CARTAN FORMULAE

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In this note we present a method for determining the value of a higher order operation on a cup product $a \cup b$ in terms of operations on a cupped with operations on b. Indeed, up to certain cup products of lower order operations, we completely determine these formulae.

Previous attempts in this direction have been limited to secondary operations with Z_2 for coefficients, and the methods have been to use functional operations [1], [2], [8], (which greatly increase the indeterminacy) or to use cochain operations [5] which are cumbersome—and often lead to incorrect results. (L. Kristensen points out that the main theorem and illustrative example of [5] are, in fact, incorrect, though it is not too difficult to correct them.)

Our method, on the other hand, seems very direct and elementary. We regard the existence of a Cartan formula as equivalent to the existence of a special kind of mapping

$$f : X \# Y \to Z$$

for certain spaces X, Y, Z, and the decomposition of an operation on a cup product is obtained by finding $f^*(\Phi(\iota))$ where Φ is the operation in question and ι is a fundamental class in $H^*(Z)$.

The evaluation of f^* is now obtained inductively by considering fiberings

$$F \to E \to X, \qquad G \to E' \to Y, \qquad H \to E'' \to Z,$$

determining the fiber K in the map

$$E \# E' \to X \# Y$$

and studying the lifting problem

$$\begin{tikzcd} K \arrow[r, "\bar{f}|"] \arrow[d] & H \arrow[d] \\ E \# E' \arrow[r, "\bar{f}"] \arrow[d] & E'' \arrow[d] \\ X \# Y \arrow[r, "f"] & Z. \end{tikzcd}$$

If H is a product of Eilenberg-MacLane spaces then the lifting problem is easy to solve and modulo certain restrictions which are easily stated $\bar{f}\,|^*$ is arbitrary. Thus $\bar{f}^*$ is essentially determined and in this way we obtain our main results.

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These results are given in Section 3, as is the precise definition of a Cartan formula. Section one is devoted to notation and elementary remarks about the types of spaces and spectra which we need, while section two gives the results about K and $H^*(K)$ which we need. Sections four and five are devoted to applications. In four we prove a sharpened version of a result of O. Valdivia [8] (which in turn sharpens a formula of Adém and Gitler [3]), which at present seems to be the most useful Cartan formula for secondary operations. Section five on the other hand shows how we can obtain Cartan formulae for some third and higher order operations, by means of a specific tower for which the third order Cartan formula is completely determined.

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1. Preliminaries: the universal examples for stable conditions

1.1. A script letter $\mathfrak{X} = \{X_i\}$ will always denote an infinite sequence of spaces $X_1, X_2, X_3, \cdots$, satisfying the following.

(i) X_j is $j-1$-connected and there is a distinguished class $\bar{\iota} \in H^*(X_j, Z_p)$ (note that $\bar{\iota}$ may be a cohomology vector rather than a single class).

(ii) There is a homotopy equivalence

$$\lambda_j : X_j \to \Omega X_{j+1}$$

and $\lambda^*(\sigma(\bar{\iota}_{j+1})) = \bar{\iota}_j$ where $\sigma : H^i(X) \to H^{i-1}(\Omega X)$ is the "suspension" homomorphism,

(iii) $\bar{\iota}$ satisfies a set of stable conditions **R** (for example $Sq^k(\bar{\iota}) = 0$ or $\varphi(\bar{\iota}) = 0$ where φ is a vector of stable higher order operations). Moreover, given any cohomology vector $\bar{a} \in H^*(Y, Z_p)$ (Y a CW complex) where a satisfies **R** then there is an X_j and a map $(\bar{a}) : Y \to X_j$ so $(\bar{a})^*(\bar{\iota}) = a$.

We call $\mathfrak{X}$ a universal example for **R**, and will sometimes write $\mathfrak{X} = \mathfrak{U}(\mathbf{R})$.

For example **R** might be the condition $Sq^2a + Sq^4b = 0$, then $\mathfrak{U}(R)_j$ is the fiber in the map

$$\mathfrak{U}(\mathbf{R})_j \to K(Z_2, j) \times K(Z_2, j+2) \xrightarrow{Sq^4\iota_1 + Sq^2\iota_2} K(Z_2, j+4)$$

where $K(Z_2, j)$ is an Eilenberg-MacLane space.

A map $\theta : \mathfrak{U}(\mathbf{R}) \to \mathfrak{U}(\mathbf{S})$ is defined as a sequence of maps

$$\theta_i : \mathfrak{U}(\mathbf{R})_i \to \mathfrak{U}(\mathbf{S})_i$$

satisfying the consistency condition that the diagrams

1.1.2
$$\begin{tikzcd} \mathfrak{U}(\mathbf{R})_i \arrow[r, "\theta_i"] \arrow[d, "\lambda"] & \mathfrak{U}(\mathbf{S})_i \arrow[d, "\lambda"] \\ \Omega\mathfrak{U}(\mathbf{R})_{i+1} \arrow[r, "\Omega\theta_{i+1}"] & \Omega\mathfrak{U}(\mathbf{S})_{i+1} \end{tikzcd}$$

homotopy commute.

1.2. The cohomology of $\mathfrak{X}$ is given by

1.2.1 $$H^i(\mathfrak{X}; Z_p) = \lim_{n\to\infty} H^{n+i}(X_n ; Z_p)$$

(using $(\mathrm{adj}\ \lambda_n) : \Sigma X_n \to X_{n+1}$ to define the limit). Note that 1.1 (i), (ii) imply $H^i(\mathfrak{X}; Z_p) = H^{n+i}(X_n ; Z_p)$ for $n \geq 2i + 1$. There is a unique class $\bar{\iota} \in H^*(\mathfrak{X}, Z_p)$ (called the fundamental class of $\mathfrak{X}$) so that its restriction to $H^*(X_n ; Z_p)$ is just $\bar{\iota}_n$, and there is a unique way for $\mathfrak{A}(p)$ (the mod p Steenrod algebra) to act on $H^*(\mathfrak{X}; Z_p)$ so the restriction to $H^*(X_n ; Z_p)$ is an $\mathfrak{A}(p)$ map. Finally, given a map $\theta : \mathfrak{X} \to \mathfrak{Y}$ there is a well defined $\mathfrak{A}(p)$ map

$$\theta^* : H^*(\mathfrak{X}; Z_p) \to H^*(\mathfrak{Y}; Z_p)$$

defined as the limit of the θ_i^*.

1.3. By a fibering

$$\mathfrak{X} \xrightarrow{\Psi} \mathfrak{Y} \xrightarrow{\theta} \mathfrak{Z}$$

we mean maps θ, Ψ so that each sequence

1.3.1 $$X_i \xrightarrow{\Psi_i} Y_i \xrightarrow{\theta_i} Z_i$$

is a fibering. Passing to cohomology we have

LEMMA 1.3.2. *Let*

$$\mathfrak{X} \xrightarrow{\Psi} \mathfrak{Y} \xrightarrow{\theta} \mathfrak{Z}$$

be a fibering, then there is an $\mathfrak{A}(p)$ *map* t *of degree* $+1$

$$t : H^*(\mathfrak{X}; Z_p) \to H^*(\mathfrak{Z}; Z_p)$$

and the sequence

$$\cdots \xrightarrow{t} H^*(\mathfrak{Z}; Z_p) \xrightarrow{\theta^*} H^*(\mathfrak{Y}; Z_p) \xrightarrow{\Psi^*}$$
$$H^*(\mathfrak{X}; Z_p) \xrightarrow{t} H^*(\mathfrak{Z}; Z_p) \xrightarrow{\theta^*} \cdots$$

is exact.

(In stable dimensions the Leray-Serre spectral sequences reduce to exact sequences and the consistency condition 1.1.2 assures t is well defined on passing to limits.)

1.4. The smash product $\mathfrak{X} \# \mathfrak{Y}$ is defined as the set of pairs $X_i \# Y_j$ $(1 \leq i, j < \infty)$. There are inclusions

1.4.1 $I_1 : (\Omega X_i) \# Y_j \subset \Omega(X_i \# Y_j)$, $I_2 : X_i \# (\Omega Y_j) \subset \Omega(X_i \# Y_j)$

defined by $I_1(f, y)t = (f(t), y)$, $I_2(x, g)t = (x, g(t))$ and we set

1.4.2 $$\lambda^1_{i,j} = I_1(\lambda_i \# \mathrm{id}_j), \qquad \lambda^2_{i,j} = I_2(\mathrm{id}_i \# \lambda_j).$$

These mappings evidently satisfy the iterative condition

1.4.3 $$\lambda^2_{i+1,j} \circ \lambda^1_{i,j} = \lambda^1_{i,j+1} \circ \lambda^2_{i,j}.$$

1.5. The cohomology of $\mathfrak{X} \# \mathfrak{Y}$ is given by

1.5.1 $$H^i(\mathfrak{X} \# \mathfrak{Y}; Z_p) = \lim_{r,s\to\infty} H^{r+s+i}(X_r \# X_s; Z_p).$$

(Using (adj $\lambda^1_{i,j}$), (adj $\lambda^2_{i,j}$) the limit makes good sense.)

LEMMA 1.5.2.

$$H^{i+r+s}(X_r \# Y_s, Z_p) = H^i(\mathfrak{X} \# \mathfrak{Y}; Z_p)$$

for $i < \min(2r + s, 2s + r)$

(This is evident.)
1.5.2 implies

COROLLARY 1.5.3.

$$H^*(\mathfrak{X} \# \mathfrak{Y}; Z_p) \cong H^*(\mathfrak{X}; Z_p) \otimes H^*(\mathfrak{Y}; Z_p)$$

as an $\mathfrak{A}(p)$ *module and there is a fundamental class* $\bar{\iota}_{\mathfrak{X}\#\mathfrak{Y}}$ *in* $H^*(\mathfrak{X} \# \mathfrak{Y}; Z_p)$ *which under the isomorphism corresponds to*

$$\bar{\iota}_{\mathfrak{X}} \otimes \bar{\iota}_{\mathfrak{Y}}.$$

2. The smash product of two fiberings

2.1. Let

$$\mathfrak{F} \xrightarrow{\Psi} \mathcal{E} \xrightarrow{\theta} \mathcal{B}, \qquad \mathfrak{F}' \xrightarrow{\Psi'} \mathcal{E}' \xrightarrow{\theta'} \mathcal{B}'$$

be two fiberings. In this section we convert the map

$$\theta \# \theta' : \mathcal{E} \# \mathcal{E}' \to \mathcal{B} \# \mathcal{B}'$$

into a fibering, and evaluate the structure of the exact sequence corresponding to 1.3.2.

DEFINITION 2.1.1. *Let*

$$F \xrightarrow{\partial} E \xrightarrow{\Pi} B, \qquad F' \xrightarrow{\partial'} E' \xrightarrow{\Pi'} B'$$

be two Serre fiberings; then set

$$F(\Pi, \Pi') = E \# F' \cup_{F\#F'} F \# E'.$$

Let

$$p_1 : E \# F' \to F(\Pi, \Pi'), \qquad p_2 : F \# E' \to F(\Pi, \Pi')$$

$$i_1 : F \# F' \to E \# F', \qquad i_2 : F \# F' \to F \# E'$$

be the evident inclusions. Then we have the Meyer-Victoris sequence

2.1.2 $$\cdots \xrightarrow{\delta} H^*(F(\Pi, \Pi')) \xrightarrow{p_1^* \oplus p_2^*} H^*(E \# F') \oplus H^*(F \# E') \xrightarrow{i_1^* - i_2^*} H^*(F \# F') \xrightarrow{\delta}$$

and one obtains

LEMMA 2.1.3. $H^*(F(\Pi, \Pi'); Z_p)$ *is additively isomorphic to* $\mathcal{R} \oplus \mathcal{S}$ *where* $\mathcal{S}$

is kernel $(i_1^* - i_2^*)$ *and* $\mathcal{R}$ *is*

$$H^*(F \# F', Z_p)/\mathrm{im}(i_1^* - i_2^*)$$

with dimension augmented by one. (Over $\mathcal{A}(p)$ $\mathcal{R}$ *is a submodule and, using the projection*

$$\mathcal{R} \oplus \mathcal{S} \to \mathcal{S} \subset H^*(E \# F') \oplus H^*(F \# E'),$$

$\mathcal{S}$ *becomes a quotient* $\mathcal{A}(p)$ *module of* $H^*(F(\Pi, \Pi'))$.

2.2. Convert the map $\Pi \# \Pi' : E \# E' \to B \# B'$ into a fibering by regarding $B \# B'$ as the mapping cylinder M of $\Pi \# \Pi'$. $\Pi \# \Pi'$ then becomes the inclusion

$$I : E \# E' = 0 \times E \# E' \subset M.$$

$E \# E'$ is equivalent to $E_{E \# E', M}$ (the set of paths in M with initial point in $E \# E'$) and I is equivalent to the fibering

$$\rho : E_{E \# E', M} \to M$$

given by endpoint projection. There is an inclusion

$$j : F(\Pi, \Pi') \subset F_{E \# E, M}$$

(where $F_{E \# E, M}$ is the fiber of ρ) given by $j(x)t = (t, x)$ and we have

THEOREM 2.2.1. *Suppose* F, E, B *all n-connected,* F', E', B' *all m-connected* $(m, n > 2)$, *then* j *is a weak homotopy equivalence in dimensions less than* $k = \min(2n + m, 2m + n)$.

Proof. In dimensions less than $2n$, B is weakly equivalent to E/F, while in dimensions less than $2m$, B' is weakly equivalent to E'/F'. Thus

$$B \# B' \simeq_w E/F \# E'/F'$$

in dimensions less than k. On the other hand

$$E \# E'/F(\Pi, \Pi') = (E/F) \# (E'/F'),$$

so in the range of dimensions less than k we have the homology exact sequence

2.2.2
$$\xrightarrow{t} H^*(B \# B') \xrightarrow{(\Pi \# \Pi')^*} H^*(E \# E') \xrightarrow{i^*} H^*(F(\Pi, \Pi')) \xrightarrow{t} \cdots.$$

Now the theorem follows from the 5 lemma and the fact that the diagram

2.2.3
$$\begin{array}{ccc} F(\Pi, \Pi') & \xrightarrow{I\,|} & F \\ \downarrow & & \downarrow \\ E \# E' & \xrightarrow{I} & E_{E \# E', M} \\ \downarrow & & \downarrow \\ B \# B' & \xrightarrow{J} & M \end{array}$$

homotopy commutes where $J : B \# B' \to M$ is the standard inclusion.

2.3. Actually the proof of 2.2.1 shows more on close examination. Using the representation of $H^*(F(\Pi, \Pi'))$ given in 2.1.3 we have

LEMMA 2.3.1. *In 2.2.2, $t(r) = t(a) \otimes t(b)$ where $r \in \mathcal{R}$ and r can be written $\delta(a \otimes b)$. Also, the representation of $\mathcal{S}$ can be chosen so that*

$$t(s) = t(a) \otimes b'$$

if $s = a \otimes b \in H^*(F \# E')$ and $\Pi'^*(b') = b$. Similarly

$$t(s) = \pm a' \otimes t(b)$$

if $s \in H^*(E \# F')$ and $\Pi^*(a') = a$.

(Perhaps the easiest proof of this is to prove the dual statement for homology by seeing how the elements dual to $\mathcal{R}$ are built up, and then using 2.2.3. A similar argument will prove the statements for $\mathcal{S}$.)

Remark 2.3.2. Note that t is a monomorphism on $\mathcal{R}$, as well as an $\mathcal{A}(p)$ map. This fact when combined with 2.2.2, 2.3.1 is enough to determine the $\mathcal{A}(p)$ structure of $H^*(F(\Pi, \Pi'))$.

2.4. We convert the map $\mathcal{E} \# \mathcal{E}' \to \mathcal{B} \# \mathcal{B}'$ into a fibering by converting each of the maps $E_i \# E_j' \to B_i \# B_j'$ into a fibering with fiber $F'_{i,j}$ by the process outlined in 2.2. Then 2.2.1 in the limit gives the structure of the stable fiber (by 1.5.2), and 2.3.1 gives the map t in the exact sequence 1.3.2. Moreover, given a fibering

$$\mathfrak{F}'' \xrightarrow{\varphi''} \mathcal{E}'' \xrightarrow{\theta''} \mathcal{B}''$$

and maps $\bar{u} : \mathcal{E} \# \mathcal{E}' \to \mathcal{E}''$, $u : \mathcal{B} \# \mathcal{B}' \to \mathcal{B}''$ so the diagram

2.4.1
$$\begin{tikzcd}
\mathcal{E} \# \mathcal{E}' \arrow[r, "\bar{u}"] \arrow[d, "\theta \# \theta'"] & \mathcal{E}'' \arrow[d, "\theta''"] \\
\mathcal{B} \# \mathcal{B}' \arrow[r, "u"] & \mathcal{B}''
\end{tikzcd}$$

commutes we have the $\mathcal{A}(p)$ map of long exact sequences

2.4.2
$$\begin{tikzcd}
\to H^*(\mathcal{B}'') \arrow[r, "\theta''"] \arrow[d, "u^*"] & H^*(\mathcal{E}'') \arrow[r, "\Psi''^*"] \arrow[d, "\bar{u}^*"] & {} \\
\to H^*(\mathcal{B} \# \mathcal{B}') \arrow[r, "(\theta \# \theta')^*"] & H^*(\mathcal{E} \# \mathcal{E}') \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
H^*(\mathfrak{F}'') \arrow[r, "t"] \arrow[d, "\bar{u}|^*"] & H(\mathcal{B}'') \to \arrow[d, "u^*"] \\
H^*(\mathfrak{F} \# \mathcal{E}' \cup \mathcal{E} \# \mathfrak{F}') \arrow[r, "t"] & H^*(\mathcal{B} \# \mathcal{B}') \to
\end{tikzcd}$$

where the bottom row is the limit of 2.2.2. In particular knowledge of $u\,|^*$ and u^* determine $\bar{u}^*$ up to elements in the image of $(\theta \;\#\; \theta')^*$.

Remark 2.4.3. In order for 2.4.1, 2.4.2 to make good sense we require that a map

2.4.4 $$W : \mathfrak{X} \;\#\; \mathcal{Y} \to \mathcal{Z}$$

satisfy

$$W_{i,j} : X_i \;\#\; Y_j \to Z_{i+j} \quad \text{and} \quad \lambda W_{i,j} \simeq \Omega W_{i+1,j}\, \lambda^1_{i,j} \simeq \Omega W_{i,j+1}\, \lambda^2_{i,j}$$

and when talking of maps of type 2.4.4 we shall automatically assume this in the sequel.

3. Cartan formulae

3.1. A Cartan formula for a higher order operation Φ is a set of operations Φ_i', Φ_i'' (of varying order) such that

3.1.1 $$\Phi(a \cup b) = \Sigma\Phi'(a) \cup \Phi''(b)$$

modulo the indeterminacy on both sides. More exactly 3.1.1 means that whenever both sides are defined the intersection of the two sets $\Phi(a \cup b)$ and $\Sigma\Phi_i'(a) \cup \Phi_i''(b)$ is never empty. Of course implicit in 3.1.1 is the fact that the Φ_i' are defined on a and the Φ_i'' on b. Thus a formula of type 3.1.1 makes no sense unless we first specify the *kinds* of a, b for which we want it to hold. As a result we redefine the notion of Cartan formula as follows:

DEFINITION 3.1.2. A Cartan formula of type **R**, **S**, **T** is a mapping

$$\varphi : \mathcal{U}(\mathbf{R}) \;\#\; \mathcal{U}(\mathbf{S}) \to \mathcal{U}(\mathbf{T})$$

which satisfies the condition

$$\varphi^*(\bar{\iota}_{\mathbf{T}}) = \bar{\iota}_{\mathbf{R}} \otimes \bar{\iota}_{\mathbf{S}}.$$

Set $\mathbf{R} = \{\text{the } \Phi_i' \text{ are defined on } \bar{\iota}_{\mathbf{R}}\}$, $\mathbf{S} = \{\text{the } \Phi_i'' \text{ are defined on } \bar{\iota}_{\mathbf{S}}\}$ and $\mathbf{T} = \{\Phi \text{ is defined on } \iota_{\mathbf{T}}\}$. Then 3.1.1 is equivalent to saying there is a map

$$\varphi : \mathcal{U}(\mathbf{R}) \;\#\; \mathcal{U}(\mathbf{S}) \to \mathcal{U}(\mathbf{T})$$

with $\varphi^*(\bar{\iota}_{\mathbf{T}}) = \bar{\iota}_{\mathbf{R}} \otimes \bar{\iota}_{\mathbf{S}}$ and $\varphi^*\Phi(\bar{\iota}_{\mathbf{T}}) = \Sigma\Phi_i'\, \bar{\iota}_{\mathbf{R}} \otimes \Phi_i''\bar{\iota}_{\mathbf{S}}$. On the other hand, given φ satisfying 3.1.2, and suppose $\Phi(\bar{\iota}_{\mathbf{T}}) \not\simeq 0$; then

$$\varphi^*\Phi(\bar{\iota}_{\mathbf{T}}) = \Sigma\Phi_i'(\bar{\iota}_{\mathbf{R}}) \otimes \Phi_i''(\bar{\iota}_{\mathbf{S}})$$

gives a Cartan formula of type 3.1.1. The two definitions are thus equivalent.

3.2. The results of Section 2 can be used to obtain information about Cartan formulae for fibrations. Suppose

3.2.1 $$\mathfrak{F} \xrightarrow{\Psi} \mathcal{E} \xrightarrow{\theta} \mathcal{B}, \quad \mathfrak{F}' \xrightarrow{\Psi'} \mathcal{E}' \xrightarrow{\theta'} \mathcal{B}', \quad \mathfrak{F}'' \xrightarrow{\Psi''} \mathcal{E}'' \xrightarrow{\theta''} \mathcal{B}''$$

are fiberings with the fiber $\mathfrak{F}''$ a generalized Eilenberg-MacLane spectrum, then we have

THEOREM 3.2.2. *Suppose there is a Cartan formula*

$$\varphi : \mathfrak{B} \# \mathfrak{B}' \to \mathfrak{B}'';$$

then a necessary and sufficient condition that φ *lift to a Cartan formula* $\bar{\varphi} : \mathcal{E} \# \mathcal{E}' \to \mathcal{E}''$ *is that* $(\theta \# \theta')^* \varphi^* t(\bar{\iota}_{\mathfrak{F}''}) = 0$. *Moreover, given any element* $\bar{a} \in H^*(\mathfrak{F} \# \mathcal{E}' \cup \mathcal{E} \# \mathfrak{F}'; Z_p)$ *so* $t(\bar{a}) = \varphi^* t(\bar{\iota}_{\mathfrak{F}''})$ *there is a lifting* $\bar{\varphi}$ *with* $\varphi \mid^* (\bar{\iota}_{\mathfrak{F}''}) = \bar{a}$.

(The fiberings $F_i'' \to E_i'' \to B_i''$ are all principal so we can vary any lifting by a map into the fiber. This gives us the freedom to take $\bar{a} = \varphi \mid^* (\bar{\iota}_{\mathfrak{F}''})$. On the other hand the obstruction to lifting is exactly $\varphi^*(t(\bar{\iota}_{\mathfrak{F}''}))$.)

Passing to cohomology we have

COROLLARY 3.2.3. *Under the assumptions of* 3.2.2 *let there be given an* $\mathfrak{A}(p)$ *map*

$$u : H^*(\mathfrak{F}''; Z_p) \to H^*(\mathfrak{F} \# \mathcal{E}' \cup \mathcal{E} \# \mathfrak{F}'; Z_p)$$

so that $tu(\bar{\iota}_{\mathfrak{F}''}) = \varphi^* t(\bar{\iota}_{\mathfrak{F}''})$, *then there is a lifting* $\bar{\varphi}$ *so* $\bar{\varphi} \mid^* = u$. *Moreover,* u *determines* $\bar{\varphi}^*$ *up to elements in* $\operatorname{im}(\theta \# \theta')^*$.

3.3. To compute a Cartan formula of type 3.1.1 we proceed as follows. Recall that

$$H^*(\mathfrak{F} \# \mathcal{E}' \cup \mathcal{E} \# \mathfrak{F}; Z_p) = \mathfrak{R} \oplus \mathcal{S}$$

by 2.1.3 and from 2.3.2 the kernel of t is contained in $\mathcal{S}$. Hence, if we are interested in $\bar{\varphi}^*(\Phi)$ where $\Psi^*(\Phi)$ is nonzero it is enough to determine $u\Psi^*(\Phi)$ in $\mathcal{S}$ regarded as a quotient module of $\mathfrak{R} \oplus \mathcal{S}$. By 3.2.3 this determines $\bar{\varphi}^*(\Phi)$ up to something coming from $H^*(\mathfrak{B} \# \mathfrak{B}'; Z_p)$.

THEOREM 3.3.1. *Suppose* $t_1 : H^*(\mathfrak{F}) \to H^*(\mathfrak{B})$, $t_2 : H(\mathfrak{F}') \to H^*(\mathfrak{B}')$ *are both epimorphic away from* $\iota_{\mathfrak{B}}, \iota_{\mathfrak{B}'}$, (i.e., $\operatorname{im} H^*(\mathfrak{B})$ *in* $H^*(\mathcal{E})$ *is just* $\iota_{\mathcal{E}}$, $\operatorname{im} H^*(\mathfrak{B}')$ *in* $H(\mathcal{E}')$ *is* $\iota_{\mathcal{E}'}$); *then given a Cartan formula* $\bar{\varphi}$ *lifting* φ *we have*

$$\bar{\varphi}^*(\Phi) = \Phi' \otimes \iota_{\mathcal{E}'} + \iota_{\mathcal{E}} \otimes \Phi''$$

with zero indeterminacy.

Proof. The map $H^*(\mathfrak{B} \# \mathfrak{B}') \to H^*(\mathcal{E} \# \mathcal{E}')$ is zero away from the fundamental class so $\bar{\varphi}^*$ is determined by $\bar{\varphi} \mid^*$. Now note that $t(\mathfrak{R})$ is all of $H^*(\mathfrak{B} \# \mathfrak{B}')$ except

$$\iota_{\mathfrak{B}} \otimes H^*(\mathfrak{B}') \oplus H^*(\mathfrak{B}) \otimes \iota_{\mathfrak{B}'}$$

hence the only elements which matter are those of the form $f \otimes \iota_{\mathcal{E}'}$ or $\iota_{\mathcal{E}} \otimes g$, and $\bar{\varphi} \mid^* (\Phi)$ has restriction to the fiber of the form

3.3.2 $$\Sigma[\mathcal{P}^{K(j)}(f_j \otimes \iota_{\mathcal{E}'}) + \mathcal{P}^{L(j)}(\iota_{\mathcal{E}} \otimes g_j)]$$

for some f_j, g_j and elements $\mathcal{P}^{K(j)}$, $\mathcal{P}^{(L_j)}$ in $\mathfrak{A}(p)$. However, since $\theta^*(\iota_{\mathfrak{B}}) = \iota_{\mathcal{E}}$,

$\theta^*(\iota_{\mathcal{B}'}) = \iota_{\mathcal{E}'}$ it follows that $\mathcal{P}^K\iota_{\mathcal{E}} = 0$ for $\mathcal{P}^K$ of positive degree, and similarly for $\iota_{\mathcal{E}'}$. Hence 3.3.2 can be written

$$\Sigma(\mathcal{P}^{K(j)}f_j) \otimes \iota_{\mathcal{E}'} + \iota_{\mathcal{E}} \otimes (\mathcal{P}^{L(j)}g_j)$$

so 3.3.1 follows.

Remark 3.3.3. Theorem 3.3.1 generalizes a result of Adém [1], and is relevant, for example, to the cohomology operations defined by using an Adams resolution of the stable sphere.

4. The Cartan formula of Adém, Gitler and Valdivia

4.1. Corresponding to the relation

4.1.1 $$Sq^1Sq^{2n} + (Sq^3 + Sq^2Sq^1)Sq^{2n-2} + Sq^{2n}Sq^1$$

in $\mathcal{A}(2)$, there is a secondary operation of degree $2n$ which we denote by Φ_{2n}. It is defined on any class a which satisfies $Sq^1(a) = Sq^{2n-2}a = Sq^{2n}(a) = 0$, and its values are taken in

$$H^*(X)/Sq^1H^*(X) + (Sq^3 + Sq^2Sq^1)H^*(X) + Sq^{2n}H^*(X).$$

The universal example for Φ_{2n} are the fibers in the maps

4.1.2 $$E_m^{2n} \to K(Z_2, m) \xrightarrow{Sq^1, Sq^{2n-2}, Sq^{2n}} K(Z_2; m+1, m+2n-2, m+2n)$$

Thus we have the fibering

4.1.3 $$K(Z_2;\ m,\ m+2n-3,\ m+2n-1) \to E_m^{2n} \to K(Z_2;\ m)$$

and $t(\iota_m) = Sq^1\iota$, $t(\iota_{m+2n-3}) = Sq^{2n-2}\iota$, $t(\iota_{m+2n-1}) = Sq^{2n}\iota$. This determines the stable cohomology of $\mathcal{E}^{2n}$.

4.2. In general there is no Cartan formula of type

$$\mathcal{E}^{2r} \ \# \ \mathcal{E}^{2s} \to \mathcal{E}^{2(r+s)}.$$

However, there are circumstances in which Cartan formulae $\mathcal{E}' \ \# \ \mathcal{E}'' \to \mathcal{E}^{2n}$ are defined.

DEFINITION 4.2.1. $\mathcal{E}^{2n}(k)$ is the set of fibers

$$E_m^{2n}(k) \to K(Z_2;\ m)$$

$$\xrightarrow[Sq^1,\ Sq^{2k},\ Sq^{2k+2},\ \cdots,\ Sq^{2n-2},\ Sq^{2n}]{} K(Z_2;\ m+1, m+2k, \cdots m+2n).$$

We have mappings

4.2.2 $$\mu_{k,s} : \mathcal{E}^{2n}(k) \to \mathcal{E}^{2s}$$

for $s > k$ where $\mu_{k,s}\,|^*\ \iota_{2s-2} = \iota_{2s-2}$, $\mu_{k,s}\,|^*\ \iota_{2s} = \iota_{2s}$. $\mu_{k,s}^*(\Phi_{2s})$ will again be denoted by Φ_{2s} in what follows.

THEOREM 4.2.3. *There is a Cartan formula*

$$\varphi_k : \mathcal{E}^{2n}(k) \;\#\; \mathcal{E}^{2n}(n-k-1) \to \mathcal{E}^{2n}$$

and modulo the indeterminacy of 3.2.3 *we have*

$$\varphi^*(\Phi_{2n}) = \Phi_{2n} \otimes \iota + \Sigma\Phi_{2r} \otimes Sq^{2n-2r}\iota + \Sigma Sq^1\Phi_{2r} \otimes Sq^{2n-2r-1}\iota$$
$$+ \Sigma Sq^{2l+1}\iota \otimes Sq^1\Phi_{2(n-l-1)} + \Sigma Sq^{2l}\iota \otimes \Phi_{2(n-l)} + \iota \otimes \Phi_{2n}\,.$$

Proof. Evidently $Sq^{2n-2}(\iota \otimes \iota) = Sq^{2n}(\iota \otimes \iota) = Sq^1(\iota \otimes \iota) = 0$ under our hypothesis and 3.2.2 guarantees the existence of a Cartan formula $\bar{\varphi}$. Now note that we can choose $\bar{\varphi}\,|$ so that the projection of $\bar{\varphi}\,|^*\,(\iota_{2n})$ on $\mathcal{S}$ has the form

4.2.4
$$\Sigma(\iota_{2s} \otimes Sq^{2n-2s}\iota + Sq^1\iota_{2s} \otimes Sq^{2n-2s-1}\iota) + \Sigma(Sq^{2k+2j}\iota \otimes \iota_{2(n-k+j)}$$
$$+ Sq^{2k+1j-1}\iota \otimes Sq^1\iota_{2(n-k-j)}) + \iota_{2n} \otimes \iota + \iota \otimes \iota_{2n}$$

and we have a similar formula for $\bar{\varphi}\,|^*\,(\iota_{2n-2})$. From here on in the proof we look only at terms of the form $Sq^a\iota_{2k} \otimes Sq^b\iota$ for convenience, since the formulae are essentially symmetric. We have

$$Sq^1(\iota_{2n}) \to \Sigma Sq^1\iota_{2s} \otimes Sq^{2n-2s}\iota + \Sigma\iota_{2s} \otimes Sq^{2n-2s+1}\iota$$
$$(Sq^3 + Sq^2Sq^1)(\iota_{2n-2}) \to \Sigma\,(Sq^3 + Sq^2Sq^1)\iota_{2s-2} \otimes Sq^{2n-2s}\iota$$
$$+ \Sigma\iota_{2s} \otimes (Sq^3 + Sq^2Sq^1)Sq^{2n-2s-2}\iota.$$

Similarly

$$Sq^{2n}\iota_1 \to \Sigma Sq^{2s}\iota_1 \otimes Sq^{2n-2s}\iota + Sq^{2s-1}\iota_1 \otimes Sq^{2n-2s+1}\iota.$$

Adding the terms together we have

$$\Sigma\,(Sq^1\iota_{2s} + (Sq^3 + Sq^2Sq^1)\iota_{2s-2} + Sq^{2s}\iota_1) \otimes Sq^{2n-2s}\iota$$
$$+ \Sigma\iota_{2s} \otimes (Sq^1Sq^{2n-2s} + (Sq^3 + Sq^2Sq^1)Sq^{2n-2s-2})\iota$$
$$+ \Sigma Sq^1\iota_{2s-2} \otimes (Sq^3 + Sq^2Sq^1)Sq^1(Sq^{2n-2s-2})\iota$$
$$+ \Sigma\,(Sq^1Sq^{2s-2}\iota_1 + Sq^1(Sq^3 + Sq^2Sq^1)\iota_{2s-4} \otimes Sq^{2n-2s+1}\iota.$$

Now the second and third sums are contained in $\mathcal{R}$, so they can be ignored. Moreover,

$$Sq^1Sq^{2s-2}\iota_1 + Sq^1(Sq^3 + Sq^2Sq^1)\iota_{2s-4}$$

is the restriction of $Sq^1\Phi_{2s-2}$ to the fiber. Similarly

$$(Sq^1\iota_{2s} + (Sq^3 + Sq^2Sq^1)\iota_{2s-2} + Sq^{2s}\iota_1)$$

is the restriction of Φ_{2s} to the fiber and by symmetry the proof is complete.

Remark 4.2.5. Theorem 4.2.3 was first proved by O. Valdivia in his thesis [9] by use of functional operations. Consequently his indeterminacy is much greater, and the proof much longer. A formula somewhat more restrictive than this was given by Adém and Gitler in [3], but their indeterminacy, too, was much larger than that appearing here.

Remark 4.2.6. We can prove that there are choices of the Φ_{2s} so the formula of 4.2.3 becomes exact by using the fact that Φ_{2s} can be chosen to vanish identically on a $2s - 1$ class on which it is defined (see for example [4]).

5. A second application

5.1. Let E_n be the universal example for the stable conditions that (Sq^1, Sq^3) vanish on an n-dimensional cohomology class. There are fiberings

5.1.1 $$K(Z_2, n, n+2) \xrightarrow{\Psi_n} E_n \xrightarrow{\theta_n} K(Z_2; n)$$

with $t(\iota_n) = Sq^1(\iota)$, $t(\iota_{n+2}) = Sq^3(\iota)$ and we have

THEOREM 5.1.2. *As a module over* $\mathcal{A}(2)$ $H^*(\mathcal{E})$ *has four generators* $\iota, u, \bar{u}, v$ *with*

$$\Psi^*(u) = Sq^1(\iota_0), \Psi^*(\bar{u}) = Sq^1(\iota_2), \qquad \Psi^*(v) = Sq^3(\iota_2) + Sq^5(\iota_0).$$

Moreover, a basic set of relations over $\mathcal{A}(2)$ *is*

$$Sq^1\iota = Sq^3\iota = 0, \qquad Sq^1u = Sq^1\bar{u} = Sq^1v = 0, \qquad Sq^3v = Sq^5\bar{u} + Sq^7u.$$

Proof. $\mathcal{E}$ represents the first stage in an Adams resolution of

$$\lim_{n\to\infty} B_{U[2n, 2n+2\cdots\infty]}$$

by Strong's result [7], and 5.1.2 now follows by computing Tor^1, Tor^2 for a resolution of $\Psi(Sq^1, Sq^{01})$ {see §2 of [6] for further details on this kind of argument}.

5.2. We now have

THEOREM 5.2.1. *There is a Cartan formula*

$$\varphi : \mathcal{E} \not\!\!\times \mathcal{E} \to \mathcal{E}$$

and

$$\varphi^*(u) = u \otimes \iota + \iota \otimes u,$$
$$\varphi^*(\bar{u}) = \bar{u} \otimes \iota + Sq^2\iota \otimes u + Sq^2\iota \otimes u + \iota \otimes \bar{u},$$
$$\varphi^*(v) = v \otimes \iota + \bar{u} \otimes Sq^2\iota + Sq^2\iota \otimes \bar{u} + \iota \otimes v.$$

Proof. $\varphi\,|^*$ may be chosen so

$$\iota_0 \to \iota_0 \otimes \iota + \iota \otimes \iota_0,$$
$$\iota_2 \to \iota_2 \otimes \iota + \iota_0 \otimes Sq^2\iota + Sq^2\iota \otimes \iota_0 + \iota \otimes \iota_2.$$

Now, discounting elements in $\mathcal{R}$,

$$Sq^3(\iota_2) \to Sq^3\iota_2 \otimes \iota + Sq^1\iota_2 \otimes Sq^2\iota + Sq^2\iota \otimes Sq^1\iota_2 + \iota \otimes Sq^3\iota_2 + Sq^3\iota_0 \otimes Sq^2\iota + Sq^2\iota \otimes Sq^3\iota_0$$

$$Sq^5(\iota_0) \to Sq^5\iota_0 \otimes \iota + Sq^3\iota_0 \otimes Sq^2\iota + Sq^2\iota \otimes Sq^3\iota_0 + \iota \otimes Sq^5\iota_0$$

and adding we establish 5.2.1 up to an element coming from the base. However, the image of $H^*(K(Z_2, n), Z_2)$ in $H^*(\mathcal{E}_n; Z_2)$ is

$$\iota,\ 0,\ Sq^2\iota,\ 0,\ Sq^4\iota,\ 0,\ Sq^4Sq^2\iota,\ Sq^6\iota$$

in dimensions less than 7. Hence there is no cohomology coming from the base in dimensions 1, 3, 5 which are the dimensions of u, $\bar{u}$, and w respectively. The proof is complete.

5.3. 5.2.1 together with 3.2.2 implies that if H_n represents the next stage in the Adams resolution of $\lim B_{U[2n\cdots\infty]}$, i.e., kill u, $\bar{u}$, v, there is a Cartan formula

5.3.1 $$\bar{\varphi} : \mathcal{H} \,\#\, \mathcal{H} \to \mathcal{H}$$

lifting φ. Explicitly we have

THEOREM 5.3.2. *$H^*(\mathcal{H}, Z_2)$ has generators ι, $\bar{\nu}$, $\tilde{\nu}$, $\bar{\omega}$ over $\mathcal{A}(2)$ with*

$$\Psi^*\nu = Sq^1(\iota_0), \qquad \Psi^*(\bar{\nu}) = Sq^1(\iota_2), \qquad \Psi^*(\tilde{\nu}) = Sq^1(\iota_4)$$

$$\Psi^*(\bar{\omega}) = Sq^3\iota_4 + Sq^5\iota_2 + Sq^7\iota_0$$

and relations

$$Sq^1\iota = Sq^1\nu = Sq^1\bar{\nu} = Sq^1\tilde{\nu} = Sq^1\bar{\omega} = 0,$$

$$Sq^3\iota = 0, \qquad Sq^3\bar{\omega} = Sq^5\tilde{\nu} + Sq^7\bar{\nu} + Sq^9\nu.$$

(The proof, involving the calculation of $\mathrm{Tor}^3_{\Lambda(Sq^1, Sq^{01})}(Z_2, Z_2)$, is analogous to the proof of 5.1.2.)

From 5.3.2 we now obtain the tertiary Cartan formulae of

THEOREM 5.3.3.

$$\bar{\varphi}^*(\nu) = \nu \otimes \iota + \iota \otimes \nu,$$

$$\bar{\varphi}^*(\bar{\nu}) = \bar{\nu} \otimes \iota + \nu \otimes Sq^2\iota + Sq^2\iota \otimes \nu + \iota \otimes \bar{\nu},$$

$$\bar{\varphi}^*(\tilde{\nu}) = \tilde{\nu} \otimes \iota + \bar{\nu} \otimes Sq^2\iota + Sq^2\iota \otimes \bar{\nu} + \iota \otimes \tilde{\nu},$$

$$\bar{\varphi}^*(\bar{\omega}) = \bar{\omega} \otimes \iota + \tilde{\nu} \otimes Sq^2\iota + Sq^2\iota \otimes \tilde{\nu} + \iota \otimes \bar{\omega}$$

hold for some choice of $\bar{\varphi}$.

Proof.

$$\varphi\,|^*\,(\iota_0) = \iota_0 \otimes \iota + \iota \otimes \iota_0,$$

$$\varphi\,|^*\,(\iota_2) = \iota_2 \otimes \iota + \iota_0 \otimes Sq^2\iota + Sq^2\iota \otimes \iota_0 + \iota \otimes \iota_2,$$

$$\varphi\,|^*\,(\iota_4) = \iota_4 \otimes \iota + \iota_2 \otimes Sq^2\iota + Sq^2\iota \otimes \iota_2 + \iota \otimes \iota_4.$$

Now, as in the proof of 5.2.1, the formulae of 5.3.3 hold modulo possibly $Sq^7\iota \otimes \iota$ or $\iota \otimes Sq^7\iota$ but Sq^7 is in the indeterminacy of $\bar{\omega}$ and we can get rid of it without changing any of the lower images.

5.3.3 in turn shows there is a 4th order Cartan formula. The author does not know how much further this process will continue.

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