COMBINATORIAL THEORY OF LOCAL COMPLEXES

BY

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1. Introduction

Peter Hilton has conjectured that combinatorial theorems, analogous to those of J. H. C. Whitehead [13] for CW-complexes, could be proven for local CW-complexes. The purpose of this article is to prove that this conjecture is correct with emphasis being placed on the two most important combinatorial results—the cellular approximation and homotopy theorems. Before proceeding any further with the introduction let us recall some definitions and results of localization in homotopy theory [3], [8], [9], [12].

In the sequel all of the spaces and maps are from the pointed homotopy category, the set P will always denote a subset of the primes, and $P' \subset \mathbf{Z} - \{0\}$ will be the multiplicative set generated by the primes not in P. A connected space Y, with the homotopy type of a CW-complex, is *nilpotent* if $\pi_1 Y$ is nilpotent and each $\pi_n Y$, $n \geq 2$, is a nilpotent $\pi_1 Y$-module. A nilpotent space Y is P-local if $\pi_n Y$ is P-local for all $n \geq 1$; and a map $e: X \to Y$ between nilpotent spaces *P-localizes* if Y is P-local and e has the universal property, i.e.,

$$e^{\#}: [Y, Z] \to [X, Z]$$

is a bijection for all P-local Z. For N the homotopy category of nilpotent spaces there exists a *P-localization functor* $L: N \to N$ and a *natural* transformation $e: 1 \to L$ such that $e(X): X \to LX = X_P$ P-localizes.

Since the n-sphere S^n, for $n > 0$, is a nilpotent space, it has a localization S_P^n called the P-local n-sphere. In fact, $S_P^n \simeq M(\mathbf{Z}_P, n)$, where $\mathbf{Z}_P$ is the P-localization of the integers $\mathbf{Z}$ and $M(\ ,\ )$ is the Moore space. The cone over S_P^n is referred to as the P-local $(n+1)$-cell, and it is denoted by e_P^{n+1}.

A CWP-complex, as defined in Section 4, is a Hausdorff space built inductively from a point by attaching P-local cells using maps of the P-local spheres into the lower "P-local skeletons" [12]. D. Sullivan has shown in [12] that if X is a 1-connected CW-complex with one zero cell, then there is a CWP-complex X_P and a "cellular" map $e: X \to X_P$ such that

(i) e induces a one-to-one correspondence between the cells of X and the local cells of X_P, and
(ii) e P-localizes.

Received November 2, 1973.

[1] Most of the results of this paper are contained in a portion of a doctoral dissertation [11] submitted to the University of Washington by the author and supervised by P. J. Hilton.

As a simple example, if $X = S^n \bigcup_\alpha e^m$, where $m > n$, then X_P and $e: X \to X_P$ are obtained as indicated in the following diagram

$$\begin{array}{ccccc} S^{m-1} & \xrightarrow{\alpha} & S^n & \longrightarrow & X \\ \Big\downarrow e & & \Big\downarrow e & & \Big\downarrow e \\ S_P^{m-1} & \xrightarrow{L\alpha} & S_P^n & \longrightarrow & X_P = S_P^n \bigcup\limits_{L\alpha} e_P^m. \end{array}$$

This result of Sullivan's generates motivation for studying the intrinsic properties of the CWP-complexes, since the localization of a 1-connected CW-complex is a CWP-complex.

One approach to proving the cellular approximation theorem for CWP-complexes would be to explicitly apply the classical one for CW-complexes, but this method generates at least two serious questions. Given a CWP-complex is it a CW-complex, and if so, how is the CW-structure related to the CWP-structure? It can be proven that a CWP-complex has the homotopy type of a CW-complex, but there is no guarantee that it has a CW-structure. To avoid this problem, one possible approach would be to redefine a CWP-complex in the following manner. First, suppose, for $n > 0$, S_P^n is a CW-complex having only one 0-cell, an infinite number of n-cells, and an infinite number of $(n + 1)$-cells, which is possible since $S_P^n \simeq M(\mathbf{Z}_P, n)$. Now use for S_P^n the CW-complex mentioned above and let the definition of a CWP-complex be as before, with the exception that the attaching maps are required to be cellular. With this new definition both of the questions are answered, but at the same time it now becomes clear that the classical cellular approximation theorem will not directly imply the corresponding theorem for CWP-complexes since the n-skeleton of a CWP-complex can contain both n- and $(n + 1)$-Euclidean cells. The reader is referred to 2.1, 4.1, and 4.5 for the definition of the n-skeleton of a CWP-complex. As an example, let $X = S_P^n \vee S_P^{n+1}$ and $f: X \to X$ be a cellular CW-complex map. Here the n- and $(n + 1)$-skeleta of X as a CWP-complex are S_P^n and X, respectively; whereas, the $(n + 1)$-skeleton of X as a CW-complex is not X and contains $(n + 1)$-Euclidean cells from both the n- and the $(n + 1)$-skeleta of X as a CWP-complex; and hence it cannot be concluded that $f(S_P^n) \subset S_P^n$.

For another point of interest, recall that the homotopy theorem follows from the approximation theorem for CW-complexes, since $X \times I$ is a CW-complex with $X \times \dot{I}$ a subcomplex. However, for X a CWP-complex, $X \times I$ does not have an obvious CWP-structure leaving the original structure on $X \times \dot{I}$ unchanged. Thus, the homotopy theorem for CWP-complexes does not follow as an immediate corollary from the CWP-approximation theorem. To avoid this awkwardness, we prove the main theorems for a general class of spaces with filtrations, which includes the CWP-complexes.

Sections 2 through 4 discuss the cellular approximation and homotopy theorems for filtrations and CWP-complexes, while Section 5 deals with questions related to the direct limit of the homotopy and homology groups over the finite subcomplexes of a CWP-complex. Also, in Section 5, the question of the existence of a CW-complex such that its localization is the given CWP-complex is answered.

2. Filtrations

Throughout this section, as well as sequel sections, all spaces are supposed path connected and Hausdorff.

DEFINITION 2.1. A family $\{X^k\}_{k=0}^{\infty}$ of closed subspaces of X is *a filtration of* X provided $X^k \subset X^{k+1}$ for all $k \geq 0$, $X = \bigcup X^k$, and X has the weak topology determined by $\{X^k\}$. X^k will be called the *k-skeleton of* X and X the *limit space*. X is said to be *filtered* by $\{X^k\}$, and the *dimension of* X is $\inf\{k\colon X^k = X\}$ and is denoted by $\dim X$. A filtration $\{A^k\}$ of $A \subset X$ is a subfiltration of $\{X^k\}$, if $A^k = A \cap X^k$ and A is closed. Given filtrations $\{X^k\}$, $\{Y^k\}$ for X, Y, a map $f\colon X \to Y$ is *cellular* if $f(X^k) \subset Y^k$ for all $k \geq 0$.

If $\{X^k\}$, $\{Y^k\}$ are filtrations then $\{X^k \times Y^k\}$ may not be a filtration of $X \times Y$, since the weak topology determined by the family of subspaces may not agree with the product topology. However, a theorem of J. H. C. Whitehead [5; p. 262] implies that $\{X^k \times I\}$ is a filtration of $X \times I$, since I is compact. Another example, which is important in the sequel, is the following. Let $\{A^k\}$ be a subfiltration of $\{X^k\}$. Then

$$\{X^{n-1} \times I \cup X^n \times \dot{I}\}_{n=0}^{\infty}$$

is a filtration of $X \times I$ and $\{A^{n-1} \times I \cup X^n \times \dot{I}\}_{n=0}^{\infty}$ is a subfiltration with its limit space $A \times I \cup X \times \dot{I}$, where X^{-1} and A^{-1} are empty sets.

It is obvious that in general a map between two filtered spaces cannot be approximated by a cellular map. So, some additional conditions must be imposed on the filtrations in order to guarantee such an approximation.

DEFINITION 2.2. The pair (X, A) is *simple* if A is a closed subset of X, (X, A) is n-simple for all $n > 1$, and $\iota_{\#}\colon \pi_1 A \twoheadrightarrow \pi_1 X$, where ι is the inclusion map. The *homology dimension of* X is

$$\inf\{k\colon H_l X = 0 \quad \text{for all } l > k\}.$$

DEFINITION 2.3. Let $\{X^k\}$ be a filtration of X. Then $\{X^k\}$ is

(1) *hep* if (X^k, X^{k-1}) has the homotopy extension property (HEP) for all $k \geq 1$,

(2) *partially hep* if (X, X^{k-1}) has the HEP for all $k \geq 1$,

(3) *connected* if (X, X^k) is k-connected for all $k \geq 1$,

(4) *s-connected* if X^k is s-connected for all $k \geq 0$,

(5) *simple* if (X, X^k) is simple for all $k \geq 1$,

(6) *relative CW* if (X^k, X^{k-1}) is a relative CW-complex for all $k \geq 1$,

(7) *an R-filtration* (resp., *a free R-filtration*) if, for each $k \geq 1$, X^k/X^{k-1} has homology dimension $\leq k$ and $\tilde{H}_k(X^k/X^{k-1})$ is an R-module (resp., a free R-module), here R is a commutative ring with an identity,

(8) *Moore* if $X^k/X^{k-1} \simeq M(G_k, k)$ for each $k \geq 1$, where G_k is a group.

PROPOSITION 2.4. *If $\{X^k\}$ is a filtration of X and $C \subset X$ is compact, then $C \subset X^k$ for some k.*

Proof. The proof is analogous to that given by J. H. C. Whitehead in [13] for CW-complexes. □

The following is a list of easily proven facts.

PROPOSITION 2.5.

(1) *A hep filtration is a partial hep filtration.*

(2) *A relative CW-filtration is hep and hence partially hep.*

(3) *A 1-connected filtration has a 1-connected limit space, and hence it is simple.*

(4) *A hep 1-connected Moore filtration is connected.*

(5) *If $\{A^k\}$ is a subfiltration of $\{X^k\}$ and $(X^k, X^{k-1} \cup A)$ has the HEP for all $k \geq 1$, then $\{X^k \cup A\}$ is a hep filtration of X.*

Completely analogous to the situation for CW-complexes, we can define a chain complex for each filtration $\{X^k\}$ of a space X, and hence obtain what we will call the *homology of the filtration.* For each $n \geq 0$, let $\mathscr{C}_n(X, X^0) = H_n(X^n, X^{n-1})$, where H_n is the singular homology functor; and let

$$\partial_n\colon \mathscr{C}_n(X, X^0) \to \mathscr{C}_{n-1}(X, X^0)$$

be the boundary map for the triple (X^n, X^{n-1}, X^{n-2}). It is then clear that $(\mathscr{C}(X, X^0), \partial)$ is a chain complex. The following theorem gives the relationship between the homology of the filtration $\{X^k\}$ and the singular homology of the pair (X, X^0) for a special case.

THEOREM 2.6. *If $\{X^k\}$ is a hep Moore filtration of X then*

$$H_*(\mathscr{C}(X, X^0), \partial) \cong H_*(X, X^0).$$

Proof. The proof is analogous to the argument given for CW-complexes. □

Also, we can associate with each filtration of a space a homology spectral sequence obtained from the exact couple

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

where $D = \{H_q X^p\}$, $E = \{H_q(X^p, X^{p-1})\}$, and the maps α, β, γ are induced from i_*, j_*, and ∂ in the exact sequence

$$\longrightarrow H_q X^{p-1} \xrightarrow{i_*} H_q X^p \xrightarrow{j_*} H_q(X^p, X^{p-1}) \xrightarrow{\partial} H_{q-1}(X^{p-1}) \longrightarrow.$$

Notice that, for a fixed q, $H_q(X) \cong \varinjlim H_q X^p$ and $H_q(X^p, X^0) = 0$ for small enough p. Hence, the spectral sequence converges with the limit being the bigraded module associated to the filtration $\{F^p\}$ of $H(X, X^0)$ defined by im $[H(X^p, X^0) \to H(X, X^0)] = F^p$.

3. Cellular approximation and homotopy theorems for filtrations

We begin by quoting the foundation stone of this section, a theorem by Sze-Tsen Hu [10].

THEOREM 3.1. *Let f: $(X, A) \to (Y, B)$, where (X, A) is a relative CW-complex and (Y, B) is simple. Then f can be deformed into B relative to A* (rel A), *if $H^k(X, A: \pi_k(Y, B)) = 0$ for all $k > 1$.*

THEOREM 3.2. (Cellular Approximation). *Let $\{A^k\}$, $\{X^k\}$, and $\{Y^k\}$ be filtrations of A, X, and Y with the following properties.*

(1) *$\{A^k\}$ is a subfiltration of $\{X^k\}$.*
(2) *$\{Y^k\}$ is simple.*
(3) *$(X^k, X^{k-1} \cup A^k)$ is a relative CW-complex for all $k \geq 1$.*
(4) *$H^r(X^k, X^{k-1} \cup A^k; \pi_r(Y, Y^k)) = 0$ for all $r > 1$ and $k \geq 1$.*

Then for any map f: $(X, X^0) \to (Y, Y^0)$ that is cellular on A, there exists g: $(X, X^0) \to (Y, Y^0)$, a cellular map homotopic to f rel $(X^0 \cup A)$.

Proof. We will construct a sequence of homotopies

$$H_k\colon (X, X^0) \times I \to (Y, Y^0), \quad k = 0, 1, \ldots,$$

relative to A (i.e., $H_k(a, t) = f(a)$, $(a, t) \in A \times I$) for each $k \geq 0$ satisfying the following conditions.

(1) $H_0(\ \ , 0) = f$.
(2) $H_k(\ \ , 1) = H_{k+1}(\ \ , 0)$ for $k \geq 0$.
(3) H_k is a homotopy rel X^{k-1} (i.e., $H_k(x, t) = H_{k-1}(x, 1)$ for all $(x, t) \in X^{k-1} \times I$).
(4) $H_k(X^k \times 1) \subset Y^k$.

Let $\bar{H}_0\colon X^0 \times I \cup A \times I \to Y$ be defined by $\bar{H}_0(\ \ , t) = f$ for all $t \in I$. $\bar{H}_0$ can be extended to $(X, X^0) \times I$ over f since $(X, X^0 \cup A)$ has the HEP (note 2.5, parts 1 and 5), Let H_0 denote this extension. Assume H_{n-1} has been constructed, $n \geq 1$. Let $g_{n-1} = H_{n-1}(\ \ , 1)$ and $\bar{g}_{n-1} = g_{n-1} \mid X^n$, $\bar{g}_{n-1}$: $(X^n, X^{n-1} \cup A^n) \to (Y, Y^n)$. By the hypothesis and 3.1 there exist h_n and a homotopy

$$\bar{H}_n\colon h_n \simeq \bar{g}_{n-1} \quad \text{rel } (X^{n-1} \cup A^n)$$

with $h_n(X^n) \subset Y^n$. Let $\tilde{H}_n\colon (X^n \cup A) \times I \to Y$ be defined by $\tilde{H}_n \mid X^n \times I = \bar{H}_n$ and $\tilde{H}_n(x, t) = f(x)$, $(x, t) \in A \times I$. Use the HEP of the pair $(X, X^n \cup A)$ to extend $\tilde{H}_n$ to H_n on $X \times I$ over g_{n-1}. Clearly, H_n has the desired properties.

Define $H\colon (X, X^0) \times I \to (Y, Y^0)$ as follows:

$$H(x, t) = \begin{cases} H_k\left(x, \dfrac{t - (1 - 1/k)}{1/k - 1/(k+1)}\right) & \text{for } 1 - \dfrac{1}{k} \le t \le 1 - \dfrac{1}{k+1},\ k \ge 1 \\ H_k(x, 1) & \text{if } x \in X^k. \end{cases}$$

$g = H(\ \ , 1)$ is the desired map. □

If for each $n \ge 1$, (A^n, A^{n-1}) is a subcomplex of (X^n, X^{n-1}), as relative CW-complexes, then $(X^n, X^{n-1} \cup A)$ is a relative CW-complex. So, condition (3) in 3.2 could be eliminated if the subfiltration $\{A^k\}$ satisfied this stronger condition.

COROLLARY 3.3. (Homotopy Theorem). *Let the hypothesis be the same as in* 3.2 *except for* (4), *which is replaced by*

(4′) $\tilde{H}^{r-1}(X^n/X^{n-1} \cup A^n; \pi_r(Y, Y^{n+1})) = 0$ *for all* $r > 1$ *and* $n \ge 0$.

Now, if $f, g\colon (X, X^0) \to (Y, Y^0)$ are cellular maps with homotopy F, and if F is cellular on $A \times I$ with respect to the filtration

$$\{A^{n-1} \times I \cup A^n \times \dot{I}\}_{n=0}^{\infty},$$

then there exists $G\colon (X, X^0) \times I \to (Y, Y^0)$, a homotopy of f and g, which is cellular with respect to the filtration $\{X^{n-1} \times I \cup X^n \times \dot{I}\}$. In fact,

$$G \simeq F \quad \text{rel}\,(A \times I \cup X \times \dot{I}).$$

Proof. Notice first that F is cellular on $A \times I \cup X \times \dot{I}$ with respect to the filtration $\{A^{n-1} \times I \cup X^n \times \dot{I}\}$, which is a subfiltration of $\{X^{n-1} \times I \cup X^n \times \dot{I}\}$. Also, observe that

$$(X^n \times I \cup X^{n+1} \times \dot{I}, (X^{n-1} \times I \cup X^n \times \dot{I}) \cup (A^n \times I \cup X^{n-1} \times \dot{I}))$$

is a relative CW-complex for all $n \ge 0$. Therefore, the conclusion follows from 3.2 after noticing that

$$H^r(X^n \times I \cup X^{n+1} \times \dot{I}, (X^{n-1} \times I \cup X^n \times \dot{I}) \cup (A^n \times I \cup X^{n+1} \times \dot{I}); \pi_r(Y, Y^{n+1}))$$

and

$$\tilde{H}^r(\Sigma(X^n/X^{n-1} \cup A^n); \pi_r(Y, Y^{n+1}))$$

are isomorphic. □

The following technical lemma will be used for specific filtrations that are considered later, in order to show they satisfy conditions (4) and (4′) of 3.2 and 3.3, respectively. For the rest of the paper R will denote the P-localization of the integers.

LEMMA 3.4. *Suppose that X has homology dimension $\le n$ (resp., $X = M(G, n)$), where $n \ge 1$, and that $\{G_k\colon k \ge 1\}$ is a family of abelian groups with*

$G_k = 0$ *for* $k \leq n$ *(resp.,* $G_n = 0$*). Then* $H^{n+1}(X, G_{n+1}) = 0$, *and hence* $H^k(X, G_k) = 0$ *for all* $k \geq 1$, *provided at least one of the following holds:*

(1) $\text{Ext}\,(H_nX, G_{n+1}) = 0$,
(2) $H_n(X, R)$ *is a free R-module and* G_{n+1} *is an R-module,*
(3) H_nX *is a free R-module and* G_{n+1} *is an R-module.*

Proof. If condition (1) holds, the result follows immediately from the universal coefficient theorem for cohomology. Now we will prove that

$$(3) \Rightarrow (2) \Rightarrow (1).$$

The universal coefficient theorem for homology yields $H_k(X, R) \cong H_kX \otimes R$, for each k, since R is torsion free; and if A is an R-module, then

$$A \cong A \otimes_R R \cong A \otimes R.$$

Thus (3) $\Rightarrow$ (2). For R-modules A and B with A free, $\text{Ext}\,(A, B) = 0$ [1]. Hence, (2) $\Rightarrow$ (1). □

The conclusion of Lemma 3.4 does not hold for arbitrary rings R. As an example, let $R = \mathbf{Z}/3$ and let (3) of 3.4 be satisfied. From the universal coefficient theorems and the hypothesis of the lemma,

$$H^{n+1}(X, G_{n+1}) \cong \text{Ext}_R\,(H_n(X, R), G_{n+1}) \oplus \text{Hom}_R\,(H_{n+1}(X, R), G_{n+1})$$

and $H_{n+1}(X, R) \cong \text{Tor}\,(H_nX, R)$. Since, H_nX is a free $\mathbf{Z}/3$-module, it is isomorphic to $\oplus_\Gamma \mathbf{Z}/3$. Hence,

$$H_{n+1}(X, R) \cong \oplus_\Gamma \mathbf{Z}/3 \quad \text{and} \quad \text{Hom}_R\,(H_{n+1}(X, R), G_{n+1}) \cong \Pi_\Gamma G_{n+1}.$$

Therefore, $H^{n+1}(X, G_{n+1}) = 0$ if and only if $G_{n+1} = 0$ or $H_nX = 0$.

At this point we give an application of 3.2 and 3.3, but first we need to set the stage.

DEFINITION 3.5. The 1-connected CW-complex K is said to be *normal* if it has a 1-connected CW filtration $\{K^n\}$ such that $K^1 = K^0 = *$, each K^n is a subcomplex, $H_r(K^n) = 0$ for all $r > n$ and $\iota_*\colon H_r(K^n) \cong H_r(K)$ for $r \leq n$, where $\iota\colon K^n \to K$ is the inclusion map.

THEOREM 3.6. *Every* 1-*connected CW-complex is homotopic to a normal complex.*

The reader is referred to [7; p. 53] for the proof of 3.6 and for further details of this decomposition. Such a decomposition yields a 1-connected, simple, connected, CW, Moore $\mathbf{Z}$-filtration (see Definition 2.3). Naturally, the question of "approximating" an arbitrary map between two normal complexes arises. In general this cannot be done, but if a condition is imposed on the homology of X and Y the question is answered in the affirmative (see also [4]).

THEOREM 3.7. *Let X, Y be 1-connected CW-complexes, and let $\{X^n\}$, $\{Y^n\}$ be normal homology decompositions of X, Y respectively.*

(1) *If* $\operatorname{Ext}(H_nX, H_{n+1}Y) = 0$ *for all* $n \geq 2$, *any map* $f\colon X \to Y$ *is homotopic to a cellular map* rel $*$.

(2) *If* $\operatorname{Ext}(H_nX, H_{n+2}Y) = 0$ *for all* $n \geq 2$, *then for every pair of homotopic cellular maps,* $f, g\colon X \to Y$, *there exists a cellular homotopy between them.*

Proof. (1) According to Theorem 3.2 it suffices to show that

$$H^r(X^n, X^{n-1}; \pi_r(Y, Y^n)) = 0 \quad \text{for all } r > 1 \text{ and } n \geq 1.$$

By the exact homology sequence for (Y, Y^n) and the definition of a normal homology decomposition, $H_r(Y, Y^n) = 0$ for $r \leq n$ and $H_{n+1}(Y, Y^n) \cong H_{n+1}(Y)$. From the Hurewicz Theorem, $\pi_{n+1}(Y, Y^n) \cong H_{n+1}(Y, Y^n)$. So, an application of 3.4 gives $H^r(X^n, X^{n-1}; \pi_r(Y, Y^n)) = 0$.

The proof of (2) is similar to (1). □

The following result gives two alternative conditions for the cohomology hypothesis in 3.2 and 3.3.

THEOREM 3.8. *Hypotheses* (4) *in* 3.2 *and* (4′) *in* 3.3 *can be replaced by either of the following:* $\{X^n \cup A\}$ *is a free R-filtration (resp., Moore free R-filtration),* $\{Y^n\}$ *is connected (resp.,* $\pi_n(Y, Y^n) = 0$ *for all* $n \geq 1$) *and* $\pi_{n+1}(Y, Y^n)$ *is an R-module for each* $n \geq 1$.

Proof. We will prove the theorem when $\{X^n \cup A\}$ is a free R-filtration and $\{Y^n\}$ is connected. The other proof is entirely analogous. First we will show that condition (4) in 3.2 is satisfied. From homology theory,

$$H^r(X^n, X^{n-1} \cup A^n; \pi_r(Y, Y^n)) \cong \tilde{H}^r(X^n \cup A/X^{n-1} \cup A; \pi_r(Y, Y^n)).$$

By the hypothesis, $X^n \cup A/X^{n-1} \cup A$ has homology dimension $\leq n$ with its nth homology a free R-module. Furthermore, $\pi_r(Y, Y^n) = 0$ for $k \leq n$ and $\pi_{n+1}(Y, Y^n)$ is an R-module for each $n \geq 1$. Hence, for each $n \geq 1$ an application of 3.4 shows that

$$H^r(X^n, X^{n-1} \cup A^n; \pi_r(Y, Y^n)) = 0 \quad \text{for every } r > 1.$$

That condition (4′) in 3.3 is satisfied follows from the work above and the fact that

$$\{X^{n-1} \times I \cup X^n \times \dot{I}) \cup (A \times I \cup X \times \dot{I})\}$$

is a free R-filtration (resp., a Moore free R-filtration). In fact, Z^{n+1}/Z^n is homeomorphic to $\Sigma(X^n \cup A/X^{n-1} \cup A)$, where

$$Z^n = (X^{n-1} \times I \cup X^n \times \dot{I}) \cup (A \times I \cup X \times \dot{I}).$$ □

In closing this section, we give two more results analogous to the ones for CW-complexes. They will be corollaries of the following theorem.

THEOREM 3.9. *Let $\{X^k\}$ be a relative CW filtration of X, (Y, B) a simple pair, and $H^r(X^k, X^{k-1}; \pi_r(Y, B)) = 0$ for all $r > 1$ and $k \geq 1$. Then any map $f: (X, X^0) \to (Y, B)$ can be deformed into B* rel X^0.

Proof. The proof is similar to that of 3.2. □

COROLLARY 3.10. *If X has a relative CW free R-filtration (resp., Moore free R-filtration) of dimension $\leq n$, (Y, B) is simple and n-connected (resp., $\pi_n(Y, B) = 0$), and $\pi_{n+1}(Y, B)$ is an R-module then every map $f: (X, X^0) \to (Y, B)$ can be deformed into B* rel X^0 *(X^0 is the 0-skeleton).*

COROLLARY 3.11. *If X has a relative CW free R-filtration and (Y, B) is both simple and n-connected for all $n \geq 1$, then any map*

$$f: (X, X^0) \to (Y, B)$$

can be deformed into B rel X^0.

4. Relative cellular complexes and CWP-complexes

The definition of a CWP-complex will be given in this section, and the cellular approximation and homotopy theorems will be proven. These theorems are essentially just corollaries of previous results.

DEFINITION 4.1. Let $\phi_\alpha : Y_\alpha \to X$, $\alpha \in \Gamma$, be a family of continuous maps. Then a space Z is said to be obtained from X by attaching cones on the Y_α if $Z = X \bigcup_\phi C(\vee Y_\alpha)$, where ϕ is the coproduct map $\langle \phi_\alpha \rangle : \vee Y_\alpha \to X$.

DEFINITION 4.2. *A relative cellular complex (X, A) modeled on Y and starting at $l > 0$* is a topological space X with a closed subspace A, and a filtration $\{(X, A)^k\}_{k=0}^{\infty}$ such that

(1) $(X, A)^k = A$ for all $0 \leq k \leq l - 1$, and

(2) for $k \geq l$, $(X, A)^k$ is obtained from $(X, A)^{k-1}$ by attaching cones on $\Sigma^{k-l} Y$, where $\Sigma^0 Y = Y$.

When no confusion can occur, we will abbreviate in one or more of the following ways. The k-skeleton $(X, A)^k$ will be denoted by X^k, X will denote (X, A); and l will not be mentioned.

Every relative CW-complex is an example of a relative cellular complex, which is modeled on S^0 starting at $l = 1$.

PROPOSITION 4.3. *If X is a relative cellular complex modeled on Y then the family of skeleta $\{X^k\}$ is a hep filtration of X. Furthermore, if Y is a CW-complex then $\{X^k\}$ is relative CW.*

Proof. See [11]. □

DEFINITION 4.4. A *subcomplex* (Z, B) of a relative cellular complex (X, A) modeled on Y is a relative cellular complex modeled on Y such that Z is a closed subspace of X and $(Z, B)^k = Z \cap (X, A)^k$ for all $k \geq 0$. Notice that the skeleton-filtration of (Z, B) is a subfiltration of the skeleton-filtration of (X, A).

THEOREM 4.5. *A relative cellular complex (X, A) modeled on a CW-complex Y has the homotopy type of a relative CW-complex (Z, A).*

Proof. See [11]. □

The following example shows that the cellular approximation theorem is not necessarily valid in the category of relative cellular complexes modeled on Y. In fact, Y may be taken to be a Moore space. In the example $A = *$ and Y is a Moore space with its group a solid ring [2]. However, if in addition to being solid we also require R to be torsion free, it will be proven that any map between any two 1-connected relative cellular complexes modeled on $M(R, 1)$ and starting at $l = 2$ can be approximated. Concerning such a ring, Bousfield and Kan have proven that $R \cong \mathbf{Z}_P$ for some P a subset of the primes [2].

Let $\bar{S}^1 = M(\mathbf{Z}/3, 1)$. $\Sigma^2\bar{S}^1$ and $\Sigma^3\bar{S}^1$ are cellular complexes modeled on $\bar{S}^1$ and starting at $l = 2$. Denote them by $\bar{S}^3$ and $\bar{S}^4$. Any cellular map from $\bar{S}^3$ to $\bar{S}^4$ has to be nonessential, since the 3-skeleton of $\bar{S}^4$ as a cellular complex modeled on $\bar{S}^1$ is the point $*$. So, if $[\bar{S}^3, \bar{S}^4] \neq 0$, then there are maps from $\bar{S}^3$ to $\bar{S}^4$ which are not homotopic to a cellular map. From the Puppe sequence,

$$S^3 \to \bar{S}^3 \to S^4 \to S^4 \to \bar{S}^4 \to S^5 \to \quad \text{where } \bar{S}^1 = S^1 \bigcup_3 e^2,$$

we obtain the following exact sequence

$$[S^3, \bar{S}^4] \leftarrow [\bar{S}^3, \bar{S}^4] \leftarrow [S^4, \bar{S}^4] \leftarrow [S^4, \bar{S}^4] \leftarrow [\bar{S}^4, \bar{S}^4] \leftarrow [S^5, \bar{S}^4] \leftarrow.$$

$\pi_3(\bar{S}^4) = 0$ by the cellular approximation theorem for CW-complexes, $\pi_4(\bar{S}^4) \cong \mathbf{Z}/3$ by the Hurewicz theorem, $[\bar{S}^4, \bar{S}^4] \neq 0$, since $\bar{S}^4$ is not contractable, and $[S^5, \bar{S}^4] = 0$ by [6; p. 133]. Therefore,

$$0 \neq [\bar{S}^4, \bar{S}^4] \rightarrowtail \mathbf{Z}/3 \to \mathbf{Z}/3 \twoheadrightarrow [\bar{S}^3, \bar{S}^4]$$

implies $[\bar{S}^3, \bar{S}^4] \cong \mathbf{Z}/3$.

DEFINITION 4.6. Let P be a subset of the primes. A *relative* CWP-*complex*, (X, A), is a relative cellular complex modeled on S_P^1 starting at $l = 2$, with the space A being 1-connected (A will often be a single point).

Even though S_P^1 is not 1-connected, the mapping cone of any map $Q \to A$, with Q connected, is 1-connected, so we do not leave the category of 1-connected spaces in this construction.

PROPOSITION 4.7. *The skeleton-filtration, $\{X^n\}$, of a relative CWP-complex is a hep,* 1-*connected, connected, simple, Moore, relative CW, free $\mathbf{Z}_P$-filtration.*

Proof. This proof follows from previous results and the fact that $X^n/X^{n-1} \cong \bigvee S_P^n$. □

From 2.6, 4.5, and 4.7 every relative CWP-complex has the homotopy type of a relative CW-complex and the homology of its skeleton-filtration coincides with its singular homology.

THEOREM 4.8. *The homology of a relative CWP-complex is P-local.*

Proof. From the remark above, its homology can be computed using the chain complex of the skeleton-filtration. But since this chain complex has P-local groups the homology is P-local [1]. □

THEOREM 4.9. (Cellular approximation theorem for relative CWP-complexes). *Any map f: $(X, A) \to (Y, B)$ between relative CWP-complexes that is cellular on the subcomplex $(\bar{X}, \bar{A})$ is homotopic* rel $(\bar{X} \cup A)$ *to a cellular map*

$$g\colon (X, A) \to (Y, B).$$

Proof. It suffices to show that the properties (1) through (4) are satisfied in Theorem 3.2. From the definition of a subcomplex it follows that the skeleton-filtration of $(\bar{X}, \bar{A})$ is a subfiltration of the skeleton-filtration of (X, A). According to 4.7, $\{Y^n\}$ is simple. And $(X^k, X^{k-1} \cup \bar{X}^k)$ is obviously a relative CW-complex for all $k \geq 1$, since $(\bar{X}, \bar{A})$ was a subcomplex of (X, A). We will now use Theorem 3.8 to prove property (4) of 3.2. First observe that the skeleton-filtration of $(X, A \cup \bar{X})$ is a Moore free $\mathbf{Z}_P$-filtration. By 4.7, the filtration of (Y, B) is connected and hence $\pi_n(Y, Y^n) = 0$ for all $n \geq 1$. Also,

$$\pi_{n+1}(Y, Y^n) \cong H_{n+1}(Y, Y^n),$$

and hence it is P-local by 4.8 for all $n \geq 1$. □

THEOREM 4.10. (Homotopy theorem for relative CWP-complexes.) *Let*

$$f, g\colon (X, A) \to (Y, B)$$

be cellular maps between relative CWP-complexes. Furthermore, suppose $F\colon f \simeq g$ such that F is cellular on $(\bar{X}, \bar{A}) \times I$, where $(\bar{X}, \bar{A})$ is a subcomplex of (X, A). Then there exists a cellular homotopy G between f and g such that $G \simeq F$ relative to

$$(\bar{X} \times I \cup X \times \dot{I}).$$

Proof. The proof follows from 3.3 and 3.8. □

Of course, there are analogous corollaries to 3.9 and 3.10 for the relative CW-complexes.

Remark 4.11. With these tools now available for CWP-complexes the analogs of theorems presented in J. H. C. Whitehead's work [13] can be proven. In particular, an exact couple can be constructed which gives rise to a spectral sequence with an edge being the P-local version of Whitehead's "A certain exact sequence" [14].

5. Special CWP-complexes

We deal with the following questions in this section.

(1) Are the homotopy (resp., homology) groups of a CWP-complex obtained from the direct limit of the homotopy (resp., homology) groups of the finite CWP-subcomplexes?

(2) Given a finite CWP-complex X does there exist a finite CW-complex Y such that $LY = X$, i.e., does there exist $\kappa: Y \to X$ that P-localizes?

If one attempts to prove assertion (1) the following question arises.

(3) If $f: X \to Y$ is a map of CWP-complexes with X finite, then does there exist a $g \simeq f$ such that $g(X)$ is contained in a finite CWP-subcomplex?

The following example shows that (1) and (3) are both false for CWP-complexes, but in the sequel it will be shown that they are both valid for special CWP-complexes to be defined later.

Example. Let

$$g = \iota_1 \circ 2: S^n \xrightarrow{2} S^n \xrightarrow{\iota_1} \bigvee_{r=1}^{\infty} S_P^n, \quad n \geq 2,$$

where ι_1 is the inclusion in the first factor. Let $\phi: S_P^n \to \bigvee_1^\infty S_P^n$ be such that $\phi(S_P^n)$ is not contained in a finite wedge of local n-spheres and $\phi \simeq g_P$ (such a map exists). Let $X = (\bigvee_1^\infty S_P^n) \bigcup_\phi e_P^{n+1}$. Notice that the only finite CWP-subcomplexes of X are finite wedges of local n-spheres. We now proceed to show that there exists an $h: S_P^n \to X$ such that $h(S_P^n)$ is not contained in a finite subcomplex of X, and furthermore, any h' with $h'(S_P^n)$ in a finite subcomplex is not homotopic to h. This will give a counter example for question (3).

First observe that $X \simeq ((\bigvee_1^\infty S^n) \bigcup_g e^{n+1})_P$, and since $(\bigvee_1^\infty S^n) \bigcup_g e^{n+1}$ is 1-connected,

$$\pi_n\left(\bigvee_1^\infty S^n\right) \twoheadrightarrow \pi_n\left(\left(\bigvee_1^\infty S^n\right) \bigcup_g e^{n+1}\right) \cong \frac{\pi_n(\bigvee_1^\infty S^n)}{\langle g \rangle} \quad \text{by [11; 2.10].}$$

Take $P = \{2\}$. Then

$$\bigoplus_{r=1}^{\infty} \mathbf{Z}_P \cong \pi_n\left(\bigvee_1^\infty S_P^n\right) \twoheadrightarrow \pi_n(X) \cong \left(\bigoplus_{r=2}^{\infty} \mathbf{Z}_P\right) \oplus \mathbf{Z}/2.$$

Let $[h] \in [S_P^n, X] \cong \pi_n X$ represent the element of order 2. Then every $[h'] \in [S_P^n, X]$ such that $h'(S_P^n)$ is contained in a finite subcomplex of X (a finite wedge of local n-spheres) has infinite order. Hence, $h \not\simeq h'$.

Also, it is obvious that $\varinjlim \pi_n X_\gamma \cong \pi_n(\bigvee_1^\infty S_P^n) \neq \pi_n X$, where the limit is over the finite subcomplexes of X. Hence, question (1) is also false for CWP-complexes.

Recall that the n-skeleton, X^n, of a CWP-complex is of the form

$$X^{n-1} \bigcup_{\langle \phi_\alpha^n \rangle} \left(\bigvee_{\alpha \in \Gamma} e_P^n \right),$$

where X^{n-1} is the $(n-1)$-skeleton and $X^0 = X^1 = *$. The ϕ_α^n are called the attaching maps.

DEFINITION 5.1. A CWP-complex is *special* if the image of each attaching map is contained in a finite subcomplex.

Notice that every finite CWP-complex is automatically a special CWP-complex.

PROPOSITION 5.2. *Every compact subset of a special CWP-complex is contained in a finite subcomplex.*

Proof. Let C be a compact subset of the special CWP-complex X. By 2.4 $C \subset X^k$ for some k, where X^k is the k-skeleton of X as a CWP-complex. The proof is by induction on k. For $k = 0$ or 1 the result is obvious. Suppose the proposition is true for $k - 1 \geq 0$ and let $C \subset X^k$. $C \cap X^{k-1}$ is compact so by the induction hypothesis it is contained in a finite subcomplex. If

$$C \cap ({}_\alpha e_P^k - {}_\alpha S_P^{k-1}) \neq \emptyset$$

for only a finite number of α, then C is contained in a finite subcomplex, since $\phi_\alpha^k({}_\alpha e_P^k)$ is contained in a finite subcomplex for each α. So, pick exactly one x_α from $C \cap ({}_\alpha e_P^k - {}_\alpha S_P^{k-1})$ for each α provided it is not empty and assume $A = \{x_\alpha\} \subset C$ is infinite. Now $A \cap X^{k-1} = \phi$ and $A \cap {}_\alpha e_P^k$ is the singleton set $\{x_\alpha\}$ or ϕ, which is closed in ${}_\alpha e_P^k$. Hence, A is closed in X^k [5; p. 128]. Similarly, any subset of A is also closed. Therefore, A is an infinite discrete closed subset of C. But this cannot occur with C compact. This completes the induction. $\square$

Let $\{Y_\alpha : \alpha \in \Gamma\}$ be any family of subspaces of Y. Order Γ by inclusion and for $\alpha \leq \beta$ let $\iota^{\alpha\beta}: Y_\alpha \to Y_\beta$ and $\iota^\alpha: Y_\alpha \to Y$ be inclusions.

PROPOSITION 5.3. *For any space X, $\{[X, Y_\alpha]; \iota_\#^{\alpha\beta}\}$ is a direct system of sets, and if X is a co-H-group it is a direct system of groups. Furthermore, there exists a morphism*

$$\psi: \varinjlim\, [X, Y_\alpha] \to [X, Y], \quad \psi\{a\} = \iota_\#^\alpha a, \quad \textit{where } a \in [X, Y_\alpha].$$

PROPOSITION 5.4. *For any $k > 0$, $\{H_k X_\alpha; \iota_*^{\alpha\beta}\}$ is a direct system of abelian groups, and there exists a homomorphism*

$$\psi: \varinjlim\, H_k X_\alpha \to H_k X, \quad \psi\{a\} = \iota_*^\alpha a \quad \textit{for } a \in H_k X_\alpha.$$

THEOREM 5.5. *If X is a finite CW-complex and Y is a SCWP-complex (special*

CWP-complex), then $\psi\colon \varinjlim\,[X, Y_\alpha] \cong [X, Y]$, *where* $\{Y_\alpha\}$ *is the family of finite subcomplexes of* Y. *If* X *is a co-H-group then* ψ *is an isomorphism.*

Proof. To prove ψ is a bijection we will show that there exists

$$\phi\colon [X, Y] \to \varinjlim\,[Y, Y_\alpha]$$

such that $\phi\psi = 1$ and $\psi\phi = 1$. For $[f] \in [X, Y]$, Proposition 5.2 implies there exists $g\colon X \to Y_\alpha$, for some α, such that $\iota^\alpha g \simeq f$. Define $\phi[f] = \{[g]\}$. First we will show ϕ is well defined. Suppose, $F\colon \iota^\alpha g \simeq \iota^\beta h$. Since $X \times I$ is compact, $F(X \times I) \subset Y_\gamma$, for some γ. Without loss of generality, assume $Y_\alpha, Y_\beta \subset Y_\gamma$. Therefore, $\iota^{\alpha\gamma} g \simeq \iota^{\beta\gamma} h$, and hence, $\{[g]\} = \{[h]\} = \phi[f]$. Let $\{a\} \in \varinjlim\,[X, Y_\alpha]$ with $a = [g] \in [X, Y_\alpha]$. Then $\phi\psi\{a\} = \phi\iota^\alpha_\# a = \phi[\iota^\alpha \circ g] = \{[g]\} = \{a\}$. For $[f] \in [X, Y]$ and $[g] \in [X, Y_\alpha]$ such that $[f] = \iota^\alpha_\#[g]$, $\psi\phi[f] = \psi\{[g]\} = \iota^\alpha_\#[g] = \{f\}$. □

COROLLARY 5.6. *If* Y *is a SCWP-complex, then* $\psi\colon \varinjlim\,\pi_k Y_\alpha \cong \pi_k Y$, *where* $\{Y_\alpha\colon \alpha \in \Gamma\}$ *is the family of finite subcomplexes of* Y.

THEOREM 5.7. *If* Y *is a SCWP-complex, then* $\psi\colon \lim H_k Y_\alpha \cong H_k Y$ *for each* $k > 0$, $\{Y_\alpha\}$ *is the family of finite subcomplexes of* Y.

Proof. As in Theorem 5.5, we define $\phi\colon H_k Y \to \varinjlim\,H_k Y_\alpha$, $\phi\psi = 1$ and $\psi\phi = 1$. Let $u \in H_k Y$. By the compact support theorem for singular homology there exist a compact subspace Y_1 of Y and $v \in H_k Y_1$ such that $i_* v = u$, where $i\colon Y_1 \subseteq Y$. By 5.2, $Y_1 \subset Y_\alpha$ for some α. Let $j\colon Y_1 \subseteq Y_\alpha$ and $\phi u = \{j_* v\}$. First we will show ϕ is well defined. Let Y_1, Y_2 be compact subspaces of Y and $v_1 \in H_k Y_1$, $v_2 \in H_k Y_2$ such that $i_{1*} v_1 = u = i_{2*} v_2$, where $i_s\colon Y_s \subseteq Y$ for $s = 1, 2$. Let

$$j_1\colon Y_1 \subseteq Y_\alpha \quad\text{and}\quad j_2\colon Y_1 \subseteq Y_\beta \quad \text{for } \alpha, \beta \in \Gamma.$$

Consider the following diagram of inclusions

$$\begin{tikzcd}
Y_1 \arrow[dr, "\kappa_1"] \arrow[ddr, "i_1"'] & & Y_2 \arrow[dl, "\kappa_2"'] \arrow[ddl, "i_2"] \\
& Y_1 \cup Y_2 \arrow[d, "\varepsilon"] & \\
& Y &
\end{tikzcd}$$

Clearly, $v = \kappa_{1*} v_1 - \kappa_{2*} v_2 \in \ker \varepsilon_*$. So, by a corollary to the compact support theorem there exists a compact set $B \supseteq Y_1 \cup Y_2$ such that $l_* v = 0$, where $l\colon Y_1 \cup Y_2 \subseteq B$. Pick γ such that $Y_\alpha \cup Y_\beta \cup B \subseteq Y_\gamma$. The diagram

$$\begin{tikzcd}
H_kY_1 \arrow[dd, "j_{1*}"'] \arrow[dr, "\kappa_{1*}"] & & H_kY_2 \arrow[dd, "j_{2*}"] \arrow[dl, "\kappa_{2*}"'] \\
& H_k(Y_1 \cup Y_2) \arrow[d, "l_*"] & \\
H_kY_\alpha \arrow[dr, "\iota_*^{\alpha\gamma}"'] & H_kB \arrow[d, "\bar{\varepsilon}_*"] & H_kY_\beta \arrow[dl, "\iota_*^{\beta\gamma}"] \\
& H_kY_\gamma \arrow[d, "\iota_*^{\gamma}"] & \\
& H_kY &
\end{tikzcd}$$

induced by inclusion maps, is commutative; and

$$\iota_*^{\alpha\gamma} j_{1*} v_1 - \iota_*^{\beta\gamma} j_{2*} v_2 = \bar{\varepsilon}_* l_* \kappa_{1*} v_1 - \bar{\varepsilon}_* l_* \kappa_{2*} v_2 = \bar{\varepsilon}_* l_* v = 0.$$

Therefore, $\iota_*^{\alpha\gamma} j_{1*} v_1 = \iota_*^{\beta\gamma} j_{2*} v_2$ and hence $\{j_{1*}v_1\} = \{j_{2*}v_2\} = \phi u$. It now follows from the definitions of ϕ and ψ that $\psi\phi = 1$ and $\phi\psi = 1$. □

Remark 5.8. Since every compact subset of a CWP-complex is contained in some local n-skeleton, arguments similar to those in 5.5 and 5.7 yield

$$\psi\colon \varinjlim \pi_k X^n \cong \pi_k X \quad \text{and} \quad \psi\colon \varinjlim H_k X^n \cong H_k X$$

for every CWP-complex X, where the direct limit is taken over the local skeleton.

The following theorem gives us a method of replacing a CWP-complex with a SCWP-complex.

THEOREM 5.9. *For each CWP-complex X there exist a SCWP-complex Y and an equivalence $f\colon X \to Y$ such that (f_n, f_{n-1}), mapping (X^n, X^{n-1}) to (Y^n, Y^{n-1}), is an equivalence of pairs for each n, where f_n is the restriction of f to the local n-skeleton. Furthermore, the set of local n-cells in X is in one-to-one correspondence with the set of local n-cells in Y for each $n \geq 0$.*

Proof. We will construct Y inductively. Let $Y^i = X^i$ and $f_i = 1$ for $i = 0, 1, 2$. Suppose, Y^k and f_k have been constructed for $2 \leq k \leq n - 1$. Now

$$X^n = X^{n-1} \bigcup_{\langle\phi_\alpha\rangle} \left(\bigvee_{\Gamma} {}_\alpha e_P^n\right).$$

Let

$$\tilde{Y}^n = Y^{n-1} \bigcup_{\langle f_{n-1}\phi_\alpha\rangle} \left(\bigvee_{\Gamma} {}_\alpha e_P^n\right).$$

Then there exists $g_n\colon X^n \to \tilde{Y}^n$ such that

$$(g_n, f_{n-1})\colon (X^n, X^{n-1}) \to (\tilde{Y}^n, Y^{n-1})$$

is an equivalence of pairs [7; p. 40]. For each α,

$$e^{\#}[f_{n-1}\phi_\alpha] = [\bar{\psi}_\alpha] \quad \text{with } e\colon S^{n-1} \to S_P^{n-1}.$$

By Proposition 5.2, $\bar{\psi}_\alpha(S^{n-1})$ is contained in a finite subcomplex, Y_0, of Y^{n-1}. Hence, by the commutativity of the diagram

$$\begin{tikzcd} {[S_P^{n-1}, Y_0]} \arrow[r, "e\#", "\cong"'] \arrow[d] & {[S^{n-1}, Y_0]} \arrow[d] \\ {[S_P^{n-1}, Y^{n-1}]} \arrow[r, "e\#", "\cong"'] & {[S^{n-1}, Y^{n-1}]}, \end{tikzcd}$$

$f_{n-1}\phi_\alpha \simeq \psi_\alpha$ for some ψ_α such that $\psi_\alpha(S_P^n) \subset Y_0$. Let

$$Y^n = Y^{n-1} \bigcup_{\langle\psi_\alpha\rangle} \left(\bigvee_{\Gamma} {}_\alpha e_P^n\right).$$

As above there exists $h_n\colon \tilde{Y}^n \to Y^n$ with $(h_n, 1)\colon (\tilde{Y}^n, Y^{n-1}) \to (Y^n, Y^{n-1})$ an equivalence of pairs. Let $f_n = h_n g_n$. This completes the induction step. Let $Y = \bigcup Y^n$ and let $f\colon X \to Y$ be defined by the family $\{f_n\}$. Since $\varinjlim \pi_k X^n \cong \pi_k X$ and $\varinjlim \pi_k Y^n \cong \pi_k Y$ for all k, $f_{\#}\colon \pi_k X \to \pi_k Y$ is an isomorphism for all $k \geq 0$; and hence, f is an equivalence since X, Y have the homotopy type of CW-complexes (see 4.5). □

We will now proceed to answer the third question stated in the introduction, and then we will give a generalization of 5.5 which will then answer the first question raised.

LEMMA 5.10. *For each* $\alpha \in [S_P^{m-1}, X_P]$ *there exists an* $\alpha' \in [S^{m-1}, X]$ *such that* $X_P \bigcup_\alpha e_P^m \cong X_P \bigcup_{\alpha_P} e_P^m$ *relative to* X_P.

Proof. By [12], we have the following commutative diagram:

$$\begin{tikzcd} \pi_{m-1}X \arrow[r, "e\#"] \arrow[dr, "-\otimes 1"'] & \pi_{m-1}X_P \arrow[d, "\psi", "\cong"'] & {[S_P^{m-1}, X_P]} \arrow[l, "e\#"', "\cong"] \\ & \pi_{m-1}X \otimes \mathbf{Z}_P. & \end{tikzcd}$$

So, there exist $\alpha' \in \pi_{m-1}X$ and $s \in P'$ such that $\psi e^{\#}\alpha = \alpha' \otimes 1/s$, which implies $\psi e^{\#} s\alpha = \alpha' \otimes 1 = \psi e_{\#}\alpha'$. Hence, $e^{\#} s\alpha = e_{\#}\alpha'$. But since localization is a functor, $\alpha'_P = s\alpha$. So, as maps,

$$\begin{tikzcd} S_P^{m-1} \arrow[r, "\alpha_P'"] \arrow[d, "s"'] & X_P \arrow[d, equal] \\ S_P^{m-1} \arrow[r, "\alpha"] & X_P \end{tikzcd}$$

commutes up to homotopy. Furthermore, since $s \in P'$, the map induced by it is an equivalence [11; Section 3.4]. Therefore,

$$X_P \bigcup_{\alpha_{P'}} e_P^m \simeq X_P \bigcup_{\alpha} e_P^m$$

relative to X_P [7; p. 40]. □

THEOREM 5.11. *For each CWP-complex X there exist a* 1*-connected CW-complex Y and an equivalence* $f\colon X \to Y_P$ *such that the map* (f_n, f_{n-1}) *from* (X^n, X^{n-1}) *to* $((Y_P)^n, (Y_P)^{n-1})$ *is an equivalence of pairs for each n, where* f_n *is the restriction of f to the local n-skeleton. Furthermore, the set of local n-cells in X is in one-to-one correspondence with the set of local n-cells in* Y_P*, and hence with the set of n-cells in Y, for each* $n \geq 0$.

Proof. The proof is similar to that of 5.9. Let

$$Y^0 = Y^1 = \bar{Y}^0 = \bar{Y}^1 = X^0 = X^1 = *, \quad Y^2 = \bigvee_{\Gamma} S^2$$

and

$$\bar{Y}^2 = X^2 \simeq \bigvee_{\Gamma} S_P^2.$$

Let $\bar{f}_i = 1\colon X^i \to \bar{Y}^i$ for $i = 0$, 1, and 2. Suppose, Y^i, $\bar{Y}^i$, and $\bar{f}_i$ have been constructed for $2 \leq i < n$. Now

$$X^n = X^{n-1} \bigcup_{\langle\phi_\alpha\rangle} \left(\bigvee_{\Gamma} {}_\alpha e_P^n\right).$$

Let

$$Z^n = \bar{Y}^{n-1} \bigcup_{\langle\bar{f}_{n-1}\phi_\alpha\rangle} \left(\bigvee_{\Gamma} {}_\alpha e_P^n\right).$$

As in 5.9 there exists an equivalence $(g_n, \bar{f}_{n-1})$ from (X^n, X^{n-1}) to $(Z^n, \bar{Y}^{n-1})$. By an obvious extension of Lemma 5.10 there exist maps $\psi_\alpha\colon S^{n-1} \to Y^{n-1}$, for each α, and an h_n such that

$$(h_n, 1)\colon (Z^n, \bar{Y}^{n-1}) \to (\bar{Y}^n, \bar{Y}^{n-1})$$

is an equivalence, where $\bar{Y}^n$ equals

$$Y_P^{n-1} \bigcup_{\langle\psi_{\alpha P}\rangle} \left(\bigvee_{\Gamma} {}_\alpha e_P^n\right).$$

Notice that in the definition of $\bar{Y}^n$, $\psi_{\alpha P}$ is a map and not a class of maps. Let

$$Y^n = Y^{n-1} \bigcup_{\langle\psi_\alpha\rangle} \left(\bigvee_{\Gamma} {}_\alpha e^n\right).$$

Clearly, $Y_P^n \simeq \bar{Y}^n$. Let $\bar{f}_n = h_n g_n$, $Y = \bigcup Y^n$, $\bar{Y} = \bigcup \bar{Y}^n$ and $\bar{f}\colon X \to \bar{Y}$ be the map defined by the family $\{\bar{f}_n\}$. As in 5.9, $\bar{f}$ is an equivalence. Now observe that from the definition of the localization of a 1-connected CW-complex given by D. Sullivan there exists an equivalence $k: \bar{Y} \to Y_P$ such that

$$(k_n, k_{n-1}): (\bar{Y}^n, Y^{n-1}) \to ((Y_P)^n, (Y_P)^{n-1})$$

is an equivalence for each n, where k_n is the restriction of k to the local n-skeleton. $f = k\bar{f}$ is a map that satisfies the conclusion. $\square$

So, for X a finite CWP-complex there exists a finite CW-complex Y such that $Y_P \simeq X$, i.e., there exists a map $k: Y \to X$ that P-localizes.

THEOREM 5.12. *If X is a finite CWP-complex and Y is a SCWP-complex, then ψ: $\varinjlim [X, Y_\alpha] \cong [X, Y]$, where the limit is over the finite subcomplexes of Y.*

Proof. By 5.11, there exists a finite CW-complex Z and a map $\kappa: Z \to X$ that localizes. Let $\kappa^{\alpha\#}: [X, Y_\alpha] \xrightarrow{\sim} [Z, Y_\alpha]$ and $\kappa^{\#}: [X, Y] \xrightarrow{\sim} [Z, Y]$ be the equivalences induced by κ. $\{\kappa^{\alpha\#}\}$ is a map of the direct system $\{[X, Y_\alpha]; \bar{\imath}^{\alpha\beta}_{\#}\}$ into the direct system $\{[Z, Y_\alpha]; \imath^{\alpha\beta}_{\#}\}$. These systems were defined above. With ψ, $\bar{\psi}$ as defined in 5.3 the following diagram commutes.

$$\begin{tikzcd} \varinjlim [X, Y_\alpha] \arrow[r, "\varinjlim \kappa^{\alpha\#}"] \arrow[d, "\bar{\psi}"'] & \varinjlim [Z, Y_\alpha] \arrow[d, "\psi"] \\ {[X, Y]} \arrow[r, "\kappa\#"] & {[Z, Y]} \end{tikzcd}$$

By the universal property and properties of direct limits, $\kappa^{\#}$ and $\varinjlim \kappa^{\alpha\#}$ are equivalences, and ψ is an equivalence by 5.5. Therefore, $\bar{\psi}$ is one too. $\square$

COROLLARY 5.13. *If f: $X \to Y$ is a map between the finite CWP-complex X and the SCWP-complex Y then there exists $g \simeq f$ with $g(X)$ contained in a finite subcomplex of Y.*

COROLLARY 5.14. *If $f \simeq g$: $X \to Y$ are maps of SCWP-complexes such that X is finite and $f(X)$, $g(X) \subset Y_0$, where Y_0 is a finite subcomplex of Y, then there exists G: $f \simeq g$ such that $G(X \times I)$ is contained in a finite subcomplex of Y containing Y_0.*

6. P-local 1-connected CW-complexes and CWP-complexes

Recall that X is P-local if $\pi_* X$ is P-local. Let $\mathscr{P} - \mathscr{CW}_1$ denote the full subcategory of the homotopy category $\mathscr{H}_1$ consisting of all spaces with the homotopy type of a 1-connected P-local CW-complex, and let $\mathscr{CWP}$ denote the full subcategory consisting of all spaces with the homotopy type of a CWP-complex.

By 2.6, $H_* X$ is P-local for every CWP-complex; and hence, $\pi_* X$ is P-local by [12].

THEOREM 6.1. *If X is 1-connected and P-local then there exists $\bar{X} \in \mathscr{CWP}$ and a weak homotopy equivalence $\bar{f}$: $\bar{X} \to X$. Furthermore, if $\bar{\bar{X}}$ is another space* *in* $\mathscr{CWP}$ *and* $\bar{f}\colon \bar{\bar{X}} \to X$ *is a weak homotopy equivalence, then there exists a homotopy equivalence* $h\colon \bar{X} \to \bar{\bar{X}}$ *such that* $\bar{f}h \simeq \bar{f}$.

Proof. The proof uses the CW approximation to a space and 4.5. □

THEOREM 6.2. $\mathscr{P} - \mathscr{CW}_1 = \mathscr{CWP}$ *as categories.*

Proof. The proof follows from 6.1 and 4.5. □

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