CONVEX-SUPPORTING DOMAINS ON SPHERES

BY

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A convex function on a riemannian manifold is a real-valued function whose restriction to every geodesic is convex. They occur in abundance and have many structural implications on manifolds with nonpositive curvature [3]. There is an extensive theory of convex functions on linear spaces [4], [9]. Convex functions form an important link in the analysis of open manifolds having nonnegative curvature [7] or nonnegative ricci curvature [8]. W. B. Gordon has found two applications to mechanics surprisingly opposite in consequence: a trajectory cannot stay in a compact domain supporting a function which is strictly convex with respect to the jacobi metric [5]; and (much deeper) if a potential function is convex and is strictly convex on a geodesic through a minimum point, then every neighborhood of that point has nontrivial closed trajectories ([6], see also [1]).

A sublevel set of a convex function is totally convex, that is, contains every geodesic arc whose ends are in the set. In Section 2, a theorem is formulated which gives necessary and sufficient conditions for a smooth filtration by totally convex sets to be the sublevel filtration of some smooth convex or strictly convex function. Use of the theorem is illustrated in Section 4 by constructing convex functions on subsets of spheres. In order to show that certain limitations on our choices of subsets are necessary, Section 3 considers the boundary behavior of convex functions.

Our manifolds will be assumed to be C^2. This is sufficient to obtain the usual properties of geodesics, and curvature plays no part in the generalities of this paper. By a *smooth* convex function we mean one which is C^2. In many contexts mere continuity is more natural. A *domain* is an open subset plus a subset of its boundary.

1. Convex functions

If M is a riemannian manifold, the function $f: M \to R$ is *convex* if for every geodesic $\gamma: [0, 1] \to M$, f satisfies

$$f\gamma(s) \leq (1 - s)f\gamma(0) + sf\gamma(1), \quad 0 \leq s \leq 1.$$

It is equivalent to say that f is continuous and satisfies the midpoint inequality ($s = \frac{1}{2}$). Speaking more geometrically, f is convex if the epigraph of f is totally convex, where the epigraph is defined to be the subset $\{(p, c) : c \geq fp\}$ of $M \times R$.

Strict convexity is defined by requiring the strict convexity inequality. In

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the case that f is smooth, convexity is equivalent to $(f \circ \gamma)'' \geq 0$ for all geodesics γ, or to positive semidefiniteness of the hessian $\nabla^2 f$ at each point. Strict convexity in this context ($\nabla^2 f$ positive definite) is a slightly stronger condition on a smooth function than the one above. (Consider x^4 on R.) Note that the convexity inequality still makes sense if $+\infty$ is allowed in the range of f; we will need to consider such extended convex functions in Section 3.

It is well-known that convex functions on R have monotonic one-sided derivatives everywhere. Thus we may define an action of tangent vectors on convex functions by taking the one-sided derivative along the corresponding geodesic. That is, if $x \in M_p$ (the tangent space at p) and f is convex, let γ be the geodesic having $\gamma'(0) = x$ and define

$$xf = \lim_{t \to 0^+} (f\gamma(t) - fp)/t.$$

A point p is a *critical point* of f if $xf \geq 0$ for every $x \in M_p$. The following properties of a convex function are easy consequences of the definition and known properties of convex functions on R:

(1) The sublevel sets are totally convex.

(2) $xf + (-x)f \geq 0$.

(3) A local minimum is a critical point.

(4) A critical point is a minimum point of the restriction of the function to any geodesic through the point. Hence a critical point is an absolute minimum of the restriction of the function to a normal coordinate neighborhood of the point.

(5) The set of critical points is closed.

(6) The critical set is totally convex. Hence also each of its components; there is no geodesic from one of these components to another; there is no geodesic from a critical point to a noncritical point having the same f-value.

(7) Each level set $f^{-1}c$ is decomposed into components, each of which is either a component of the critical set or a convex topological hypersurface, disjoint from the critical set, with f increasing on the outside.

By "L is a convex topological hypersurface" we mean that for every $p \in L$ there is an open half-space H of M_p and a neighborhood U of the origin such that $\exp(U \cap H)$ does not meet L and the projection of $L \cap \exp U$ onto $\exp(\partial H \cap U)$ is a homeomorphism. The *projection* here is orthogonal projection with respect to normal coordinates at p. In case L is smooth this means [10] that the second fundamental form with respect to an "outward-pointing" unit normal vector field is positive semidefinite. The link with convexity of f when L is the level hypersurface of a smooth function f is provided by the expression for the second fundamental form in terms of the hessian of f. The latter is a symmetric (0, 2) tensor field $\nabla^2 f$ defined by

$$\nabla^2 f(X, Y) = XYf - (\nabla_X Y)f$$

for vector fields X, Y on M. Letting $N = \operatorname{grad} f / |\operatorname{grad} f|$ be the normal field and X a vector field orthogonal to N, the second fundamental form of

L is given by

$$\langle X, \nabla_X N\rangle = -\langle \nabla_X X, N\rangle,$$

whereas

$$\nabla^2 f(X, X) = -(\nabla_X X)f = -|\operatorname{grad} f| \langle \nabla_X X, N\rangle.$$

Finally, we collect some properties, which will not be needed elsewhere in this paper, concerning first differentiability of convex functions. The *differential* df_p on each tangent space M_p is the positively homogeneous function defined by $df_p x = xf$. Then df_p is convex on the linear space M_p. (See, for example, [7] which shows equivalently that the epigraph of f has convex tangent cone at each boundary point (p, fp).) The following are consequences of the convexity of df_p:

(1) For $x \in M_p$ in a neighborhood of $0 \in M_p$,

$$(f \circ \exp)x - fp - df_p x = o(\|x\|).$$

(2) If γ is any C^1 path satisfying $\gamma'(0) = x \in M_p$, then

$$\lim_{t \to 0^+} (f\gamma(t) - fp)/t$$

exists and is equal to xf.

(3) If $df_p x_i + df_p(-x_i) = 0$ for d independent directions $x_i \in M_p$, where d = dimension M, then f is differentiable at p.

(4) f is differentiable almost everywhere on M. (A proof may be modelled after [4] or [9].)

2. Main theorem

We enquire when a family of hypersurfaces can be the level hypersurfaces of a convex function. The theorem below says, in particular, that a smoothly parametrized family of strictly convex hypersurfaces L_c is the family of level hypersurfaces of a smooth strictly convex function if and only if certain information about the relative positions of the L_c's is continuously bounded.

Now fix $t: M \to R$ to be a smooth function on a connected riemannian manifold M. We assume these necessary conditions: The only critical points are local minima. For every c in the range of t, $t^{-1}c$ is a union of subsets K_c and L_c, where K_c is in the critical set, t is everywhere noncritical on L_c, and L_c is a convex hypersurface with outward-pointing normal $\operatorname{grad} t / |\operatorname{grad} t| = N$. That is, the restriction of $\nabla^2 t$ to the tangent spaces of L_c is positive semidefinite.

The assumptions on t may be equivalently stated: The only critical points are local minima and the sublevel sets are totally convex. Verifying the equivalence in one direction total convexity of each sublevel set

$$M_b = \{p \in M: tp \le b\}$$

follows because if a geodesic arc γ with ends in M_b were to leave M_b, then at a point (necessarily noncritical) were $t\gamma$ first took its maximum c, γ would be tangent to L_c, contradicting convexity of L_c. The reverse implication follows because in a sufficiently small neighborhood of a noncritical point, total convexity of M_c implies that a geodesic tangent to L_c cannot enter the interior of M_c. We observe that for any smooth function whose critical points are local minima, each component of the critical set is also a component of the corresponding level set. If in addition the sublevel sets are totally convex, then the function must be constant on geodesics between critical points, so total convexity of the critical set is automatic.

A measure of how badly t fails to be convex is given by the negativeness of the following function defined on the values of c having nonvacuous L_c:

$$\mu(c) = \inf \{\tau(p), \tau(p) - \nabla^2 t(x, N_p)^2/dt(N_p)^2 \nabla^2 t(x, x) : p \in L_c, \\ x \text{ is a unit tangent to } L_c \text{ at } p \text{ and is orthogonal to the nullspace of } \nabla^2 t \text{ on } L_{cp}\},$$

where $\tau(p) = \nabla^2 t(N_p, N_p)/dt(N_p)^2$. (The only time the first, $\tau(p)$, of the pair of numbers comes into play is when the set of x's for the second is vacuous; that is, when $\nabla^2 t$ vanishes on L_{cp}.)

THEOREM 1. *There is a smooth function f*: range $t \to R$ *such that* $f' > 0$ *and* $f \circ t$ *is convex if and only if the following conditions are satisfied:*

(a) $\nabla^2 t(x, x) = 0$ *implies* $\nabla^2 t(x, N_p) = 0$ *for* $x \in L_{cp}$.

(b) *The function μ is bounded below by a continuous real-valued function defined on the range of t.*

Moreover, if the L_c are strictly convex (so (a) *is vacuously true) and the critical points of t are nondegenerate, then we may write "$f \circ t$ is strictly convex".*

Remarks. (1) Since $\nabla^2 t$ is semidefinite on L_{cp}, condition (a) means that the nullspace of $\nabla^2 t$ on L_{cp} lies in the nullspace of $\nabla^2 t$.

(2) The nullspace of $\nabla^2 t$ on L_{cp} is the *relative nullity space* for L_c in M, consisting of $x \in L_{cp}$ satisfying $\langle \nabla_x N, y \rangle = 0$ for all $y \in L_{cp}$. Geodesics in L_c whose tangents are relative nullity directions are also geodesics in M. L_c is *strictly* convex if its relative nullity spaces are all zero. Since

$$\nabla^2 t(x, N_p) = x \,|\, \text{grad}\, t \,|,$$

condition (a) says that any relative nullity direction x satisfies $x \mid \text{grad } t \mid = 0$. In other words, nearby L_c's remain at the same distance when we move in such directions.

(3) When $\nabla^2 t$ vanishes on L_c, then L_c is totally geodesic. If this happens on an open subset, then condition (a) implies that N is parallel along its integral curves in that subset, and hence M is locally a riemannian product of L_c and an interval.

(4) In applications it is often possible to verify (b) by showing that μ is continuous and finite.

(5) This theorem was discovered independently of the version for linear spaces [4]. We thank Ralph Alexander for pointing out this reference, as well as for suggestions concerning Section 3.

Proof of Theorem 1. Let X be a vector field orthogonal to N. For any $f:R \to R$ we calculate

$\nabla^2(f\circ t) = \nabla([f' \circ t]dt) = f'' \circ t(dt)^2 + f' \circ t\nabla^2 t$ on $X + uN$, $u \in R$:

$$\nabla^2(f \circ t)(X + uN, X + uN) = u^2(dt(N)^2 f'' \circ t + f'\circ t\nabla^2 t(N, N)) + 2uf'\circ t\nabla^2 t(N, X) + f'\circ t\nabla^2 t(X, X).$$

In order for this to be nonnegative when $f' > 0$ and $\nabla^2 t(X, X) \geq 0$, the polynomial in u must have a nonnegative coefficient of u^2, that is,

$$dt(N)^2(f'' \circ t + [f'\circ t]\tau) \geq 0,$$

and it must have nonnegative discriminant:

$$D(X, N) = f'' \circ t\, dt(N)^2\nabla^2 t(X, X) + f'\circ t[\nabla^2 t(N, N)\nabla^2 t(X, X) - \nabla^2 t(X, N)^2] \geq 0.$$

Now suppose that conditions (a) and (b) are satisfied. If $\nabla^2 t(X, X) = 0$, then condition (a) gives $D(X, N) = 0$. But $\tau \geq \mu$, so both requirements for the polynomial in u to be nonnegative are satisfied provided that $f'' + \mu f' \geq 0$. If $\nabla^2 t(X, X) > 0$, then removing the positive factor $dt(N)^2\nabla^2 t(X, X)$ from $D(X, N)$ reduces the second requirement to $f'' + \mu f' \geq 0$. By (b), μ has a smooth lower bound h defined on range t, so choose f to be a solution of the differential equation $f'' + hf' = 0$ such that $f' > 0$. Then the requirements for $\nabla^2(f \circ t)$ to be positive semidefinite are satisfied at all the noncritical points of t. But the critical points of t are local minima, so also the critical points of $f \circ t$. Hence $\nabla^2(f \circ t)$ is positive semidefinite everywhere. The case for strict convexity follows in the same way by taking h to be a strict lower bound for μ.

Now suppose that $\nabla^2(f \circ t)$ is positive semidefinite. Then if $\nabla^2 t(X, X) = 0$, $D(X, N)$ reduces to $-f' \circ t\nabla^2 t(X, N)^2$. Thus condition (a) follows from $D(X, N) \geq 0$. The requirements for $\nabla^2(f \circ t)$ to be positive semidefinite clearly show that $-f''/f'$ is lower bound for μ, so (b) also is true.

3. Boundary behavior of convex functions on hemispheres

Let f be a convex function on an open d-dimensional unit hemisphere H with boundary $(d - 1)$-sphere ∂H. Since a convex function on a finite interval has a limit in $R \cup \{+\infty\}$ at each end, the convex function f on H determines a *radial limit function* $\partial f:\partial H \to R \cup \{+\infty\}$, which gives the limits of f at ∂H along geodesics radiating from the center $o \in H$. A sequence p_i in H is said to converge to $q \in H$ *nontangentially* if the initial directions of geodesic arcs from q to p_i approach no directions tangent to ∂H. This is the same kind of limit considered in the theory of harmonic functions, but it seems there can be no logical implications either way: the intersection of harmonic and convex functions on a domain in the sphere consists of constant functions. Moreover, the subharmonic functions include the convex functions, but the theory of non-tangential limits is known not to extend to subharmonic functions.

The following lemma obviously holds in the larger context of convex functions on the interior of a manifold with boundary. In that context we define $\partial f(q)$ to be the limit of f along any geodesic segment ending in q and lying in the interior of M except for q. The proof of the lemma shows that this limit is independent of the choice of segment, since neither of two such segments is tangential.

LEMMA 1. *Along any sequence converging nontangentially to $q \in \partial H$, f has limit $\partial f(q)$. The minimum of the limiting values for f at q is $\partial f(q)$.*

Proof. We suppose that $\{p_i\}$ and $\{p_i'\}$ converge to q and satisfy $\lim f(p_i) > \lim f(p_i')$, and show that $\{p_i\}$ must converge tangentially. Otherwise, $\{p_i\}$ lies in a geodesic wedge W radiating from q with closure in $H \cup \{q\}$. Now let c be any number satisfying $0 < c < \pi$. By substituting a subsequence of $\{p_i'\}$ for $\{p_i'\}$ if necessary, we may arrange that the sequence of open geodesic arcs γ_i of length c, where γ_i starts at p_i' and passes through p_i, has a subsequence converging to some arc γ in the closure of W. The restriction of f to γ_i has slope

$$[f(p_i) - f(p_i')]/d(p_i, p_i')$$

from p_i' to p_i, and beyond p_i it remains at least that steep. Since this quotient approaches ∞, the value of f at a point of γ_i approaches ∞ as γ_i approaches γ. Since we have arranged for γ to lie in H, the contradiction follows by continuity of f.

LEMMA 2. *∂f is convex on any minimizing geodesic arc in ∂H.*

Proof. Suppose $\alpha \subset \partial H$ is an open great semicircle with endpoints q', q''. We choose in H some open 2-dimensional hemisphere H^2 bounding on α, and approach α by means of semicircles α_i in H^2 having endpoints q', q''. Since the movement of a point $p_i \in \alpha_i$ to its limit $q \in \alpha$ is not tangential to ∂H, the values $f(p_i)$ approach $\partial f(q)$. Thus ∂f on α is expressed as the limit of a sequence of convex functions, and is convex itself. Convexity of ∂f on the closure of α follows from the fact that $\partial f(q')$, $\partial f(q'')$ give convex extensions of f to the endpoints of α_i for each α_i, and hence give a convex extension of ∂f to the endpoints of α.

COROLLARY. *If ∂f is finite on an antipodal pair of points, then ∂f is constant.*

Proof. Suppose ∂f is finite on antipodal points q', q'' in ∂H. Then for any $q \in \partial H$, since ∂f is convex on the closed semicircle $q'qq''$, we have

$$\partial f(q) \leq \max\{\partial f(q'), \partial f(q'')\}.$$

Thus ∂f is a finite convex function on ∂H, hence is constant.

The following thorem uses the notion of convex subset. Our definition is as follows: A subset C of a riemannian manifold M is *convex* if for every pair of points in C, M contains exactly one minimizing arc between them and that arc lies in C.

THEOREM 2. *Let f be a real-valued convex function on an open hemisphere H with boundary sphere ∂H. Let C be the subset of ∂H on which the radial limit function ∂f is finite.*

(a) *If ∂f is not constant, then C is convex in ∂H.*

(b) *∂f is convex on the interior of C.*

(c) *Everywhere in ∂H except at included boundary points of C, ∂f is a continuous extension of f. At included boundary points of C, ∂f gives the limits of f along sequences not tangential to ∂H.*

Proof. (a) When ∂f is not constant, C does not contain antipodal pairs, so ∂H contains just one minimizing arc between any pair of points in C. By Lemma 2, this arc lies in C.

(b) This claim follows immediately from the preceding one and Lemma 2.

(c) First suppose that $\partial f(q) = \infty$ By Lemma 1, ∞ is the minimum limit value for f at q, so all limit values are ∞; that is, ∂f continuously extends f at q. Now let $q \in \partial H$ be a point to which f does not extend continuously Then q lies in C. If we apply the technique of the proof of Lemma 1 without the requirement that the arcs γ_i lie in a wedge, a subsequence will converge (possibly tangentially) to some arc in ∂H. However, we conclude that on every neighborhood (in S^d) of q the values of f are unbounded A sequence of geodesic arcs from o through points on which f is unbounded will have limiting arcs on which the radial limit is ∞. Thus ∂f will take the value ∞ in every neighborhood of q in ∂H; that is, q is a boundary point of C.

In view of Theorem 2 (a), we include some remarks about convex subsets of the sphere: If a subset C of the sphere S^d is convex under our definition, then its closure is the intersection of closed hemispheres. That is, C locally possesses supporting great hyperspheres at boundary points; such a local support element is easily seen to determine a closed hemisphere containing C. We define a *half-closed* hemisphere of S^d inductively to be one that intersects its boundary sphere in a half-closed $(d-1)$-hemisphere, where a half-closed 0-hemisphere is a point. Then any convex subset of S^d lies in a half-closed hemisphere. This is immediate by induction on d, since we know a convex subset of S^d lies in a closed hemisphere and has convex intersection with the boundary $(d - 1)$-sphere. The half-closed hemispheres are the maximal convex sets and the complement of a half-closed hemisphere is a half-closed hemisphere.

Theorem 2 may be interpreted in terms of the graph of f. Part (c) says that the graph of ∂f fails to be the boundary of the graph of f in $S^d \times (R \cup \{\infty\})$ only in its behavior at $C \cap \partial C$. It is an easy consequence of convexity that at $q \in C \cap \partial C$, f takes all limit values between $\partial f(q)$ and ∞. Thus the boundary of the graph of f consists of the graph of ∂f plus the vertical rays starting at $(q, \partial f(q))$, for all $q \in C \cap \partial C$.

Examples. If M is a riemannian manifold, the distance r from a point p is convex on every ball centered at p for which all of the sublevel sets of r are convex. The minimum radius of the largest such ball has been estimated as $\pi/2k$, where k^2 is an upper bound on curvature [2, p. 249]. This estimate is achieved on ordinary spheres. Thus r is convex on the hemisphere with pole p, but not any larger open set. Furthermore, r^2 and $(\tan r)^2$ are strictly convex and smooth. This shows that the radial limit for smooth strictly convex functions can be constant and either finite or infinite.

Another extreme is an example for which the set C of Theorem 2 is a single point. Choose rectangular coordinates $x, y, \cdots$ for the ambient euclidean space of the unit sphere S^d having the origin at the center of the sphere. Then by direct calculation it can be shown that the restriction of $(1 - y)/x$ to the hemisphere on which $x > 0$ is strictly convex. The level hypersurfaces are small $(d - 1)$-spheres tangent to the boundary at $(1, 0, \cdots)$ and the radial limit is infinite except at $(1, 0, \cdots)$ where it has value 0.

In Section 4 we construct examples for which C is the interior to any $(d - 2)$-sphere in ∂H. In the case of S^2 it is possible to visualize the level curves of strictly convex functions on a hemisphere H which realize all the possible types and permissible lengths of intervals in ∂H for the set C. The formal technique required is to apply Theorem 1 to families of level curves which are patched together to satisfy strict convexity and the desired edge behavior. Continuity of μ is assured for families which can be continued transversely across the boundary. In order to obtain an included endpoint q of C, we make the level curves enter q tangentially, locally congruent to the level curves of $(1 - y)/x$ on one side of its finite point. Since $(1 - y)/x$ is convex we know that μ is not forced to $-\infty$ near q.

4. Convex-supporting domains

We define a domain in a riemannian manifold to be *convex-supporting* if there is a real-valued function on the domain which is strictly convex along every geodesic arc lying in the domain. Use of Theorem 1 is illustrated below (Proposition 2) by the construction of a dense convex-supporting domain in S^d. Proposition 3 shows that S^d with a closed $(d - 1)$-hemisphere removed is not convex-supporting, even though it is a nested union of open convex-supporting domains. More generally, Proposition 3 shows that convex-supporting domains need not lie in maximal ones.

A different definition of convex-supporting domain is suggested by W. B. Gordon in [5], namely, that every compact subset be contained in an open convex-supporting domain (in our sense). Since this property is preserved under nested unions of open sets, maximal open sets of this type always exist. S^2 with a closed geodesic arc of length π removed is an example.

Letting x, y be as in Section 3, for $0 < \varepsilon < 1$ we obtain an open subset D_ε be deleting from S^d the portion of the great $(d - 1)$-sphere $y = 0$ determined by the inequality $-\varepsilon \leq x$. We assume $d > 1$.

PROPOSITION 2. *D_ε supports a strictly convex function.*

Proof. Let $0 \leq \alpha < \varepsilon$. For $-1 < c < -\alpha$ the hypersurface $L_c = t^{-1}c$ is defined to be the intersection of D_ε with the d-plane (if $c \leq -\varepsilon$) or folded d-plane (if $-\varepsilon < c$; the edge here is in the $(d-2)$-plane $y = 0, x = c$) which has equation

$$x = g(c) \mid y \mid + c.$$

The smooth function g is required to satisfy

$$g(c) = 0 \quad \text{for} \quad -1 \leq c \leq -\varepsilon;$$

$$g'(c) > 0 \quad \text{for} \quad -\varepsilon < c < -\alpha;$$

and

$$\lim_{c \to -\alpha} g(c) = \infty.$$

We want to show that Theorem 1 can be applied here to show that there is a smooth function f such that $f \circ t$ is strictly convex on D_ε. Thus we make the following observations:

(1) t has a single minimum point where $x = -1$; since $t = x + 1$ in a neighborhood ($x < -\varepsilon$), this minimum point is nondegenerate.

(2) Except at the points $\mid y \mid = 1$ and $x = -1$, we can take a coordinate system which includes x, y as two of the coordinate functions. Then the implicit determination of t by $g(t) \mid y \mid + t - x = 0$ gives

$$\partial t/\partial x = [g'(t) \mid y \mid + 1]^{-1} \neq 0.$$

At the points $\mid y \mid = 1$ we can still use x as a coordinate, and since y is critical there, $\partial y/\partial x = 0$, so the same formula prevails for $\partial t/\partial x$. Thus t has only $x = -1$ as a critical point.

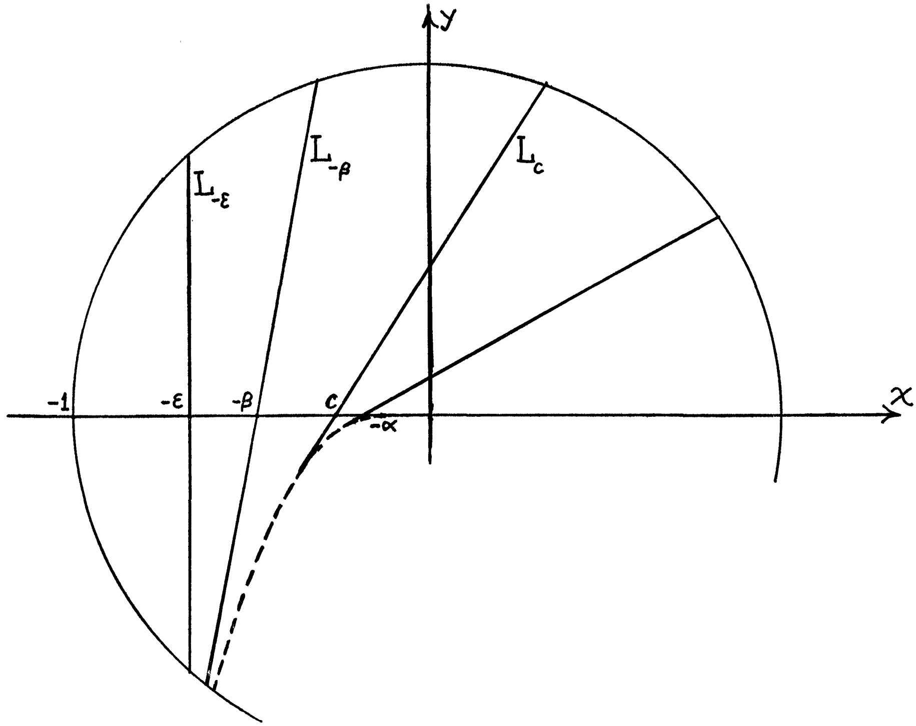
(3) Locally the L_c's are small $(d-1)$-spheres, so they are strictly convex.

(4) For $-1 \leq c \leq -\varepsilon$, $t = x = -\cos r$ where r is the distance in S^d from the point $x = -1$. But r is convex with strictly convex level hypersurfaces and $-\cos r$ is a strictly convex function of r, so t is strictly convex in this range. By continuity of $\nabla^2 t$, t is strictly convex on a neighborhood of $L_{-\varepsilon}$. Such a neighborhood will include all L_c for $-\varepsilon \leq c < -\beta$ for some $\beta < \varepsilon$. Hence $\mu(c) > 0$ for $c < -\beta$.

(5) It remains to verify that μ is smoothly bounded below for $c \geq -\beta$. We do this by observing that the restriction of t to the upper (or lower) hemisphere $y > 0$ can be extended smoothly in the region in question across the boundary into the lower hemisphere. We simply use the same formula without the absolute value signs on y: namely $g(t)y + t - x = 0$. By the implicit function theorem this equation defines t as a smooth function of x, y as long as $g'(t)y + 1 > 0$, that is, $y > -1/g'(t)$. Since $-1/g'(t)$ is negative for $t \geq -\beta$, this allows the extension across $y = 0$. (See figure.)

Now the $\mu(c)$ on the original L_c is the restriction of the defining formula to a compact part $(y \geq 0)$ of the new L_c, so that the infimum is actually a minimum and it is a continuous function of c.

Remark. The radial limit of the restriction of $f \circ t$ to a hemisphere must be infinite exactly on the complement of D_α.



PROPOSITION 3. *Every convex-supporting domain of S^d which contains some D_ε also lies in the image under a rotation of some D_ε, and is therefore not maximal.*

Proof. Suppose $D \subset S^d$ supports a nonconstant convex function f and contains D_ε. We denote the $(d-1)$-sphere $y = 0$ by ∂H. Then D contains the two open d-hemispheres H $(y > 0)$ and H' $(y < 0)$ bounded by ∂H. The corresponding radial limit functions on ∂H are denoted by ∂f and $\partial f'$. Let C be the subset of ∂H on which both ∂f and $\partial f'$ are finite. If $C = \partial H$, then both ∂f and $\partial f'$ would be finite everywhere and hence take a constant value c which is an upper bound for f. But then any interior point of $\partial H \cap D$ would be a maximum point for f, so f would be constant. Thus C is the intersection of two sets, one of which is convex. Hence there is a half-closed hemisphere in ∂H not intersecting C.

Observe that the closure of $\partial H \cap D$ is contained in C. Indeed, if p is a limit point of $\partial H \cap D$, then a geodesic arc through p perpendicular to ∂H may be approached by geodesic arcs in D, and the limit of convex functions is convex.

Now we show that there is a closed hemisphere outside of K, the closure of $\partial H \cap D$. Suppose that a half-closed hemisphere which does not meet K consists of the point where $x_1 = 1$, the half-circle on which $x_2 > 0$ in the x_1x_2-plane, the 2-hemisphere on which $x_3 > 0$ in the $x_1 x_2 x_3$-space, etc After a rotation of the $x_1 x_2$ axes which puts the new point $x_1 = 1$ at half the distance of the old one from K, we find that the new closed half-circle $x_2 \geq 0$ does not meet K. Then by a rotation of the $x_3 x_3$ axes we arrange that the new closed 2-hemisphere $x_3 \geq 0$ does not meet K. At the i-th stage we rotate the $x_i x_{i+1}$ axes so as to arrange that the new closed i-hemisphere $x_{i+1} \geq 0$ does not meet K.

Finally, if $\sin^{-1}\beta$ is the distance (in S^d) from K to such a closed hemisphere in ∂H, then D is contained in the image of D_β under a rotation.

References

1. M. S. Berger, *Periodic solutions of second order dynamical systems and isoperimetric variational problems*, Amer. J. Math., vol. 93 (1971), pp. 1–10.
2. R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.
3. R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., vol. 145 (1969), pp. 1–49.
4. W. Fenchel, *Convex cones, sets, and functions*, mimeographed notes, Princeton University, 1953.
5. W. B. Gordon, *Convex functions and harmonic maps*, Proc. Amer. Math. Soc., vol. 33 (1972), pp. 433–437.
6. ———, *A theorem on the existence of periodic solutions to hamiltonian systems with convex potential*, J. Differentiol Equations, vol. 10 (1971), pp. 324–335.
7. J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math., vol. 96 (1972), pp. 413–443.
8. ———, *The splitting theorem for manifolds of non-negative Ricci curvature*, J. Differential Geometry, vol. 6 (1971), pp. 95–118.
9. R. T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, 1970.
10. R. L. Bishop, *Infinitesimal convexity implies local convexity*, to appear.

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