x = 0$, then there is an l-modification of $(M, \mathfrak{F})$ killing $\pi_* h^{-1}(x)$, where h is the Hurewicz isomorphism.*

LEMMA 2. *Suppose x as in Lemma 1, and there is $z \in H_l(\hat{M})$ such that $x \cdot z = 1$, $z \cdot \rho z = 0$. Let $(M', \mathfrak{F}')$ be the result of an l-modification killing $\pi_* h^{-1}(x)$. Then $\pi_i(M') \approx \pi_i(P)$ for $i < l$ and $H_l(\hat{M}')$ is isomorphic to* $(\ker x \cap \ker \rho x)/(Zx \oplus Z\rho x)$.

Proof. There will be two disjoint spheres S_1^l, $S_2^l \subset \hat{M}$ interchanged by ρ, and two disjoint spheres S_1^{l-1}, $S_2^{l-1} \subset \hat{M}'$ interchanged by ρ' so that $\hat{M} - S_1^l - S_2^l$ and $\hat{M}' - S_1^{l-1} - S_2^{l-1}$ are diffeomorphic as Z_2 spaces. Moreover, the following sequences are exact sequences of Z_2-modules

$$0 \to H_l(\hat{M}^-) \to H_l(\hat{M}) \xrightarrow{x \oplus \rho x} H_l(\hat{M}, \hat{M}^-) \to 0$$

$$0 \to H_{l+1}(\hat{M}', \hat{M}'^-) \xrightarrow{x \oplus \rho x} H_l(\hat{M}'^-) \to H_l(\hat{M}') \to 0,$$

which proves the lemma.

THEOREM 2. *Suppose k and l are even and $(M, \mathfrak{F})$ is a closed $k\xi$-manifold of dimension $2l$ such that $\pi_i(M) \approx \pi_i(P)$ for $i < l$. Then if* rank $\pi_l(M) > 2$, *there is a $k\xi$-cobordism from $(M, \mathfrak{F})$ to $(M', \mathfrak{F}')$ such that $\pi_i(M') \approx \pi_i(P)$ and* rank $\pi_l(M') <$ rank $\pi_l(M)$.

Proof. Let

$$\hat{M} \xrightarrow{\pi} M$$

be the double cover of M. Then $\pi_l(M) = H_l(\hat{M})$ which is free of finite rank. Let $\rho : \hat{M} \to \hat{M}$ be the covering transformation. Let

$$\Gamma_+ = \{x \in H_l(\hat{M}) \mid \rho x = x\} \quad \text{and} \quad \Gamma_- = \{x \in H_l(M) \mid \rho x = -x\}.$$

Then $H_l(\hat{M}) \otimes Q = (\Gamma_+ \otimes Q) \oplus (\Gamma_- \otimes Q)$ and $\Gamma_+ \cap \Gamma_- = 0$ and $\Gamma_+ \perp \Gamma_-$ with respect to the intersection pairing. Thus $0 \to \Gamma_+ \oplus \Gamma_- \to H_l(\hat{M}) \to$ fin grp $\to 0$ is exact.

Notice that the Lefschetz trace formula requires tr $\rho \mid H_l(\hat{M}) = -2$, so rank $\Gamma_+ = r$ and rank $\Gamma_- = r + 2$ for some r. Since Γ_+ and Γ_- are each divisible, they are each a direct summand of $H_l(\hat{M})$. We will need some of $H_*(M:B)$ where B is any of the bundles of coefficients Z_2, Z, $\bar{Z}$, $\overline{Z + Z}$.

Z_2: We use the exact sequence $0 \to Z_2 \to \overline{Z_2 + Z_2} \to Z_2 \to 0$ to obtain

$$\cdots \to H_i(M; Z_2) \to H_i(\hat{M}; Z_2) \xrightarrow{\pi_*} H_i(M; Z_2) \to H_{i-1}(M; Z_2) \to \cdots .$$

Thus $H_i(M; Z_2) = Z_2$ for $i < l$ and $H_l(M; Z_2) = (r + 2)Z_2$.

Z: We use the exact sequence

$$0 \to Z \xrightarrow{2} Z \to Z_2 \to 0$$

as above to obtain

$$\cdots \to H_i(M) \xrightarrow{2} H_i(M) \to H_i(M; Z_2) \to H_{i-1}(M) \to \cdots$$

so

$$H_1(M) = H_3(M) = \cdots = H_{l-1}(M) = Z_2,$$
$$H_2(M) = H_4(M) = \cdots = H_{l-2}(M) = 0, \quad H_l(M) = rZ + Z_2$$

0 (There is no odd torsion.)

$\bar{Z}$: From

$$0 \to Z \xrightarrow{2} Z \to Z_2 \to 0,$$

we obtain

$$\cdots \to H_i(M; \bar{Z}) \xrightarrow{2} H_i(M; \bar{Z}) \to H_i(M; Z_2) \to H_{i-1}(M; \bar{Z}) \to \cdots$$

so

$$H_i(M; \bar{Z}) = 0 \text{ for } i \text{ odd} < l, \quad H_i(M; \bar{Z}) = Z_2 \text{ for } i \text{ even} < l.$$

Then use $0 \to \bar{Z} \to \overline{Z + Z} \to Z \to 0$ to obtain

$$\cdots \to H_i(M; \bar{Z}) \to H_i(\hat{M}) \xrightarrow{\pi_*} H_i(M) \to H_{i-1}(M; \bar{Z}) \to \cdots$$

from which follows

$$\begin{array}{ccccccccc} H_{l+1}(M) & \to & H_l(M;\bar{Z}) & \to & H_l(\hat{M}) & \xrightarrow{\pi_*} & H_l(M) & \to & H_{l-1}(M;\bar{Z}) \\ \| & & & & & & & & \| \\ H^{l-1}(M) = 0 & & & & & & & & 0 \end{array}$$

Since $\rho = -1$ on $C_*(M;\bar{Z})$, we have $\rho = -1$ on $H_l(M;\bar{Z})$, so

$$\begin{array}{ccc} H_l(M;\bar{Z}) & \to & H_l(\hat{M}) \\ & \searrow & \cup \\ & & \Gamma_- \end{array}.$$

Let F be an abelian group such that $\Gamma_- \oplus F = H_l(\hat{M})$. Then

$$0 \to H_l(M;\bar{Z}) \to \Gamma_- \oplus F \to rZ + Z_2 \to 0.$$

It follows that $0 \to H_l(M;\bar{Z}) \to \Gamma_- \to Z_2 \to 0$ and $H_l(M;\bar{Z}) = (r+2)Z$.

Finally, using $0 \to Z \to \overline{Z+Z} \to \bar{Z} \to 0$, we obtain

$$\begin{array}{ccccccccccc} 0 & \to & H_{l+1}(M;\bar{Z}) & \to & H_l(M) & \to & H_l(\hat{M}) & \to & H_l(M;\bar{Z}) & \to & H_{l-1}(M) \to 0 \\ & & \| & & & & & & & & \| \\ & & H^{l-1}(M;\bar{Z}) & & & & & & & & Z_2 \\ & & \| & & & & & & & & \\ & & Z_2 & & & & & & & & \end{array}$$

i.e.,

$$0 \to Z_2 \to rZ + Z_2 \to (2r+2)Z \to (r+2)Z \to Z_2 \to 0.$$

Since $C_*(M) \to C_*^+(\hat{M})$, we have $H_l(M) \to \Gamma_+$, and finally $0 \to Z_2 \to H_l(M) \to \Gamma_+ \to 0$.

Besides the groups and maps above, we will need some information on the intersection pairing in $H_l(\hat{M})$. The intersection of chains in regular position in $C_*(\hat{M})$ defines the intersection of chains in regular position in $C_*(M)$ and $C_*(M;\bar{Z}) = C_*(\hat{M}) \otimes_{Z_2} \bar{Z}$. Since the maps

$$\iota_+ : H_l(M) \to H_l(\hat{M}) \quad \text{and} \quad \iota_- : H_l(M;\bar{Z}) \to H_l(\hat{M})$$

are induced on the chain level by $x \to x + \rho_\# x$ and $x \to x - \rho_\# x$, where $x \in C_*(\hat{M})$, we find that $\iota_+ x \cdot \iota_+ y = 2x\cdot y$ and $\iota_- x \cdot \iota_- y = 2x \cdot y$ for $x, y \in H_l(M)$ or $H_l(M : \bar{Z})$.

Since the rational Pontrjagin classes of $k\xi$ are zero, it follows that the index of M is zero, so there is a basis (x_i, y_i) for a free part of $H_l(M)$ such that $x_i \cdot x_j = y_i \cdot y_j = 0$, $x_i \cdot y_j = \delta_{ij}$. It follows that r is even, say $r = 2s$, and $i = 1, \cdots, s$. For each pair we have that ι_+ of one member is indivisible—let it always be x_i. Then $\iota_+ x_i$, $\iota_+ y_i$, $i = 1, \cdots, s$, supplies a basis for Γ_+ with intersection matrix

$$\begin{bmatrix} 0 & 2 & & \\ 2 & 0 & & \\ & & \ddots & & \\ & & & 0 & 2 \\ & & & 2 & 0 \end{bmatrix}.$$

Over the rationals, the index of Γ_+ is then zero, and therefore, so is that of Γ_-, consequently also that of $H_1(M; \bar{Z})$. Let $\bar{x}_i, \bar{y}_i$ be a symplectic basis for $H_1(M; \bar{Z})$. Then $\iota_- \bar{x}_i$, $\iota_- \bar{y}_i$ is a family of elements in Γ_- with intersection matrix

$$\begin{bmatrix} 0 & 2 & & & \\ 2 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 2 \\ & & & 2 & 0 \end{bmatrix}$$

and such that their span has just two cosets in Γ_-. Let z be in the nontrivial coset. Then $2z \in \operatorname{span} \iota_- \bar{x}_i, \iota_- \bar{y}_i$ so $2z = \sum a_i \bar{x}_i + \sum b_i \bar{y}_i$ (where we abbreviate $\iota_- \bar{x}_i$ and $\iota_- \bar{y}_i$ by $\bar{x}_i, \bar{y}_i$). We may assume each a_i, b_i is 0 or 1 (by changing z), and that b_{s+1} (the last b) is 1. Then $\bar{x}_1, \bar{y}_1, \cdots, \bar{x}_s, \bar{y}_s, \bar{x}_{s+1}, z$ is a basis for Γ_-. Then $2z \cdot \bar{x}_{s+1} = \bar{y}_{s+1} \cdot \bar{x}_{s+1} = 2$, so $z \cdot \bar{x}_{s+1} = 1$. Replacing z with $z - ((z \cdot z)/z)\bar{x}_{s+1}$ we obtain a new basis for Γ_-, with intersection matrix

$$\begin{bmatrix} 0 & 2 & & & & & & a_1 \\ 0 & 2 & & & & & & b_1 \\ & & \ddots & & & & & \vdots \\ & & & 0 & 2 & & & a_s \\ & & & 2 & 0 & & & b_s \\ & & & & & & 0 & 1 \\ a_1 & b_1 & \cdots & a_s & b_s & & 1 & 0 \end{bmatrix}$$

Then, replacing $\bar{x}_i$ with $\bar{x}_i - (\bar{x}_i \cdot z)\bar{x}_{s+1}$ and $\bar{y}_i$ with $\bar{y}_i - (\bar{y}_i \cdot z)\bar{x}_{s+1}$, we finally obtain a basis x_i', y_i' with intersection matrix

$$\begin{bmatrix} 0 & 2 & & & & & \\ 2 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 2 & & \\ & & & 2 & 0 & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}$$

Let $\Lambda = (x_{s+1}', y_{s+1}')^{\perp}$. Then $\det(\cdot \mid \Lambda) = 1$, $\operatorname{index}(\cdot \mid \Lambda) = 0$, $\operatorname{tr}(\rho \mid \Lambda) = 0$ and $x_1, y_1, \cdots, x_s, y_s$ span $\Lambda_+ = \Gamma_+$ while $x_1', y_1', \cdots, x_s', y_s'$, span Λ_- $(= \{y \in \Lambda \mid \rho y = -y\})$. Together these span $\Lambda_+ \oplus \Lambda_-$, and have intersection matrix

$$\begin{bmatrix} 0 & 2 & & & \\ 2 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 2 \\ & & & 2 & 0 \end{bmatrix}$$

To determine $\Lambda/\Lambda_+ \oplus \Lambda_-$, we consider the coefficient sequence

$$0 \to Z \oplus \bar{Z} \to \overline{Z + Z} \to Z_2 \to 0.$$

It leads to

$$\begin{aligned} 0 \to H_{l+1}(M; Z_2) \to H_l(M) \oplus H_l(M; \bar{Z}) &\xrightarrow{\iota_+ \oplus \iota_-} H_l(M) \\ &\to H_l(M; Z_2) \to H_{l-1}(M) \oplus H_{l-1}(M; \bar{Z}) \to 0 \end{aligned}$$

that is

$$0 \to Z_2 \to Z_2 + 2sZ + (2s+2)Z \to (4s+2)Z \to (2s+2)Z_2 \to Z_2 \to 0$$

so, since the image of $\iota_+ \oplus \iota_-$ is $\Lambda_+ \oplus \Lambda_- \oplus (s'_{s+1}, y'_{s+1})$,

$$0 \to \Lambda_+ \oplus \Lambda_- \to \Lambda \to 2sZ_2 \to 0$$

is exact.

The next step is to make surgeries allowing us to assume that U, the maximal singular submodule of Λ containing $(x_1, \cdots, x_s, x'_1, \cdots, x'_s)$ is actually spanned by these elements. First notice that $z \in U$ if and only if $2z = \sum a_i x_i + \sum b_i x'_i$ because in general $2z \in \Lambda_+ \oplus \Lambda_-$, and y_i, y'_i cannot be in U, nor can any minimal linear combination involving them be in U. Then U is invariant under ρ. Next, suppose that there is some z not in span $(x_1, \cdots, x'_s)$. We may assume z to be indivisible. Let A be the smallest divisible module containing z and ρz. Then $A = \{\alpha \mid m\alpha = az + b\rho z\}$ so A is invariant under ρ, and, since z is indivisible, A has a basis z, u. Let the matrix with respect to this basis of $\rho \mid A$ be

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$\begin{bmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so if $a + d \neq 0$ then $b = c = 0$, and $a = d = \pm 1$. Consequently, $A \subset \Lambda_+ \cap U$ or $A \subset \Lambda_- \cap U$ and then $z \in A \subset$ span $(x_1, \cdots, x'_s)$, which is a contradiction. Thus $d = -a$ and $a^2 + bc = 1$. Then

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

has an eigenvalue -1, and A has another basis (v, w) with respect to which the matrix of $\rho \mid A$ is

$$\begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}.$$

That is, $\rho v = -v$ and $\rho w = bv + w$. Say b is even, $=2e$. Then replace (v, w) by $(v, w + ev)$. That is a new basis with respect to which $\rho \mid A$ has matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $A \subset \Lambda_+ \oplus \Lambda_-$ and $z \in \text{span}\ (x_1, \cdots, x_s')$, a contradiction again. Thus, b is odd, $=2e + 1$. Then the basis $(v, w + ev)$ realizes the matrix of $\rho \mid A$ as

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Thus there is some basis (v, w) with respect to which $\rho \mid A$ has matrix

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since A is a direct summand of Λ, there exists $\xi \in \Lambda$ such that $\xi \cdot v = 1, \xi \cdot w = 0$. Then $w \cdot w = 0$, $w \cdot \rho w = 0$, $w \cdot \xi = 0$, $\rho w \cdot \xi = (w + v) \cdot \xi = 1$, and Lemma 2 allows us to surger w, lowering the rank of $H_l(\hat{M})$ by four. Eventually, this reduction will be impossible, so we may assume $U = \text{span}\ (x_1, \cdots, x_s')$.

If U, the maximal singular submodule containing span $(x_1, \cdots, x_s')$ is span $(x_1, \cdots, x_s')$ itself, then we may complete the argument. Since U is a direct summand of Λ, we may find $\xi_1, \cdots, \xi_s, \xi_1', \cdots, \xi_s'$ such that $\xi_i \cdot \xi_j = \xi_i' \cdot \xi_j' = \xi_i \cdot \xi_j' = \xi_i \cdot x_j' = \xi_i' \cdot x_j = 0$ for all i, j and $\xi_i \cdot x_j = \xi_i' \cdot x_j' \neq \delta_{ij}$. Then x, x', ξ, ξ' form a basis for Λ since the intersection matrix for this set is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Now, $\rho \xi_i = \xi_i + v_i$ and $\rho \xi_i' = -\xi_i' + v_i'$. Then

$$0 = -x_j' \cdot \xi_i = \rho x_j' \cdot \xi_i = x_j' \cdot \rho \xi_i = x_j' \cdot \xi_i + x_j' \cdot v_i$$

so $x_j' \cdot v_i = 0$. On the other hand, $\rho v_i = -v_i$ so $2v_i = \sum a_{ij} \cdot x_j' + \sum b_{ij}\, y_j' \cdot$ Then $x_j' \cdot v_i = 0$ implies $2v_i \in \text{span}\ (x_1', \cdots, x_s')$ span $(x_1, \cdots, x_s')$, so $v_i \in U$. We have then $v_i = \sum c_{ij}\, x_j'$ and similarly $v_i' = \sum d_{ij}\, x_j'$. The basis ξ, ξ' may be altered to another basis by adding linear combinations of x, x' to each of its elements. The specific alteration we make is

$$\xi_i \to \xi_i + \sum [c_{ij}/2] x_j' \quad \text{and} \quad \xi_j' \to \xi_j' - \sum [c_{ij}/2] x_i.$$

This particular change of basis has the property that the intersection matrix with respect to the new basis is still

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Defining v_i, v_i' in terms of the new basis, we find that $v_i = \sum c_{ij} x_j'$ where each c_{ij} is 0 or 1. Thus v_i itself is 0 or indivisible. Suppose that some v_i, say v_1, is 0. Then $\Lambda/\Lambda_+ \oplus \Lambda_-$ has at most $2s - 1$ generators, so it cannot be $2sZ_2$. Thus each v_i, in particular v_1, is non-zero and indivisible.

Now, we wish to surger ξ_1. That $\xi_1 \cdot \xi_1 = 0$ is given, and from that follow

$$0 = \xi_1 \cdot \xi_1 = \rho\xi_1 \cdot \rho\xi_1 = (\xi_1 + v_1) \cdot (\xi_1 + v_1) = 2(\xi_1 \cdot v_1) + (v_1 \cdot v_1).$$

But $v_1 = \sum c_{ij} x_j'$ so $v_1 \cdot v_1 = 0$, and so $(\xi_1 \cdot v_1) = 0$. But $\xi_1 \cdot \rho\xi_1 = \xi_1 \cdot \xi_1 + \xi_1 \cdot v$ so $\xi_1 \cdot \rho\xi_1 = 0$ too. The fact that v_1 is indivisible means that there is some ζ such that $\zeta \cdot v_1 = 1$. Since $v_1 \cdot v_1 = 0$, we may assume $\zeta \cdot \zeta = 0$. Let $\zeta' = \zeta - (\zeta \cdot \xi_1)x_1$. Then $\zeta' \cdot \xi_1 = 0$ and $\zeta' \cdot v_1 = 1$ since $v_1 \cdot x_1 = 0$. In conclusion, we have $\xi_1 \cdot \xi_1 = 0$, $\xi_1 \cdot \rho\xi_1 = 1$, $\xi_1 \cdot \zeta' = 0$, $\rho\xi_1 \cdot \zeta' = \zeta' \cdot v_1 = 1$ and we may surger ξ_1, reducing the rank of $H_l(\hat{M})$ by 4.

COROLLARY. *Each $k\xi$-cobordism class $\alpha \in \pi_{2l+k}^s(T(k\xi))$, for k and l even, is represented by a $k\xi$-manifold $(M, \mathfrak{F})$ such that $\pi_i(M) \approx \pi_i(P)$ for $i < l$ and $H_l(\hat{M}) = Z + Z$.*

III. Projective spaces in M

Suppose $\hat{M}$ is a $2l$-dimensional closed, simply-cônnected manifold, l even, such that $H_0(\hat{M}) = H_{2l}(\hat{M}) = Z$, $H_l(\hat{M}) \neq 0$ and $H_i(\hat{M}) = 0$ otherwise. Let $\rho : \hat{M} \to \hat{M}$ be an orientation-preserving free action of Z_2 on $\hat{M}$, and let M be the quotient of $\hat{M}$ by that action, and $\pi : \hat{M} \to M$ the projection. Using obstruction theory and Haefliger's theorem, we may obtain an embedding $P_l \subset M$ such that $\pi_1(P_l) \approx \pi_1(M)$. This supplies an embedding $S^l \subset \hat{M}$ of an invariant sphere, on which ρ is the antipodal action. Let $\alpha \in H_l(\hat{M})$ be the class represented by S^l.

LEMMA 3. *A class $\beta \in H_l(\hat{M})$ is represented by an invariant sphere on which ρ is the antipodal action if and only if $\alpha - \beta \in (1 - \rho_*)H_l(\hat{M})$.*

Proof. Let $f : S^l \subset \hat{M}$ be the embedding representing α, and $g : S^l \subset \hat{M}$ that representing β. By obstruction theory on the associated embeddings $P_l \subset M$, we may assume that $f \mid S^{l-1} = g \mid S^{l-1}$. Let E_+ and E_- be simplicial chains representing the fundamental classes of the upper and lower hemispheres, and S^{l-1} a suitable simplicial chain representing the fundamental class of S^{l-1}. Then we may assume (by suitably choosing the simplicial subdivision of S^l) that $\partial E_+ = S^{l-1} = -\partial E_-$ and $(-1)_\# E_+ = -E_-$. Then $\alpha + \beta$ is represented by

$$(f_\# + g_\#)(E_+ + E_-) = (\hat{f}_\# E_+ + \bar{g}_\# E_-) + (\hat{f}_\# E_- + \bar{g}_\# E_+) = x - \rho_\# x,$$

where x is the cycle $\hat{f}_{\#} E_+ + \bar{g}_{\#} E_-$. Thus $\alpha + \beta \in (1 - \rho_*)H_l(\hat{M})$. But if β is represented as above, so is $-\beta$, and so $\alpha - \beta \in (1 - \rho_*)H_l(\hat{M})$. For the converse, let $f : S^l \subset M$ be an invariant embedding.

Choose basepoints in $\hat{M}$, M, S^l, P_l, S^{l-1} and P_{l-1} so that this commutative diagram preserves basepoints:

$$\begin{tikzcd} S^{l-1} \subset S^l \arrow[r, "\hat{f}"] \arrow[d, "p"] & \hat{M} \arrow[d, "\pi"] \\ P_{l-1} \subset P_l \arrow[r, "f"] & M \end{tikzcd}$$

Choose $y \in H_l(\hat{M})$ and let $\gamma \in \pi_l(\hat{M})$ be such that the Hurewicz image of γ is y. Using classical obstruction theory techniques, we may find $g : P_l \to M$ such that $g \mid P_{l-1} = f \mid P_{l-1}$, and such that $\gamma \in \pi_l(M) \approx \pi_l(\hat{M})$ is represented by the (basepoint-preserving) map $S^l \xrightarrow{h} M$, defined by $f \circ p$ on E_+, the upper hemisphere of S_l, and $g \circ p$ on E_-, the lower hemisphere of S^l. Once again, let E_+ and E_- also denote the appropriate simplicial chains, $\hat{f}$ and $\bar{g}$ the covering maps for f and g. Then $(\hat{f}_{\#} + \bar{g}_{\#})(E_+ + E_-) = x - \rho_{\#} x$ as before, where x is the cochain $\hat{f}_{\#} E_+ + \bar{g}_{\#} E_-$. But the (basepoint-preserving) map

$$S^l \xrightarrow{\hat{h}} \hat{M}$$

defined by $\hat{f}$ on E_+ and $\hat{g}$ on E_- covers h and so represents γ. Also, its Hurewicz image is clearly the class x, so if β is the Hurewicz image of the class of $\hat{g}$, we have $\alpha + \beta = y - \rho_{\#} y$. Since $f \mid P_{l-1} = g \mid P_{l-1}$, we have $g_* : \pi_1(P_l) \to \pi_1(M)$, so by Haefliger's theorem we may homotope (preserving the basepoint) g to an embedding g', Then the covering map $\hat{g}'$ of g' embeds S^l as a sphere on which ρ is antipodal, and which represents β. Then replacing β with $\beta \circ (-1)$ we obtain a class β', represented by an invariant sphere, such that $\alpha - \beta' = y - p_{\#} y$, Q.E.D.

Now we further restrict $H_l(\hat{M})$ to be $Z + Z$ and $\hat{M}$ to be s-parallelizable. In that case there is a base for $H_l(\hat{M})$, say u and v such that $u \cdot u = v \cdot v = 0$ and $u \cdot v = v \cdot u = 1$. Since ρ_* has order 2 and preserves intersection numbers, the matrix of ρ_* with respect to this basis must have the form

$$\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

The Lefschetz trace theorem imposes the condition that the trace is -2, so ρ_* is -1. Thus, in this case Lemma 1 states that β is represented by an antipodal embedded sphere if and only if $\alpha - \beta \in 2H_l(\hat{M})$.

Recall the fact (from the proof of Theorem 1) that $H^i(M; Z_2) = Z_2$ for $0 \leq i \leq 2l$ and $i \neq l$, and $H^l(M; Z_2) = Z_2 + Z_2$. Let x be the generator of $H^1(M; Z_2)$. Then it is easy to see that $x^l \neq 0$. Let $y \in H^l(M; Z_2)$ be such that x^l, y span $H^l(M; Z_2)$. If $x^{l+1} \neq 0$, then $x^{2l} \neq 0$ by duality so

$$S_q^1 : H^{2l-1}(M; Z_2) \to H^{2l}(M; Z_2)$$

is non-trivial, and M is non-orientable. But M is orientable, so $x^{l+1} = 0$. Then

$$H^*(M : Z_2) = Z_2[x : x^{l+1} = 0] \otimes E(y),$$

which enables us to obtain

LEMMA 4. *Let $\hat{f} : S^l \subset \hat{M}$ be an equivariant embedding of a sphere with respect to -1 on S^l and ρ on $\hat{M}$. Then $\hat{f}_* : H_l(S^l; Z_2) \to H_l(\hat{M}; Z_2)$ is non-zero.*

Proof. Since Z_2 is a field, it suffices to show that

$$\hat{f}^* : H^l(\hat{M}, Z_2) \to H^l(S^l; Z_2)$$

is non-zero. Let $f : P_l \subset M$ be the map covered by $\hat{f}$. Then we have the following commutative diagram (obtained by using the short exact sequence of coefficient bundles over M and P_l $0 \to Z_2 \to \overline{Z_2 + Z_2} \to Z_2 \to 0$) in which Z_2 coefficients are assumed:

$$\begin{tikzcd}
0 & H^{l+1}(M) \\
H^l(P_l) \arrow[u] & H^l(M) \arrow[u, "\delta"'] \arrow[l, "f^*"'] \\
H^l(S^l) \arrow[u] & H^l(\hat{M}) = Z_2 + Z_2 \arrow[u] \arrow[l, "\hat{f}^*"'] \\
H^l(P_l) \arrow[u, "p^*"] &
\end{tikzcd}$$

Since $p^* : H^l(P_l) \to H^l(S^l)$ is zero, we have $H^l(S^l) \approx H^l(P_l)$. On the other hand, δ in the right-hand sequence is multiplication by x, so $(x^l) = 0$, and there is $z \in H^l(\hat{M})$ carried into x^l. But $f^*x^l \neq 0$ so $\hat{f}^*z \neq 0$.

Thus we have

PROPOSTIION 1. *There is an embedding $f : S^l \subset \hat{M}$ equivariant with respect to the antipodal action on S^l and ρ on $\hat{M}$, such that $\hat{f}$ represents a generator of $H_l(\hat{M})$ with Z coefficients.*

Now consider $M - f(P_l)$. It is covered by $\hat{M} - \hat{f}(S^l)$. Since $\hat{f}(S^l)$ represents a generator of $H_l(\hat{M})$, the Z-cohomology of $\hat{M} - \hat{f}(S^l)$ is that of an l-sphere. As before, obstruction theory techniques and Haefliger's theorem combine to supply an embedding $g : P_l \subset M - f(P_l)$ such that $\hat{g} : S^l \subset \hat{M} - \hat{f}(S^l) \subset \hat{M}$ represents a generator. It is easy to check that g is a homotopy equivalence. Since the Whitehead group of Z_2 is zero, it follows that there is a diffeomorphism $E(\gamma) \to M - f(P_l)$, where γ is the normal bundle of $g(P_l)$ in M and $E(\gamma)$ its total space. Then the Thom space of γ is homeomorphic to $M/f(P_l)$.

IV. The normal bundle of M

We continue to assume, as above, that $H_0(\hat{M}) = H_{2l}(\hat{M}) = Z$, $H_l(\hat{M}) = Z + Z$, $H_i(\hat{M}) = 0$ otherwise, and that $\hat{M}$ is s-parallelizable. We will say such manifolds M are *reduced*. Then $f, g : P_l \subset M$ will be the embeddings constructed in Section III, and ξ will be the canonical line bundle over M. There is a unique (mod $2^{\varphi(l)}$) even integer k such that $f^*v(M) = g^*v(M)$ is stably equivalent to $k\xi_l$, where ξ_l is the canonical line bundle over P_l; we will say that k is the *type* of M. That such a k is well-defined is a consequence of the following lemma:

LEMMA 5. *Suppose M is reduced.*

(i) *If $f, h : P_l \to M$ are such that*

$$f_*, h_* : \pi_1(P_l) \approx \pi_1(M)$$

then $f^*v(M) = h^*v(M)$, where $v(M) \in K\tilde{O}(M)$ is the class of the stable normal bundle.

(ii) *M is diffeomorphic to $E(\gamma) \cup_\psi E(\gamma)$ where γ is an l-dimensional reduction of $(2^{\varphi(l)} - l - 1 - k)\xi_1$ and ψ is a diffeomorphism $S(\gamma) \to S(\gamma)$. If $\omega_l(\gamma) \neq 0$ then the twisted Euler class of γ is a generator. If $\omega_l(\gamma) = 0$ then the bundle $S(\gamma) \to P_l$ admits a cross section.*

Proof. Let $\tilde{\omega} : P_l \to P_l \vee S^l$ be obtained by collapsing the boundary of an l-cell in P_l. Then if f, h are maps as in i), there is a map $\bar{h} : S^l \to M$ such that $(f \vee \bar{h}) \circ \tilde{\omega}$ is homotopic to h. Thus

$$h^*(v(M)) = \tilde{\omega}^*(f^*(v(M)) \oplus \bar{h}^*v(M)).$$

But $\bar{h}$ factors through $\hat{M}$, and $v(\hat{M}) = 0$, so $\bar{h}^*v(M) = 0$, and

$$h^*(v(M)) = \tilde{\omega}_l^*(f^*(v(M)) \oplus 0) = f^*(v(M)).$$

(ii) It follows immediately from (i) that the type k of M is well-defined mod $2^{\varphi(l)}$. Let f and g be the disjoint embeddings $P_l \subset M$. Let

$$\gamma' = f^*v(f(P_l) : M) \quad \text{and} \quad \gamma'' = g^*v(g(P_l) : M).$$

Then since k is well-defined, γ' and γ'' are l-dimensional reductions of $(2^{\varphi(l)} - l - 1 - k)\xi_l$. Let $\hat{f}, \hat{g} : S^l \subset \hat{M}$ be the coverings of f, g and let $\pi : S^l \to P_l$ be the projection. Then if $\chi(\eta)$ is the (twisted) Euler class of the bundle η, we have

$$\pi^*(\chi(\gamma')) = \chi(v(\hat{f}(P_l) : \hat{M})) = \pm\hat{f}(S^l)\cdot\hat{f}(S^l) = \pm 2 \text{ or } 0$$

$$\pi^*(\chi(\gamma'')) = \chi(v(\hat{g}(P_l) : \hat{M})) = \pm\hat{f}(S^l)\cdot\hat{f}(S^l) = \pm 2 \text{ or } 0,$$

with both zero or both non-zero: Also, $\pi^* : H^l(P_l:\tilde{Z}) \to H^l(S^l:Z)$ carries the generator of $H^l(P_l:\tilde{Z})$ into twice that of $H^l(S^l:Z)$. Thus $\chi(\gamma')$ and $\chi(\gamma'')$ both generate $H^l(P_l:\tilde{Z})$ or are both zero. Since the Euler class classifies stably equivalent l-dimensional bundles over P_l, we have that $E(\gamma')$ and $E(\gamma'')$ are isomorphic to $E(\gamma)$ where γ is a fixed l-dimensional reduction of $(2^{\varphi(l)} - l - 1 - k)\xi_l$ with Euler class a generator or zero. Since ω_l is the mod 2 reduction of the Euler class, we have the first case if

$$\omega_l((2^{\varphi(l)} - l - 1 - k)\xi_l) \neq 0,$$

and the second case otherwise. Now (ii) follows immediately, using the fact that there are no non-trivial h-cobordisms when the fundamental group is Z_2.

Now we try to determine $v(M)$. Let $q : M \to M/g(P_l)$ be the collapsing map. From the remarks above it follows that there is a vector bundle A over $M/g(P_l)$ such that $k\xi \oplus q^*\alpha$ is stably equivalent to the normal bundle of M. That α is stably unique follows from

LEMMA 6.

$$K\tilde{O}^{-1}(M) \xrightarrow{g^*} K\tilde{O}^{-1}(P_l) \to 0$$

is exact.

Proof. Since $P_l \to P$ factors via g through M, it is enough to prove that

$$K\tilde{O}^{-1}(P_r) \to K\tilde{O}^{-1}(P_l) \to 0$$

is exact for large r. This fact is an immediate corollary of Adams' computation of $K\tilde{O}(P_r)$.

In what follows, we will need L_*, the multiplicative series determining the index. Thus if M is a closed oriented manifold of dimension $4r$ and $v(M)$ is its stable normal bundle, then index $(M) = L_r(p(v(M)))[M]$. If α is any bundle over M, define index $(\alpha) = L_r(p(\alpha))[M]$. Notice that if $p(\beta) = 1$, then index $(\alpha + \beta) =$ index (α).

Now we recall a suggestive theorem:

THEOREM 3 (Wall). *Let M^{2l} be a reduced manifold of type k, with $v(M)^n = k\xi + \varepsilon^{n-k}$ for some $n > 2l + k + 3$. Let β be an $(n - k)$-bundle over M such that* index $(\beta) = 0$ *and such that β is fiber-homotopically trivial. Then there is a reduced manifold M' and a homotopy equivalence $h : M' \to M$ such that $v(M')^n = h^*(k\xi + \beta)$.*

Proof. Since β is fiber-homotopically trivial, the Thom space $T(k\xi + \beta)$ is reducible. Let $S^{n+2l} \to T(k\xi + \beta)$ be a reducing map. By taking it transverse regular along M, we obtain a closed manifold M' together with a map $h : M' \to M$ of degree 1 such that $v(M') = h^*(k\xi + \beta) = k\xi' + h^*\beta$. Since $v(M) = k\xi + \varepsilon^{n-k}$, we have index $(M) = 0$. On the other hand,

$$\text{index } (M') = \text{index } (k\xi' + h^*\beta) = \text{index } h^*\beta = \text{index } (\beta) = 0.$$

It follows then from Wall [8] that we may assume h to be a homotopy equivalence. Naturally, we would like the converse to Theorem 3 to be true. Since we have

$$0 = \text{index } M = \text{index } (k\xi + \beta) = \text{index } \beta$$

we will always have index $(\beta) = 0$. However, there is an involution of a homotopy $S^l \times S^l$ such that the quotient manifold M has β not fiber homotopically trivial [10]. Then we may ask the weaker question, whether any $q^*\alpha$ with index $q^*\alpha = 0$ may appear. We do not know the answer to this question. In connection with this question, it may be shown that if $\beta = q^*\alpha$ is fiber homotopically trivial, then so is α.

V. The group $\Gamma(\gamma)$

In this section we generalize the h-cobordism groups Γ_l. We need a closed manifold P of dimensional l' and an l-plane bundle γ over P such that $|\gamma|$ is orientable. Pick an orientation of $|\gamma|$.

Define a class $\bar{\Gamma}(\gamma)$ by specifying that its members are the objects $A = (M(A), \iota_+(A), \iota_-(A))$ consisting of

(1) an oriented manifold $M(A)$

(2) an orientation-preserving embedding $\iota_+(A) : |\gamma| \to M(A)$

(3) an orientation-reversing embedding $\iota_-(A) : |\gamma| \to M(A)$

such that

$\iota_+(A)(|\gamma|) = M(A) - \iota_-(A)(P)$ and $\iota_-(A)(|\gamma|) = M(A) - \iota_+(A)(P)$.

If A, $B \in \bar{\Gamma}(\gamma)$, define $A \circ B \in \bar{\Gamma}(\gamma)$ as follows. $M(A \circ B)$ is obtained from

$$M(A) - \iota_+(A)(P) \cup M(B) - \iota_-(B)(P)$$

by identifying $\iota_+(A)(tx)$ with $\iota_-(B)(x/t)$, where $x \in S(\gamma)$ and $t > 0$. The orientation of $M(A \circ B)$ is that it inherits from $M(A) - \iota_+(A)(P)$. The embedding $\iota_-(A \circ B)$ is the composition

$$|\gamma| \xrightarrow{\iota_-(A)} M(A) - \iota_+(A)(P) \to M(A \circ B).$$

The embedding $\iota_+(A \circ B)$ is the composition

$$|\gamma| \xrightarrow{\iota_+(B)} M(B) - \iota_-(B)(P) \to M(A \circ B).$$

Then it is easy to check that there is an orientation-preserving diffeomorphism

$$\varphi : M(A \circ B) \circ C) \to M(A \circ (B \circ C))$$

such that $\varphi\ \iota_-(A \circ B) \circ C) = \iota_-(A \circ (B \circ C))$ and $\varphi \circ \iota_+((A \circ B) \circ C) = \iota_+(A \circ (B \circ C))$.

We reserve the symbol 1 for the element of $\bar{\Gamma}(\gamma)$ given by $M(1) = S(\gamma \times \varepsilon)$ las a manifold,

$$(1)(tx) = \frac{t}{1 + t^2/4}\,x, -\frac{1 - t^2/4}{1 + t^2/4} \quad \text{for } x \in S(\gamma), t > 0$$

$$(1)(tx) = \frac{t}{1 + t^2/4}\,x, \frac{1 - t^2/4}{1 + t^2/4}.$$

(These are stereographic projections.) Requiring $\iota_+(1)$ to be orientation-preserving determines the orientation of $M(1)$. Then it is easy to check that there is an orientation-preserving diffeomorphism $\varphi : M(A \circ 1) \to M(A)$ such that $\varphi \circ \iota_-(A \circ 1) = \iota_-(A)$ and $\varphi \circ \iota_+(A \circ 1) = \iota_+(A)$. There also is an orientation-preserving diffeomorphism $\psi : M(1 \circ A) \to M(A)$ such that the corresponding formulas hold. Define $A^{-1} \in \bar{\Gamma}(\gamma)$ by

$$A^{-1} = (-M(A), \iota_+(A^{-1}), \iota_-(A^{-1}))$$
$$\text{with } \iota_+(A^{-1}) = \iota_-(A) \text{ and } \iota_-(A^{-1}) = \iota_+(A).$$

In order to have an easy proof that $A \circ A^{-1}$ is somehow equivalent to 1, we add one condition to the objects of $\bar{\Gamma}(\gamma)$:

(4) There is an orientation-preserving diffeomorphism $\psi(A) : S(\gamma) \to S(\gamma)$ such that $\iota_-(A)(tx) = \iota_+(A)((1/t)\psi(A)(x))$ for $t > 0$ and $x \in S(\gamma)$. Now it is immediate that there is an orientation-preserving diffeomorphism $\varphi : M(A \circ A^{-1}) \to M(1)$ such that $\varphi \circ \iota_-(A \circ A^{-1}) = \iota_-(1)$ and $\varphi \circ \iota_+(A \circ A^{-1}) = \iota_+(1)$. Without condition (4), we would need a suitable kind of h-cobordism in place of an orientation preserving diffeomorphism φ. For our purpose however, we may settle for $\bar{\Gamma}(\gamma)$ whose objects satisfy (1), (2), (3), and (4). Now introduce an equivalence relation $\sim$ in $\bar{\Gamma}(\gamma)$ by setting $A \sim B$ if and only if there is an orientation-preserving diffeomorphism $\varphi : M(A) \to M(B)$ such that $\varphi \circ \iota_-(A) = \iota_+(B)$ and $\varphi \circ \iota_+(A) = \iota_+(B)$. Then the equivalence classes form a set $\Gamma(\gamma)$ (by abuse of language) which inherits a group structure from the operation $\circ$ on $\bar{\Gamma}(\gamma)$.

If $P = P_l$ and γ is the bundle of Section 3, we wish to determine the structure of $\Gamma(\gamma)$ more precisely. We begin with the group $k^0(T(\gamma)) \subset K\tilde{O}(T(\gamma))$ consisting of all reduced bundles with index zero. Then we define a map $\bar{\alpha} : \bar{\Gamma}(\gamma) \to k^0(T(\gamma))$ by observing that the map $\iota_+(A)$ induces a unique homotopy class of homotopy equivalences

$$M(A)/\iota_-(P_l) \xrightarrow{q} T(\gamma).$$

Then we have seen that there is a unique $\alpha \in k^0(T(\gamma))$ such that $k\xi \oplus q^*\alpha$ represents the reduced stable normal bundle of $M(A)$. Set $\bar{\alpha}(A) = \alpha$. Then we have seen that $\bar{\alpha}$ is onto, and it is easy to see that it factors through $\Gamma(\gamma)$ to define $\alpha : \Gamma(\gamma) \to k^0(T(\gamma))$.

LEMMA 7. *α is a homomorphism.*

Proof. It is enough to show that $\bar{\alpha}(AB) = \bar{\alpha}(A) + \bar{\alpha}(B)$. For any $A \in \bar{\Gamma}(\gamma)$, we have maps

$$T(\gamma) \xrightarrow{\iota_\pm} M(A)/\iota_\pm(E(\gamma)) \to M(A).$$

Since the maps ι_+, ι_- induced by ι_+ and ι_- are homotopy equivalences, we may compose their homotopy inverses with $M(A) \to M(A)/\iota_\mp(E(\gamma))$ to obtain $q_\pm : M(A) \to T(\gamma)$. Notice that $q = q_+$ above.

Writing $M(A) = E(\gamma) \cup_{\psi(A)} E(\gamma)$, we may assume $\psi(A)(*) = *$. Let p

be an arc in $E(\gamma)$ from P_l to $* \in S(\gamma)$. Then we may apply Theorem 1 to $A = P_l \cup p(I)$, $X = E(\gamma)$, $Y = S(\gamma)$ and $f = \psi(A)$ to obtain an exact sequence (noting $j = q_+$ and $j' = q_-$)

$$K\tilde{O}(M(A)) \xleftarrow{q_+^* + q_-^*} K\tilde{O}(T(\gamma)) + K\tilde{O}(T(\gamma)) \xleftarrow{\pi^* + \pi^*} K\tilde{O}(S(S(\gamma))).$$

Since each of q_+^* and q_-^* are monomorphisms, and the image of the right hand map is in the diagonal, it follows that $q_+^* = -q_-^*$.

Now consider $M(AB)$. A straightforward geometric construction supplies a map

$$\rho : M(AB) \to M(A) \cup_{P_l} M(B)$$

(where the identifying map is $\iota_-(B)\iota_+(A)^{-1} \mid \iota_+(A)(P_l)$) such that, up to homotopy, $q_-(A\,B) = q_-(A) \circ \rho$, and such that

$$\begin{aligned} v(M(AB)) &= k\xi + \rho^*(q_+(A)^*\bar{\alpha}(A) + q_+(B)^*\bar{\alpha}(B)) \\ &= k\xi + \rho^*(q_+(A)^*\bar{\alpha}(A) - q_-(B)^*\bar{\alpha}(B)) \\ &= k\xi + q_+(AB)^*\bar{\alpha}(A) - q_-(AB)^*\bar{\alpha}(B) \\ &= k\xi + q_+(AB)^*\bar{\alpha}(A) + q_+(AB)\bar{\alpha}(B) \\ &= k\xi + q_+(AB)^*(\bar{\alpha}(A) + \bar{\alpha}(B)), \end{aligned}$$

so $\bar{\alpha}(AB) = \bar{\alpha}(A) + \bar{\alpha}(B)$.

Thus we have an exact sequence

$$1 \to K \to \Gamma(\gamma) \xrightarrow{\alpha} k^0$$

of nonabelian groups, and a description of k^0 in terms of known invariants. Next, we seek a description of K. For this description we need a J-homomorphism

$$J : K\tilde{O}^{-1}(S(\gamma^+\varepsilon)) \to \pi^s_{2l+k}(T(k\xi)).$$

To define J as a map, recall that the elements of $K\tilde{O}^{-1}(S(\gamma + \varepsilon))$ corresponds to homotopy classes of maps $S(\gamma + \varepsilon) \to SO(n)$ for n large. Select a fixed isotopy class of bundle maps $\mathfrak{F}_0 : v(S(\gamma + \varepsilon))^n \to k\xi + \varepsilon^{n-k}$, which extends to $v(E(\gamma + \varepsilon))^n$. Since there is a map $E(\gamma + \varepsilon) \to S(\gamma + \varepsilon)$ such that $E(\gamma + \varepsilon) \to S(\gamma + \varepsilon) \subset E(\gamma + \varepsilon)$ is homotopic to the identity, we have

$$K\tilde{O}^{-1}(P_l) \approx K\tilde{O}^{-1}(E(\gamma + \varepsilon)) \subset K\tilde{O}^{-1}(S(\gamma + \varepsilon)),$$

so there will be exactly two classes—select one and stick to it. Then if

$$\alpha \in K\tilde{O}^{-1}(S(\gamma + \varepsilon))$$

corresponds to $\alpha : S(\gamma + \varepsilon) \to SO(n)$, let $J(\alpha)$ be the class of

$$\pi_{2l+n+n}\, T(k\xi + \varepsilon^{n-k} + \varepsilon^n)$$

represented by

$$v(S(\gamma + \varepsilon))^n + \varepsilon^n \xrightarrow{\mathfrak{F}_0 + \alpha} k\xi + \varepsilon^{n-k} + \varepsilon^n.$$

It is straightforward to check that J is then a homomorphism.

Now let $\pi^s_{2l+k}\, T(k\xi) \to \Lambda \to 0$ be the cokernel of J. Define a map

$$K \xrightarrow{\varphi} \Lambda$$

by sending $A \to \lambda$ (class of $(M(A), \mathfrak{F})$) where $\mathfrak{F}$ is any bundle isotopy class of bundle maps $v(M(A))^n \to k\xi + \varepsilon^{n-k}$ for n large—such an $\mathfrak{F}$ exists because $A \in K = \ker \alpha$. It is straightforward to check that α is well-defined, but we still have to check that φ is a homomorphism.

Let $\lambda(P_l) \subset M(A)$. Then there are two bundle homotopy classes of maps

$$v(M)^n \mid \iota_2(P_l) \to k\xi + \varepsilon^{n-k}$$

covering

$$P_l \xrightarrow{\iota_2} M(A) \to P$$

because $K\tilde{O}^{-1}(P_l) = Z_2$. But $K\tilde{O}^{-1}(P) \to K\tilde{O}^{-1}(P_l) \to 0$ is exact, and it factors through $K\tilde{O}^{-1}(M(A))$, so both bundle homotopy classes are restrictions of bundle homotopy classes $\mathfrak{F} : v(M(A))^n \to k\xi + \varepsilon^{n-k}$. Consequently, if $\mathfrak{G} : v(M(B))^n \to k\xi + \varepsilon^{n-k}$ is a bundle homotopy class, then there exist

$$\mathfrak{F} : v(M(A))^n \to k\xi + \varepsilon^{n-k} \quad \text{and} \quad \mathfrak{H} : v(M(A\cdot B))^n \to k\xi + \varepsilon^{n-k}$$

so that $(M(A\cdot B), \mathfrak{H})$ is $k\xi$-cobordant to $(M(A), \mathfrak{F})$ $(M(B), \mathfrak{G})$. Thus $\varphi(A\cdot B) = \varphi(A) + \varphi(B)$.

For the next step, set $G = \{A \mid M(A) = S(\gamma + \varepsilon)\}$. Then G is a normal subgroup of $\Gamma(\gamma)$, and in fact, a subgroup of K since $\alpha(A) = 0$ for $A \in G$. Even more is true: G is a subgroup of $\ker \varphi$. It will turn out that G is very nearly the same group as $\ker \varphi$.

THEOREM 4. *If l is even, but not of the form $2^j - 2$ and $l \geq 8$, then* $[\ker \varphi{:}G] \leq 2$.

Proof. Suppose $\varphi(A) = 0$. Then, setting $M = M(A)$, there exists a manifold E, together with $\mathfrak{G} : v(E)^n \to k\xi + \varepsilon^{n-k}$ such that $2E = M$. After a sequence of surgeries, we may assume $\pi_i(E) \approx \pi_i(P)$ for $i < l$.

We wish to factor $E \to P$ through P_l. It factors through P_{2l1}. by Poincaré duality

$$H^j(E) = H_{2l+1-j}(E, M),$$

and

$$H_{2l+1-j}(M) \approx H_{2l+1-1}(E) \to H_{2l+1-j}(E, M) \to H_{2l-j}(M) \approx H_{2l-j}(E)$$

for $2l + 1 - j < l$, i.e., $l + 1 < j$. Thus $H^j(E; Z_p) = 0$ for $j > l + 1$ and p

any prime (even or odd). Thus also $H^j(E; B) = 0$ for $j > l + 1$ and B any finite Z_2-module over Z_p. The fiber F of $P_{l+1} \to P_{2l+1}$ is l-connected, $\pi_{l+1}(F) = Z$, and $\pi_i(F)$ is finite for $l + 1 < i < 2l$. The pullback $H \to E$ of the fibration $P_{l+1} \to P_{2l+1}$ under $E \to P_{2l+1}$ has fiber F. The bundle of coefficients $(\pi_{l+1}(F))^{\sim}$ is Z with the trivial Z_2 action because Z_2 acts trivially on $\pi_{l+1}(P_{l+1})$. Consequently, the various obstructions to lifting $E \to P_{2l+1}$ to $E \to P_{l+1}$ are zero, and we may factor $E \to P$ through P_{l+1}.

Let $g : E \to P_{l+1}$ be the map found in that way. Assume g is regular at $x \in P_{l+1}$ and consider the framed submanifold $g^{-1}(x) \subset E$. Since $\pi_1(g^{-1}(x)) \to 0$, we know that $v(E) \mid g^{-1}(x)$ is trivial and index $g^{-1}(x) = 0$ if $l \equiv 0 \bmod 4$. For $l \equiv 2 \bmod 4$, W. Browder [11] has shown that $Arf(g^{-1}(x)) = 0$ provided $l \geq 8$ and $l \neq 2^j - 2$ for all j. Consequently, we may kill the lower and middle homotopy groups of $g^{-1}(x)$ by a sequence of ambient framed modifications in E.

We would like to realize these modifications through homotopies of g. We do so by regarding $1 \times g$ and $1 \times *$ as two embeddings of E in $E \times P_l$. Then since $\pi_i(E) = 0$ for $1 < i < l$ and $\pi_1(g^{-1}(x)) \to \pi_1(E)$ is the zero map, and since the modifications called for have degree $\leq l/2 + 1$, the method of [9] applies to supply a global isotopy modulo boundaries $\mathcal{G}_t : E \times P_l \to E \times P_l$ so that $\mathcal{G}_1 \circ (1 \times g)$ is transverse to $E \times *$, and the intersection of $\mathcal{G}_1 \circ (1 \times g)(E) \cap (E \times *)$ is Σ, the homotopy l-sphere obtained from $g^{-1}(x)$ by applying the foregoing modifications. Then if $\rho : E \times P_l \to P_l$ is the natural projection, $\rho \circ \mathcal{G}_t(1 \times g)$ is a homotopy from g to g', also regular at x, with $(g')^{-1}(x)$ a homotopy l-sphere. Thus we may as well assume $g^{-1}(x) = \sigma$ a homotopy l-sphere. Let V be a tubular neighborhood of Σ in E. Then the framing provides a diffeomorphism $V \approx \Sigma \times D^{l+1}$. But $\Sigma \times D^{l+1} \approx S^l \times D^{l+1}$. To perform surgery on $S^l \times 0$, embed $(E, M) \subset (R^{2l+1+k+r}, R^{2l+k+r})$ where r is large. We have

$$v(E) \xrightarrow{\mathcal{G}} k\xi_{l+1} \times r\varepsilon$$

where $\mathcal{G}$ is some pullback of $\mathcal{G} : v(E) \to k\xi \times \varepsilon$. Let $D^{l+1} \subset R^{2l+1+k+r}$ be a disc embedded so that is meets E only along S^l, with outward normal e_1, where e_1 is the field defined by $\mathcal{G}(e_1) =$ last vector of $r\varepsilon$. Then $\mathcal{G}$ supplies a bundle map

$$\mathcal{G}' : v(E_1)^{e_1} \mid S^l \to k\xi_{l+1} + (r - 1)\varepsilon \mid x = R^{k+r-1}.$$

If $\mathcal{G}'$ is regarded as a field of frames over S^l in $v(D^{l+1}) = D^{l+1} \times R^{l+r+k}$, it is a map $S^l \to V_{k+r-1, l+r+k}$, which is l-connected. Thus it extends over D^{l+1}. That is, the field $\mathcal{G}'$ extends to a field $\mathcal{G}''$ of $(k + r - 1)$-frames in $v(D^{l+1})$. Then thickening $\mathcal{G}''^1$ and rounding corners in the usual way provides an ambient $k\xi_{l+1}$-cobordism from $(E, \mathcal{G})$ to $(E_1, \mathcal{G}_1)$ such that $\varphi_1 : E_1 \to P_{l+1}$ misses x.

Thus, we may assume that $E \to P$ factors through P_l. The surgery above may have introduced a non-trivial $H_{l-1}(\hat{E})$, but since $\pi_{l-1}(P_l) = 0$, that may be surgered out.

Recapitulating, if $\mathfrak{F} : v(M)^n \to k\xi + \varepsilon^{n-k}$ represents zero, then there is

$$\mathcal{G} : v(E)^n \to k\xi_l + \varepsilon^{n-k}$$

such that

(1) $\partial E = M$
(2) $\mathcal{G} \mid v(M)^n$ is carried into $\mathfrak{F}$ under $k\xi_l + \varepsilon^{n-k}\ k\xi + \varepsilon^{n-k}$
(3) $\pi_i(E) \approx \pi_i(P_l)$ for $i < l$.

Now there are two cases (we wish to assume rank $\pi_l(E)$ is odd).

(I) Either rank $\pi_l(E)$ is odd, or there exists a closed $k\xi_l$-manifold $(X, \mathcal{K})$ of dimension $2l + 1$ such that $\pi_i(X) \approx \pi_i(P_l)$ for $i < l$ and rank $\pi_l(X)$ is even.

(II) Case (I) is false.

Assume Case (I). If $\pi_l(E)$ has even rank, replace E by the connected sum $E \# X$. The pullback $\overline{E \# X}$ of the double cover of P_l has $H_1(\overline{E \# X}) = Z$. A suitable 1-modification of $E \# X$ will kill this Z and introduce one in H_2. After a number of such modifications, we arrive at E_1 satisfying (1), (2), (3) with rank $\pi_l(E_1)$ odd.

We are now ready to apply Wall's theorem. For the Poincaré manifold in his hypothesis, we use the pair $(\mathfrak{M}, M)$ where $\mathfrak{M}$ is the mapping cylinder of $\varphi \mid M : M \to P_l$.

Claim. $(\mathfrak{M}, M)$ is an orientable Poincaré manifold.

Proof of Claim. Let η be the non-zero class of $H^2(P_l)$. Then

$$\tilde{H}(P_l, \text{pt.}) = \overline{Z_2[\eta]/\eta^{(l/2)+1}} = 0$$

where the overbar indicates the positive degree part. Also,

$$0 \to H^*(P_l) \xrightarrow{\varphi^*} H^*(M)$$

is exact. Let ζ be the non-zero class in $H^{l+1}(M)$ and let μ be a generator of $H^{2l}(M)$. Let $\delta : H^*(M) \to H^{*+1}(\mathfrak{M}, M)$. Then we have

$$H^i(\mathfrak{M}, M) = 0 \text{ for } i \leq l \text{ and for } i \text{ odd} < 2l + 1,$$

$$\delta\mu \text{ generates } H^{2l+1}(\mathfrak{M}, M) \approx Z,$$

$$\delta(\zeta\eta^i) \text{ generates } H^{l+1+2i}(\mathfrak{M}, M) \approx Z_2.$$

Let v generate $H_{2l+1}(\mathfrak{M}, M)$ so that $\delta\mu \cdot v = 1$. Then ∂v generates $H_2(M)$ and $v \cap \delta(\zeta\eta^i) = \partial(v \cap \zeta\eta^i) =$ generator of

$$H_{l-1-2i}(M) \xrightarrow[\varphi]{\approx} H_{l-1-2i}(P_l).$$

Thus, $(\mathfrak{M}, M)$ is a Poincaré manifold. Let $c : M \times I \to I$ be a collar neighborhood with $c(x, 0) = x$, let $E' = E - c(M \times (0, 1))$, and let $\psi : E' \to E$

be a diffeomorphism such that $\psi(c(x, 1)) = x$. Define $\mu : E \to \mathfrak{M}$ by

$$\mu(x) = \varphi(\psi(x)) \in P_l \subset \mathfrak{M} \text{ for } x \in E' \quad \text{and} \quad \mu(c(x, t)) = [t, \varphi(x)] \in \mathfrak{M}.$$

Then $\mu : (E, M) \to (\mathfrak{M}, M)$ is the identity on M and it is a map of degree 1 of Poincaré spaces. Since $\mathfrak{M} \to P_l$ is a homotopy equivalence, we may take $k\xi_l$ to be a bundle over $\mathfrak{M}$, which μ pulls back to the stable normal bundle of E. Stating Theorem 6.5 of [8] in the above notation, we have

THEOREM (Wall). *If* rank kernel $(H_l(\bar{E}, \bar{M}) \to H_l(\bar{\mathfrak{M}}, \bar{M}))$ *is even, then there exist μ-surgeries of l-spheres in* int (E) *modifying μ to a homotopy equivalence.*

Since μ-surgeries may be taken to be $k\xi_l$-surgeries, this theory tells us that we may assume that μ is a homotopy equivalence provided the rank of the kernel in question is even, and this is what happens in Case I.

On the covering space level we have

$$\begin{array}{c} 0 \to H_{l+1}(\bar{E}) \to H_{l+1}(\bar{E}, \bar{M}) \to H_l(\bar{M}) \to H_l(\bar{E}) \to H_l(\bar{E}, \bar{M}) \to 0 \\ \| \\ Z + Z \end{array}$$

from which we may conclude that rank $H_l(\bar{E}, \bar{M}) = \text{rank } H_l(\bar{E}) - 1$ so that it is even. On the other hand, the same exact sequence holds with $\bar{\mathfrak{M}}$ in place of $\bar{E}$, so rank $H_l(\bar{\mathfrak{M}}, \bar{M}) = \text{rank } H_l(\bar{\mathfrak{M}}) - 1 = 1 - 1 = 0$. Thus, the rank of the kernel in question is that of $H_l(\bar{E}, \bar{M})$, which is even.

Thus, we may assume that $P_l \subset E$ is a homotopy equivalence. A tubular neighborhood of P_l in E may be taken to be $E(\gamma + \varepsilon)$. Then $E - \text{int}\, E(\gamma + \varepsilon)$ is an h-cobordism from $S(\gamma + \varepsilon)$ to M. Since the Whitehead group of Z_2 is zero, that h-cobordism is trivial. In Case II, we have that rank $\pi_l(E)$ is even.

(1) Suppose that φ is a monomorphism. Then the manifold M must be $S(\gamma_k + \varepsilon)$. Suppose $\mathfrak{F}$ extends to

$$v(E')^n \xrightarrow{\mathcal{G}'} k\xi_l + \varepsilon^{n-k},$$

where $\pi_i(E') \approx \pi_i(P_l)$ for $i < l$ and rank $\pi_l(E')$ is odd. Then we may glue $(E, \mathcal{G})$ and $(E', \mathcal{G}')$ along $(M, \mathfrak{F})$ to obtain a $k\xi_l$-manifold X. On the covering space level,

$$0 \to H_{l+1}(\hat{E}) \oplus H_{l+1}(\hat{E}') \to H_{l+1}(\hat{X}) \to H_l(S^l \times S^l) \to H_l(\hat{E}) \oplus H_l(\hat{E}') \to H_l(\hat{X}) \to 0.$$

As in Case 1,

$$\text{rank } H_{l+1}(\hat{X}) = \text{rank } H_l(\hat{X}),$$
$$\text{rank } H_{l+1}(\bar{E}) = \text{rank } H_l(\bar{E}) - 1,$$
$$\text{rank } H_{l+1}(\bar{E}') = \text{rank } H_l(\bar{E}') - 1,$$

and $H_l(S^l \times S^l) \otimes Q \to H_l(\bar{E}) \otimes Q$, $H_l(\bar{E}') \otimes Q$ have the same kernel, of rank 1, so rank $H_l(\hat{X})$ is even, which contradicts the assumption that we are in Case II. Thus $\mathfrak{F}$ does not extend to E' as above, so $\mathfrak{F}$ does not extend to $E(\gamma_k \times \varepsilon)$, and so $\ker \bar{J} \neq 0$.

(2) Suppose $\varphi(A) = \varphi(B) = 0$ where $A \neq 1$ and $B \neq 1$.

Let $(E(A), \mathcal{G}(A))$ and $(E(B), \mathcal{G}(G))$ be $k\xi_l$-manifolds with

$$\pi_i(E(A)) \approx \pi_i(P_l) \quad \text{and} \quad \pi_i(E(B)) \approx \pi_i(P_l)$$

for $i < l$ and both ranks $\pi_l(E(A))$ and $\pi_l(E(B))$ even, and $\partial E(A) = M(A)$ and $\partial E(B) = M(B)$.

We have $P_l \to M(B) \to E(B) \to P_l$ homotopic to the identity so that $\pi_i(P_l) \approx \pi_i(P_l)$ for $i < l$, so $K\tilde{O}^{-1}(P_l) \approx K\tilde{O}(P_l)$ and thus $K\tilde{O}^{-1}(E(B)) \to K\tilde{O}^{-1}(P_l) \to 0$ is exact. Thus, $\mathcal{G}(B)$ may be altered so that we may take the "connected sum" of $E(A)$ and $E(B)$ along $P_l \subset M(A)$, $P_l \subset M(B)$, and so obtain $(E(A \cdot B), \mathcal{H})$ so that $\partial E(A \cdot B) = M(A \cdot B)$. On the covering space level we have,

$$0 \to H_l(S^l) \to H_l(\overline{E(A)}) \oplus H_l(\overline{E(B)}) \to H_l(\overline{E(A \cdot B)}) \to 0$$

so rank $\pi_l(E(A \cdot B))$ is odd and consequently, by an application of Wall's theorem as in Case I, we have $M(A \cdot B) = S(\gamma_k + \varepsilon)$.

Thus in Case II there is at most one non-trivial coset of G in $\ker \varphi$, so $[\ker \varphi : G] \leq 2$.

VI. Computation of $KO(T(\gamma))$

The purpose of this section is to indicate how $K\tilde{O}(T(\gamma))$ may be computed. Recall that γ is an l-plane bundle over P_l such that $\gamma + k\xi_l$ is stably equivalent to $v(P_l)$. Let $t = 2^{\varphi(l)} - 2l - 1 - k$ where $2^{\varphi(l)}$ is the order of the generator of (P_l). Then

$$\gamma + t = (2^{\varphi(l)} - l - 1 - k \ \xi_l$$

so

$$S^t T(\gamma) = T(2^{\varphi(l)} - l - k - 1)\xi_l$$

and so

$$K\tilde{O}(T(\gamma)) = K\tilde{O}^t(P_{t+2l}/P_{t+l-1}).$$

Thus we need the groups $K\tilde{O}^*(P_r)$, and the exact sequence

$$0 \to I^{t-1}_{t+l-1} \to K\tilde{O}^{t-1}(P_{t+l-1}) \to K\tilde{O}^t(P_{t+2l}/P_{t+l-1}) \to K\tilde{O}^t(P_{t+2l}) \to I^t_{t+2l} \to 0$$

which holds for $l > 6$ and $I^*_r = \text{Im } (K\tilde{O}^*(P) \to K\tilde{O}^*(P_r))$. The groups $K\tilde{O}^*(P_r)$ and I^*_r are known. See for example M. Fujii [11]. The ones we will need are:

$$K\tilde{O}^{-1}(P_r) = Z + Z_2, \quad r \equiv 3, 7 \ (8), \qquad K\tilde{O}^{-2}(P_r) = Z_2, \quad r \equiv 2, 4 \ (8),$$

$$K\tilde{O}^{-2}(P_r) = Z_2, \quad r \equiv 0, 6 \ (8), \qquad K\tilde{O}^{-3}(P_r) = Z, \quad r \equiv 1, 5 \ (8),$$

$$K\tilde{O}^{-4}(P_r) = Z_2\, \varphi(r + 4) - 3, \quad \text{all } r,$$

$$K\tilde{O}^{-5}(P_r) = Z, \quad r \equiv 3, 7 \ (8), \qquad K\tilde{O}^{-6}(P_r) = 0, \quad r \equiv 0, 6 \ (8),$$

$$K_{-}^{-6}(P_r) = Z_2, \quad r \equiv 2, 4 \ (8), \qquad K\tilde{O}^{-7}(P_r) = Z, \quad r \equiv 1, 5 \ (8),$$

$$I_r^0 = Z_2 \varphi(r), \qquad I_r^{-4} = Z_2 \varphi(r+4) - 3,$$

$$I_r^{-1} = Z_2, \quad I_r^{-5} = 0, \quad I_r^{-2} = Z_2, \quad I_r^{-6} = 0, \quad I_r^{-3} = 0, \quad I_r^{-7} = 0.$$

Now, inserting these values into the exact sequence above, we obtain $K\tilde{O}(T(\gamma))$, which we tabulate as follows:

$2l + k \equiv 0 \ (8)$:

$$K\tilde{O}(T(\gamma)) = Z + Z_2, \quad l \equiv 0, 2 \ (8), \qquad K\tilde{O}(T(\gamma)) = Z, \quad l \equiv 4, 6 \ (8).$$

$2l + k \equiv 2 \ (8)$:

$$K\tilde{O}(T(\gamma)) = Z.$$

$2l + k \equiv 4 \ (8)$:

$$K\tilde{O}(T(\gamma)) = Z + Z_2, \quad l \equiv 0, 2 \ (8), \qquad K\tilde{O}(T(\gamma)) = Z, \quad l \equiv 4, 6 \ (8).$$

$2l + k \equiv 6 \ (8)$:

$$K\tilde{O}(T(\gamma)) = Z.$$

Returning to the situation of section 4, let M be the quotient of a homotopy $S^l \times S^l$ by an involution, and let $q : M \to T(\gamma)$ be the collapse. Then it turned out that there is a unique $\alpha \in KO(T(\gamma))$ such that $v(M)$ is stably $k\xi + q^*\alpha$ where k is the type of the involution. It also turned out that index $(q^*\alpha) = 0$, so index $(\alpha) = 0$. But on $T(\gamma)$, index (α) is simply $cP_{l/2}(\alpha)[T(\gamma)]$ where $c \neq 0$ and $[T(\gamma)]$ is the generator of $H_{2l}(T(\gamma))$. Thus

$$\text{index} : K\tilde{O}(T(\gamma)) \to Z$$

is a homomorphism in this case. Moreover index is non-zero the free cyclic summand of $K\tilde{O}(T(\gamma))$, so $\alpha \in \ker$ (index) $= 0$ or Z_2. Thus we obtain two theorems by computation:

THEOREM 5. *The homomorphism $\alpha : \Gamma(\gamma)/G \to K\tilde{O}(T(\gamma))$ of Section 5 may be factored through Z_2, where $Z_2 \to K\tilde{O}(T(\gamma))$ is the unique epimorphism onto* kernel (index).

Notation. From now on we write $\alpha : \Gamma(\gamma)/G \to Z_2$. In the case that kernel (index) $= 0$, we take $\alpha = 0$.

THEOREM 6. *If $l \equiv 4, 6 \ (8)$ and M is the quotient of a homotopy $S^l \times S^l$ by an involution, then $v(M)$ is stably an even multiple of the canonical line bundle.*

Remark. This theorem is false for $l \equiv 0 \ (8)$.

VII. The classification

Let l be even, ≥ 8 and not $2^j - 2$ for any j.

Let $\rho : S^l \times S^l \# \Sigma \to S^l \times S^l \# \Sigma$ be an involution and let $M = S^l \times S^l \# \Sigma/\rho$. Then M is a reduced manifold of some type k, $0 \leq k < 2^{\varphi(l)}$, k even. Let γ be the l-plane bundle over P_l stably equivalent to $(2^{\varphi(l)} - l - 1 - k)\xi_l$, with Euler class a generator or zero as the case may be. Let $\Gamma(\gamma)$, K, G, φ, α and Λ have the same meaning as in Section 4. Then the elements of the group $\Gamma(\gamma)/G = H_k$ are in $1 - 1$ correspondence with the oriented diffeomorphism classes of reduced manifolds of type k. Thus, ρ determines a unique member of H_k, which in turn determines ρ up to weak equivalence. Thus the weak equivalence classes of involutions of homotopy $S^l \times S^l$'s with l as above are in $1 - 1$ correspondence with the elements of the graded group $\{H_0, H_2, \cdots, H_{2^{\varphi(l)}-2}\}$.

Thus, the object is to compute H_k in terms of known invariants. Our 'computation' consists of the following exact sequences

$$\begin{array}{ccccccccc}
 & & & & & & Z_2 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & 1 & \to & K/G & \to & \Gamma(\gamma)/G & \xrightarrow{\alpha} & Z_2 \\
 & & & & & & \downarrow \varphi & & & & \\
K\tilde{O}^{-1}(S(\gamma)) & \xrightarrow{J} & \pi_{2l+k}(T(k\xi_\infty)) & \xrightarrow{\lambda} & & & \Lambda & \to & 1 & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & 1. & & & &
\end{array}$$

Here φ and α denote the homomorphisms induced by φ and α above. Then the fact that α maps into Z_2 follows from Theorem 5 of Section 6. The fact that φ is an epimorphism follows immediately from Theorem 2, and the fact that the kernel of φ is an image of Z_2 follows from Theorem 6.

Remark. There appears to be no way at this level of detecting the elements of $\Gamma(\gamma)/G$ which corresponds to involutions of $S^l \times S^l$. However, the cofibration $T(k - 1)\xi_\infty \to Tk\xi_\infty \to S^k$ induces a map

$$\pi^s_{2l+k}\, T(k\xi_\infty) \xrightarrow{f} \pi^s_{2l+k}(S^k).$$

Let $\mathcal{J} \subset \Lambda^s_{2l+k}(S^k)$ be the image of the ordinary J-homomorphism. Then it is not hard to see that the elements of K/G corresponding to involutions of $S^l \times S^l$ are the elements of $\varphi^{-1}(\lambda(f^{-1}(\mathcal{J})))$.

Bibliography

1. M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc., vol. 11 (1961), pp. 261–310.
2. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Academic Press, New York, 1964.
3. M. Kervaire and J. Milnor, *Groups of homotopy spheres*, Ann. of Math., vol. 77 (1063), pp. 504–533.

4. R. LASHOF, *Poincaré duality and cobordism*, Trans. Amer. Math. Soc., vol. 109 (1963), pp. 257–277.
5. J. LEVINE, *A classification of differentiable knots*, Ann. of Math., vol. 82 (1965), pp. 15–50.
6. J. MILNOR, *A procedure for killing homotopy groups of differentiable manifolds*, Proc. Sympos. Pure Math., III, 1961.
7. C. T. C. WALL, *Surgery of non-simply connected manifolds*, Ann. of Math., vol. 84 (1966), pp. 217–276.
8. R. WELLS, *Cobordism groups of immersions*, Topology, vol. 5 (1966), pp. 281–294.
9. ———, *Modification of intersections*, Ill. J. Math., vol. 11 (1967), pp. 389–399.
10. ———, *Some examples of free involutions of homotopy $S^l \times S^l$'s*, Illinois J. Math., to appear.
11. W. BROWDER, *The Kervaire invariant for framed manifolds*, Ann. of Math., vol. 90 (1969), 157–186.
12. M. FUJII, *The KO groups of projective spaces*, Osaka Math. J., vol. 4 (1967), pp. 141–149.
13. R. DE SAPIO, *Differentiable structures on a product of spheres*, Notices Amer. Math. Soc., 1968, p. 628.

THE UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS