MEASURABLE WEAK SECTIONS

BY

G. A. EDGAR

Introduction

If S and T are sets and $p: T \to S$ is a function, then a *section* of p is a function $q: S \to T$ such that $p \circ q = 1_S$. According to the Axiom of Choice, there is a section of p if and only if p is surjective. If $\langle S, \mathscr{F} \rangle$ and $\langle T, \mathscr{G} \rangle$ are measurable spaces (or "Borel spaces"; S is a set and $\mathscr{F}$ is a sigma-algebra of subsets of S) and if $p: T \to S$ is measurable (i.e., $p^{-1}(A) \in \mathscr{G}$ for all $A \in \mathscr{F}$), the question of the existence of a *measurable* section $q: S \to T$ arises. If there is a measure μ on S, it might only be required that $p \circ q = 1$ μ-almost everywhere. We consider below a still less restrictive possibility (called a "weak section").

The existence of measurable sections and related questions have been considered by many mathematicians: a good survey with many references is Parthasarathy's book [20]. Other references not mentioned there include [7], [22, Lemma 4.1, p. 27], [2, Theorem 4, p. 135], [24], [4, Theorem 6], [19, p. 15], [3, Chapter VIII], [17]. All of these papers (except [3]) assume at least that T is metrizable space, most assume that it is also separable. ([24] assumes only that T has a base with cardinal not exceeding the first uncountable cardinal and is hereditarily Lindelof.) In the present paper, we are interested in the situation for "large" spaces T.

If S and T are measurable spaces and $p: T \to S$ is a measurable function, then a map p_* which takes finite measures on T to finite measures on S may be defined by $p_*(\lambda)(B) = \lambda(p^{-1}(B))$. In this situation, the following question has been asked: If μ is a finite measure on S, does there exist a finite measure λ on T such that $p_*(\lambda) = \mu$? This can be viewed as a problem on the extension of a measure to a larger sigma-algebra. (See [18], [15], [1], [27], [13], [14].)

It is known [27], [14] that there is a connection between these two problems. Indeed (compare Theorem 1.2, below), if q is a measurable section of p, then $\lambda = q_*(\mu)$ has the property $p_*(\lambda) = \mu$. In this paper we establish an approximate converse for this result under certain topological conditions. Roughly speaking, we show that if λ_0 is an extreme point of the set of all measures λ with $p_*(\lambda) = \mu$, then λ_0 is of the form $q_*(\mu)$ for some weak section q of p. See the precise statement and proof below (Theorem 1.3).

In the classical case of the problem of sections, the problem is related to two others, namely "selection" and "uniformization," which can be described (somewhat oversimplified) as follows. Let $T' = S \times T$ and let $u: T' \to S$ be

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the canonical projection. If $R \subseteq T'$ and $u(R) = S$, then a *selection* for R is a function $q: S \to T$ such that $\langle s, q(s) \rangle \in R$ for all $s \in S$. A *uniformization* for R is a set $R' \subseteq R$ such that $u^{-1}(s) \cap R'$ consists of exactly one point for each $s \in S$. If no measurability is involved, then the problems are related as follows. If q is a selection for R, then $\{\langle s, q(s) \rangle : s \in S\}$ is a uniformization for R. If R' is a uniformization for R, then it is the graph of a function $q: S \to T$ which is a selection for R. If $v: S \to R$ is a section of $u|_R$, then $v(S)$ is a uniformization for R and hence the graph of a selection for R. If $p: T \to S$ is a surjective function, then its graph $R = \{\langle p(t), t \rangle : t \in T\}$ is a subset of T', and a function $q: S \to T$ is a section of p if and only if it is a selection for R.

Even when measurability is involved the problems are related in the classical case where S and T are "small" spaces (say, Polish spaces). Then the graph of a measurable function is a measurable set, the range of an injective measurable function is a measurable set, etc. But in the case of "large" spaces, these connections no longer exist. The question of finding a measurable section for a measurable function $p: T \to S$ is still equivalent to finding a measurable selection for its graph, but that graph is no longer a measurable set. If $v: S \to R$ is a measurable section of $u|_R$, then $v(S)$ is a uniformization for R, but $v(S)$ is no longer a measurable set. L. D. Brown [3] has proved a measurable selection theorem for measurable sets which applies to some "large" spaces, but as noted above, this does not yield a section theorem in those spaces.

1. Statements of the theorems

We begin with some definitions.

Let $\langle S, \mathscr{F}, \mu \rangle$ be a finite measure space. We will write $\mathscr{F}^\mu$ for the completion of $\mathscr{F}$ with respect to the measure μ. If $A, B \in \mathscr{F}$, then A and B are said to be *equivalent*, written $A \equiv B\ [\mu]$, iff $\mu(A \,\Delta\, B) = 0$. The equivalence class of A is written $[A]_\mu$, and the collection of all equivalence classes (the *measure algebra*) is written $\mathscr{F}/\mu$. Recall that $\mathscr{F}/\mu$ is a σ-Boolean algebra in such a way that the canonical projection $\mathscr{F} \to \mathscr{F}/\mu$ is a σ-homomorphism. The outer (inner) measure associated with μ is denoted μ^* (μ_*). A set $R \subseteq S$ is said to be *thick* iff $\mu^*(R) = \mu(S)$. If $A \subseteq S$, we write μ_A for the *restriction* of μ to A, defined by $\mu_A(B) = \mu^*(A \cap B)$ for $B \in \mathscr{F}$.

Let $\langle S, \mathscr{F} \rangle$ and $\langle T, \mathscr{G} \rangle$ be measurable spaces. A function $u: S \to T$ is said to be *measurable* (from $\mathscr{F}$ to $\mathscr{G}$) iff $u^{-1}(B) \in \mathscr{F}$ for all $B \in \mathscr{G}$. If μ is a finite measure on $\mathscr{F}$, define a measure $u_*(\mu)$ on $\mathscr{G}$ by $u_*(\mu)(B) = \mu(u^{-1}(B))$. Recall the change-of-variables formula: $\int_T f\, d[u_*(\mu)] = \int_S f \circ u\, d\mu$ for all bounded $\mathscr{G}$-measurable functions $f: T \to \mathbf{R}$.

Let $\langle S, \mathscr{F}, \mu \rangle$ be a finite measure space, and let $\langle T, \mathscr{G} \rangle$ be a measurable space. Two measurable functions $u, v: S \to T$ are *weakly equivalent*, written $u \equiv v\ [\mu]$, iff $u^{-1}(B) \equiv v^{-1}(B)\ [\mu]$ for all $B \in \mathscr{G}$. If $p: T \to S$ and $q: S \to T$ are measurable, then q is a *weak section* of p iff $p \circ q \equiv 1_S\ [\mu]$.

PROPOSITION 1.1. *Let $\langle S, \mathscr{F}, \mu\rangle$ be a finite measure space and let $\langle T, \mathscr{G}\rangle$ be a measurable space. If $u, v\colon S \to T$ are measurable functions, then the following are equivalent:*

(a) $u \equiv v\ [\mu]$;
(b) $f \circ u = f \circ v$ [a.e. μ] *for all bounded $\mathscr{G}$-measurable functions $f\colon T \to \mathbf{R}$*;
(c) $u_*(\mu_A) = v_*(\mu_A)$ *for all $A \in \mathscr{F}$.*

Proof. (b) $\Rightarrow$ (a). Take $f = \chi_B$, the characteristic function of $B \in \mathscr{G}$.
(a) $\Rightarrow$ (c). For $B \in \mathscr{G}$, we have

$$u_*(\mu_A)(B) = \mu(A \cap u^{-1}(B)) = \mu(A \cap v^{-1}(B)) = v_*(\mu_A)(B).$$

(c) $\Rightarrow$ (b). If $A \in \mathscr{F}$, then

$$\begin{aligned}\int_A f \circ u\, d\mu &= \int f \circ u\, d\mu_A = \int f\, d[u_*(\mu_A)] = \int f\, d[v_*(\mu_A)] \\ &= \int f \circ v\, d\mu_A = \int_A f \circ v\, d\mu,\end{aligned}$$

so $f \circ u = f \circ v$ [a.e. μ]. ■

THEOREM 1.2. *Let $\langle S, \mathscr{F}, \mu\rangle$ be a finite measure space, let $\langle T, \mathscr{G}\rangle$ be a measurable space, and let $p\colon T \to S$ be a measurable function. If p has a weak section, then there is a measure λ on G such that $p_*(\lambda) = \mu$.*

Proof. Let $q\colon S \to T$ be a weak section of p, so that $p \circ q \equiv 1_S\ [\mu]$. Let $\lambda = q_*(\mu)$. Then $p_*(\lambda) = p_*(q_*(\mu)) = (p \circ q)_*(\mu) = (1_S)_*(\mu) = \mu$. ■

We are interested in the converse of this theorem which would read: If there is a measure λ on $\mathscr{G}$ such that $p_*(\lambda) = \mu$, then p has a weak section. An example in Section 4 shows that the converse is not true in this generality. Topological hypotheses will enable us to prove a partial converse (Theorem 1.3).

Let X be a completely regular space. (Throughout this paper, "completely regular" will be understood to include "Hausdorff.") We will write $\mathfrak{C}_b(X)$ for the set of all bounded, continuous, real-valued functions on X, and $\mathscr{B}(X)$ for the set of all Baire sets in X, i.e., the smallest sigma-algebra with respect to which all functions in $\mathfrak{C}_b(X)$ are measurable. A finite measure μ on $\langle X, \mathscr{B}(X)\rangle$ is *tight* iff for every $\varepsilon > 0$ there is a compact set $K \subseteq X$ with $\mu^*(K) \geq \mu(X) - \varepsilon$. We write $\mathfrak{P}(X)$ for the set of all tight probability measures on $\langle X, \mathscr{B}(X)\rangle$, and topologize $\mathfrak{P}(X)$ by the weak topology [26, p. 181] defined by $\mathfrak{C}_b(X)$. Recall [8, p. 239] that if X is a compact Hausdorff space, and μ is a probability measure on $\mathscr{B}(X)$, then μ can be uniquely extended to a regular Borel probability measure $\mu^\wedge$ such that

$$\mu^\wedge(G) = \sup\{\mu(K)\colon K \subseteq G,\ K \text{ compact Baire set}\}$$

for all open sets $G \subseteq X$ and

$$\mu^\wedge(B) = \inf\{\mu^\wedge(G)\colon G \supseteq B,\ G \text{ open}\}$$

for all Borel sets $B \subseteq X$. In particular, if G is open, then $\mu^\wedge(G) = \mu_*(G)$, so if F is closed, then $\mu^\wedge(F) = \mu^*(F)$. (In fact this extension is also valid for tight Baire measures on noncompact spaces X [11, p. 144], [25, Satz 2.2.10].)

The main result is the following.

THEOREM 1.3. *Let X and Y be completely regular spaces, and let $p\colon X \to Y$ be continuous. Let $\mu \in \mathfrak{P}(Y)$. If there is a tight measure $\lambda \in \mathfrak{P}(X)$ such that $p_*(\lambda) = \mu$, then there is a $\mathscr{B}(Y)^\mu$ to $\mathscr{B}(X)$ measurable weak section of p.*

The proof is in Section 3. Note that this is *not* a theorem about the measure algebras of X and Y: there are many possible choices for the measure λ, each having different null sets. In fact, for the choice of λ made in the proof of the theorem, the range of the weak section is λ-thick.

This theorem shows that the existence of a measurable weak section follows from the existence of a certain tight measure. The following is a criterion for the latter.

PROPOSITION 1.4. *Let X and Y be completely regular spaces, let $p\colon X \to Y$ be continuous, and let $\mu \in \mathfrak{P}(Y)$. Then there is a measure $\lambda \in \mathfrak{P}(X)$ such that $p_*(\lambda) = \mu$ if and only if Y is (according to μ) locally somewhere the image (under p) of a compact set, that is, for every $A \in \mathscr{B}(Y)$ with $\mu(A) > 0$, there is a compact set $K \subseteq X$ such that $p(K) \subseteq A$ and $\mu^*(p(K)) > 0$.*

This is proved in Section 2. The proof is largely routine.

COROLLARY 1.5. *Let X and Y be completely regular spaces and let $p\colon X \to Y$ be continuous and surjective. Suppose*

(a) *X is σ-compact; or*

(b) *X is locally compact and p is open; or*

(c) *X is a complete metric space and p is open.*

Then, for every $\mu \in \mathfrak{P}(Y)$ there exists $\lambda \in \mathfrak{P}(X)$ with $p_*(\lambda) = \mu$.

Proof. If $A \in \mathscr{B}(Y)$ and $\mu(A) > 0$, then there is a compact set $L \subseteq A$ with $\mu^*(L) > 0$ since μ is tight. In each of the three cases we will show that there is a compact set $K \subseteq X$ with $p(K) \subseteq L$ and $\mu^*(p(K)) > 0$. This will complete the proof by Proposition 1.4.

(a) X is σ-compact; say $X = \bigcup_{n=1}^\infty K_n$ where K_n is compact. Thus

$$\begin{aligned}
0 &< \mu^*(L) \\
&= \mu^*\left(\bigcup_{n=1}^{\infty} [L \cap p(K_n)]\right) \\
&= \mu^*\left(\bigcup_{n=1}^{\infty} p(K_n \cap p^{-1}(L))\right) \\
&\le \sum_{n=1}^{\infty} \mu^*(p(K_n \cap p^{-1}(L))),
\end{aligned}$$

so $\mu^*(p(K_n \cap p^{-1}(L))) > 0$ for some n. But $K_n \cap p^{-1}(L)$ is compact.

(b) For each $y \in L$, choose $x_y \in X$ with $p(x_y) = y$, and choose a compact neighborhood K_y of x_y. Since p is open, $p(K_y)$ is a neighborhood of y, so there are a finite number of points $y_1, \ldots, y_n \in L$ with $\bigcup_{i=1}^n p(K_{y_i}) \supseteq L$. Thus $\mu^*(p(K_{y_i} \cap p^{-1}(L))) > 0$ for some i. But $K_{y_i} \cap p^{-1}(L)$ is compact.

(c) We define inductively a sequence B_n of closed subsets of X such that B_n is $1/n$-bounded (i.e., B_n is covered by finitely many sets of diameter at most $1/n$), $B_{n+1} \subseteq B_n$, and $p(B_n^\circ) \supseteq L$ (B_n° denotes the interior of B_n). Take $B_0 = X$. Suppose B_{n-1} has been defined. For each $y \in L$, choose $x_y \in B_{n-1}^\circ$ with $p(x_y) = y$ and choose a closed neighborhood $U_y \subseteq B_{n-1}^\circ$ of x_y with diameter at most $1/n$. By compactness of L and openness of p, there are a finite number of points $y_1, \ldots, y_k \in L$ with $\bigcup_{i=1}^k p(U_{y_i}^\circ) \supseteq L$. Let $B_n = \bigcup_{i=1}^k U_{y_i}$. This completes the inductive definition of the B_n. Let $B = \bigcap_{n=1}^\infty B_n$. Then B is closed and totally bounded, hence compact. By completeness of X, we have $p(B) \supseteq L$. ■

2. Existence of measures

PROPOSITION 2.1. *Let X and Y be completely regular spaces and let $p: X \to Y$ be continuous. If $\mu \in \mathfrak{P}(X)$, then $p_*(\mu) \in \mathfrak{P}(Y)$. The map $p_*: \mathfrak{P}(X) \to \mathfrak{P}(Y)$ is affine and continuous.*

Proof. Suppose $\mu \in \mathfrak{P}(X)$. Let $\varepsilon > 0$. Then there is a compact set $K \subseteq X$ with $\mu^*(K) \geq 1 - \varepsilon$. But $p(K)$ is compact in Y. If $B \in \mathscr{B}(Y)$ and $B \supseteq p(K)$, then $p_*(\mu)(B) = \mu(p^{-1}(B)) \geq \mu^*(K) \geq 1 - \varepsilon$, so $p_*(\mu)^*(p(K)) \geq 1 - \varepsilon$. Therefore $p_*(\mu)$ is tight.

To show that p_* is affine, calculate

$$\begin{aligned} p_*(t\mu + (1 - t)\nu)(B) &= (t\mu + (1 - t)\nu)(p^{-1}(B)) \\ &= t\mu(p^{-1}(B)) + (1 - t)\nu(p^{-1}(B)) \\ &= tp_*(\mu)(B) + (1 - t)p_*(\nu)(B). \end{aligned}$$

To show that p_* is continuous, it suffices to show that for each $f \in \mathfrak{C}_b(Y)$, the map $\mu \mapsto \int_Y f\, d[p_*(\mu)]$ is continuous on $\mathfrak{P}(X)$. But

$$\int_Y f\, d[p_*(\mu)] = \int_X f \circ p\, d\mu,$$

and $f \circ p \in \mathfrak{C}_b(X)$, so $\mu \mapsto \int_X f \circ p\, d\mu$ is continuous. ■

THEOREM 2.2. *Let X and Y be compact Hausdorff spaces and let $p: X \to Y$ be continuous and surjective. Then $p_*: \mathfrak{P}(X) \to \mathfrak{P}(Y)$ is surjective.*

Proof. Define a linear operator $T: \mathfrak{C}_b(Y) \to \mathfrak{C}_b(X)$ by $T(f) = f \circ p$. Then T is an isometry of $\mathfrak{C}_b(Y)$ onto a closed subspace of $\mathfrak{C}_b(X)$. Let $\mu \in \mathfrak{P}(Y)$. Then $f \mapsto \int f\, d\mu$ is a linear functional of norm 1 on $\mathfrak{C}_b(Y)$, so by the Hahn-Banach theorem it can be extended to a linear functional of norm 1 on $\mathfrak{C}_b(X)$. Since $\int 1\, d\mu = 1$, the extension must be positive, and by the Riesz representation theorem the extension is of the form $g \mapsto \int g\, d\nu$ for some $\nu \in \mathfrak{P}(X)$. Then for $f \in \mathfrak{C}_b(Y)$ we have $\int f \circ p\, d\nu = \int f\, d\mu$, i.e., $p_*(\nu) = \mu$. ■

It would be desirable to prove a proposition like the above for noncompact spaces. One way to do it might be to imitate the above using one of the "strict" topologies [23] for $\mathfrak{C}_b(X)$ and $\mathfrak{C}_b(Y)$; the difficulty is in showing that $f \mapsto f \circ p$ is a homeomorphism onto its range in a strict topology. Instead of that, we will generalize Theorem 2.2 by applying the theorem to obtain Proposition 1.4. We first prove three routine lemmas.

LEMMA 2.3. *Let X be a completely regular space and let $Y \subseteq X$. Then $\mathscr{B}(Y) = Y \cap \mathscr{B}(X)$ provided one of the following holds:*

(a) *Y is Lindelof,*
(b) *Y is a zero-set,*
(c) *Y is a cozero-set,*
(d) *X is metrizable,*
(e) *X is normal and Y is an F_σ.*

Proof. We prove (and use below) only case (a). If $f \in \mathfrak{C}_b(X)$, then $f|_Y \in \mathfrak{C}_b(Y)$, so $\mathscr{B}(Y) \supseteq Y \cap \mathscr{B}(X)$. The set $\{U \subseteq X$: U is an open Baire set$\}$ is a base for the topology of X (since X is completely regular), so

$$\{Y \cap U\colon U \subseteq X \text{ is an open Baire set}\}$$

is a base for the topology of Y. The sigma-algebra generated by a subbase for the topology of a Lindelof space includes the Baire sets (see [6, Lemma 1.3]; the proof is the same as in [8, Theorem C, p. 221]), and $Y \cap \mathscr{B}(X)$ is a sigma-algebra containing the above base, so $\mathscr{B}(Y) \subseteq Y \cap \mathscr{B}(X)$. ∎

LEMMA 2.4. *Let X be a subset of a locally convex space E. Suppose X is Lindelof in the weak topology. Then the sigma-algebra of Baire sets for X in the weak topology is the smallest sigma-algebra with respect to which $h|_X$ is measurable for all $h \in E^*$.*

Proof. The set $\{X \cap h^{-1}(U)\colon h \in E^*,\ U \subseteq \mathbf{R} \text{ open}\}$ is a subbase for the weak topology on X, and consists of Baire sets, so the sigma-algebra it generates is $\mathscr{B}(X)$ as in the previous proof. ∎

LEMMA 2.5. *Let X be a completely regular space, let $\mu \in \mathfrak{P}(X)$, and let $\mathscr{K}$ be a subcollection of the collection of all compact subsets of X. If X is locally somewhere an element of $\mathscr{K}$ (i.e., for every $A \in \mathscr{B}(X)$ with $\mu(A) > 0$, there is $K \in \mathscr{K}$ with $K \subseteq A$ and $\mu^*(K) > 0$), then there exist a (possibly finite) sequence $K_1, K_2, \ldots, \in \mathscr{K}$ and a disjoint sequence $B_1, B_2, \ldots, \in \mathscr{B}(X)$ such that $B_n \supseteq K_n$, $\bigcup_{n=1}^{\infty} B_n = X$, and $\mu^*(\bigcup_{n=1}^{\infty} K_n) = 1$.*

Proof. Let k_1 be the smallest positive integer k for which there is $K \in \mathscr{K}$ with $\mu^*(K) > 1/k$; let $K_1 \in \mathscr{K}$ be such that $\mu^*(K_1) > 1/k_1$. If $\mu^*(K_1) = 1$, we are done; assume $\mu^*(K_1) < 1$. Let $B_1 \in \mathscr{B}(X)$ be such that $B_1 \supseteq K_1$ and $\mu(B_1) = \mu^*(K_1)$. Let k_2 be the smallest integer k for which there is $K \in \mathscr{K}$ with $K \subseteq X \backslash B_1$ and $\mu^*(K) = 1/k$; let $K_2 \in \mathscr{K}$ be such that $K_2 \subseteq X \backslash B_1$ and

$\mu^*(K_2) > 1/k_2$. If $\mu^*(K_1 \cup K_2) = 1$, we are done; assume $\mu^*(K_1 \cup K_2) < 1$. Let $B_2 \in \mathscr{B}(X)$ be such that $K_2 \subseteq B_2 \subseteq X \backslash B_1$ and $\mu(B_2) = \mu^*(K_2)$. Continuing in this manner, we obtain a sequence $K_1, K_2, \ldots, \in \mathscr{K}$, a disjoint sequence $B_1, B_2, \ldots, \in \mathscr{B}(X)$, and a sequence $k_1, k_2, \ldots,$ of integers such that if $K \in \mathscr{K}$ and $K \subseteq X\backslash(B_1 \cup \cdots \cup B_{n-1})$, then $\mu^*(K) \leq 1/(k_n - 1)$. I claim that $k_n \to \infty$; if not, then

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) \geq \sum_{n=1}^{\infty} 1/k_n = \infty,$$

contradicting the finiteness of μ. I claim next that $\mu(X\backslash\bigcup_{n=1}^{\infty} B_n) = 0$. If not, there is $K \in \mathscr{K}$ with $K \subseteq X\backslash\bigcup_{n=1}^{\infty} B_n$ and $\mu^*(K) > 0$. But for each n, we have $K \subseteq X\backslash(B_1 \cup \cdots \cup B_n)$, so $\mu(K) \leq 1/(k_n - 1)$, and thus $\mu^*(K) = 0$, a contradiction. Replace B_1 by $B_1 \cup (X\backslash\bigcup_{n=1}^{\infty} B_n)$. The resulting sets have the required properties. ■

Proof of Proposition 1.4. Suppose there is a measure $\lambda \in \mathfrak{P}(X)$ such that $p_*(\lambda) = \mu$. Let $A \in \mathscr{B}(Y)$ with $\mu(A) > 0$. Then $\lambda(p^{-1}(A)) > 0$, so (since λ is tight) there is a compact set $K \subseteq p^{-1}(A)$ such that $\lambda^*(K) > 0$. Note that $p(K) \subseteq A$. Now if $B \in \mathscr{B}(Y)$ and $B \supseteq p(K)$, then $p^{-1}(B) \supseteq K$, so $\mu(B) = \lambda(p^{-1}(B)) \geq \lambda^*(K)$. Hence $\mu^*(p(K)) \geq \lambda^*(K) > 0$. Thus Y is (according to μ) locally somewhere the image (under p) of a compact subset of X.

Conversely, suppose that Y is locally somewhere the image of a compact subset of X. Then by Lemma 2.5 there exist compact sets $K_1, K_2, \ldots, \subseteq X$ and disjoint Baire sets $B_1, B_2, \ldots, \subseteq Y$ such that $p(K_n) \subseteq B_n$, $\bigcup_{n=1}^{\infty} B_n = Y$, and $\mu^*(\bigcup_{n=1}^{\infty} p(K_n)) = 1$. Let μ_n be the measure induced by μ on $p(K_n)$, i.e.,

$$\mu_n(A) = \mu^*(A) \quad \text{for all } A \in \mathscr{B}(p(K_n)) = p(K_n) \cap \mathscr{B}(Y).$$

By Theorem 2.2, there exist measures λ_n on K_n with $p_*(\lambda_n) = \mu_n$. Define λ on $\mathscr{B}(X)$ by $\lambda(A) = \sum_{n=1}^{\infty} \lambda_n(K_n \cap A)$. Then $\lambda \in \mathfrak{P}(X)$ and $p_*(\lambda) = \mu$. ■

3. The selection theorem

In this section we prove Theorem 1.3. We begin with some lemmas. Recall that if X is a completely regular space, then the set $\mathfrak{P}(X)$ of tight probability measures on X is also a completely regular space in the weak topology [26, Theorem 1, p. 181]. We begin with three easy results on $\mathscr{B}(\mathfrak{P}(X))$.

LEMMA 3.1. *Let X be a completely regular space. If $f\colon X \to \mathbf{R}$ is a bounded, $\mathscr{B}(X)$ to $\mathscr{B}(\mathbf{R})$ measurable function, then the map $\mu \mapsto \int f\, d\mu$ is a bounded, $\mathscr{B}(\mathfrak{P}(X))$ to $\mathscr{B}(\mathbf{R})$ measurable function on $\mathfrak{P}(X)$.*

Proof. The class of all bounded Baire functions $f\colon X \to \mathbf{R}$ is the smallest class of functions including the bounded continuous functions and closed under bounded pointwise sequential limits [10, (11.46)]. The conclusion is true for continuous functions f, and if $f_n \to f$ boundedly, then $\int f_n\, d\mu \to \int f\, d\mu$ by the bounded convergence theorem. Hence the conclusion is true for all bounded Baire functions. ■

If $A \in \mathscr{B}(X)$ and $r \in \mathbf{R}$, define $\mathfrak{T}_A^r \subseteq \mathfrak{P}(X)$ by $\mathfrak{T}_A^r = \{\mu \in \mathfrak{P}(X): \mu(A) \geq r\}$. Here is a useful characterization of $\mathscr{B}(\mathfrak{P}(X))$ for compact X.

PROPOSITION 3.2. *Let X be a compact Hausdorff space. Then $\mathscr{B}(\mathfrak{P}(X))$ is the sigma-algebra generated by $\{\mathfrak{T}_A^r: A \in \mathscr{B}(X), r \in \mathbf{R}\}$.*

Proof. Let $\mathscr{C}$ be the sigma-algebra generated by $\{\mathfrak{T}_A^r: A \in \mathscr{B}(X), r \in \mathbf{R}\}$. By Lemma 3.1 we have $\mathscr{C} \subseteq \mathscr{B}(\mathfrak{P}(X))$. For each bounded Baire function $f: X \to \mathbf{R}$, let $f': \mathfrak{P}(X) \to \mathbf{R}$ be defined by $f'(\mu) = \int f\, d\mu$. Then f' is $\mathscr{C}$-measurable when f is the characteristic function of a Baire set. Hence f' is $\mathscr{C}$-measurable when f is a finite linear combination of such functions. Every bounded Baire function on X is a bounded pointwise limit of such finite linear combinations, so f' is $\mathscr{C}$-measurable when f is a bounded Baire function on X, hence in particular when f is a continuous function on X. Now $\mathfrak{P}(X)$ is compact since X is [26, p. 200], so by Lemma 2.4 we have $\mathscr{C} \supseteq \mathscr{B}(\mathfrak{P}(X))$. ■

LEMMA 3.3. *Let X and Y be compact Hausdorff spaces, and let $u: X \to Y$ be $\mathscr{B}(X)$ to $\mathscr{B}(Y)$ measurable. If for every $A \in \mathscr{B}(X)$ there is $A' \in \mathscr{B}(Y)$ with $u^{-1}(A') = A$, then for every $\mathfrak{A} \in \mathscr{B}(\mathfrak{P}(X))$ there is $\mathfrak{A}' \in \mathscr{B}(\mathfrak{P}(Y))$ with $u_*^{-1}(\mathfrak{A}') = \mathfrak{A}$.*

Proof. The set $\{u_*^{-1}(\mathfrak{A}'): \mathfrak{A}' \in \mathscr{B}(\mathfrak{P}(Y))\}$ is a sigma-algebra, and $\mathscr{B}(\mathfrak{P}(X))$ is generated by the sets $\mathfrak{T}_A^r$ of Proposition 3.2, so it suffices to prove the theorem for $\mathfrak{A}$ of the form $\mathfrak{T}_A^r$. Now by assumption there is $A' \in \mathscr{B}(Y)$ with $u^{-1}(A') = A$. The definitions yield $u_*^{-1}(\mathfrak{T}_{A'}^r) = \mathfrak{T}_A^r$. ■

We will prove a result (Theorem 3.5) on weak equivalence of functions which will be important in our proof of the existence of measurable weak sections, but which is of independent interest.

PROPOSITION 3.4. *Let X be a compact Hausdorff space, let $\langle S, \mathscr{F}, \mu\rangle$ be a complete probability space, and let $\Phi: \mathscr{B}(X) \to \mathscr{F}/\mu$ be a σ-homomorphism. Then there is an $\mathscr{F}$ to $\mathscr{B}(X)$ measurable function $u: S \to X$ such that $u^{-1}(A) \in \Phi(A)$ for all $A \in \mathscr{B}(X)$.*

Proof. Let $\theta: \mathscr{F}/\mu \to \mathscr{F}$ be a lifting (see [16, Theorem 3, p. 992] or [11, p. 36]). Then $\Phi' = \theta \circ \Phi: \mathscr{B}(X) \to \mathscr{F}$ is a homomorphism of Boolean algebras. For each $s \in S$, define

$$\mathscr{K}_s = \{K \subseteq X: K \text{ is compact Baire and } s \in \Phi'(K)\}.$$

Now $\mathscr{K}_s$ is closed under finite intersections and does not contain $\emptyset$; hence $\bigcap \mathscr{K}_s \neq \emptyset$. I claim $\bigcap \mathscr{K}_s$ consists of only one point. If $x_1, x_2 \in X$, then there exist disjoint open Baire sets G_i with $x_i \in G_i$ $(i = 1, 2)$. Now

$$(X\backslash G_1) \cup (X\backslash G_2) = X \in \mathscr{K}_s,$$

so at least one of the $(X \backslash G_i)$ belongs to $\mathscr{K}_s$, and the corresponding x_i is not in $\bigcap \mathscr{K}_s$. Let the unique element of $\bigcap \mathscr{K}_s$ be called $u(s)$. Thus u is a function $S \to X$. I claim next that u is measurable from $\mathscr{F}$ to $\mathscr{B}(X)$. Let G be an open Baire set in X. If $u(s) \in G$, then $\bigcap \mathscr{K}_s \subseteq G$, so (by compactness) some finite intersection of sets from $\mathscr{K}_s$ is included in G, and hence some element of $\mathscr{K}_s$ is included in G. Thus

$$u^{-1}(G) = \{s \in S: s \in \Phi'(G)\} = \Phi'(G) \in \mathscr{F}.$$

Thus u is measurable. We have just proved that $u^{-1}(G) = \Phi'(G) \in \Phi(G)$ for open Baire sets G. But $\mathscr{B}(X)$ is generated by the open Baire sets and Φ is a σ-homomorphism, so $u^{-1}(A) \in \Phi(A)$ for all $A \in \mathscr{B}(X)$. ∎

THEOREM 3.5. *Let X be a compact Hausdorff space, and let F be a closed subset of X. Let $\langle S, \mathscr{F}, \mu\rangle$ be a complete probability space, and let u: $S \to X$ be $\mathscr{F}$ to $\mathscr{B}(X)$ measurable. If F is $u_*(\mu)$-thick, then there is v: $S \to X$, also $\mathscr{F}$ to $\mathscr{B}(X)$ measurable, such that $u \equiv v\ [\mu]$ and $v(s) \in F$ for all $s \in S$.*

Proof. Note (Lemma 2.3) that $\mathscr{B}(F) = F \cap \mathscr{B}(X)$. Define Φ: $\mathscr{B}(F) \to \mathscr{F}/\mu$ as follows. For $A \in \mathscr{B}(F)$, choose $A' \in \mathscr{B}(X)$ with $A = F \cap A'$, and let $\Phi(A) = [u^{-1}(A')]_\mu \in \mathscr{F}/\mu$. To show that Φ is well defined, suppose also $A'' \in \mathscr{B}(X)$ and $A \in F \cap A''$. Then $A' \Delta A'' \in \mathscr{B}(X)$ and $(A' \Delta A'') \cap F = \emptyset$, so (since F is thick) $u_*(\mu)(A' \Delta A'') = 0$, so $\mu(u^{-1}(A') \Delta u^{-1}(A'')) = 0$ and $[u^{-1}(A')]_\mu = [u^{-1}(A'')]_\mu$. We claim that Φ is a σ-homomorphism. Indeed, $\emptyset = F \cap \emptyset$, so $\Phi(\emptyset) = [\emptyset]_\mu$; $F = F \cap X$, so $\Phi(F) = [X]_\mu$; if $A = F \cap A'$ and $B = F \cap B'$, then $A \Delta B = F \cap (A' \Delta B')$, so $\Phi(A \Delta B) = \Phi(A) \Delta \Phi(B)$; if $A_n = F \cap A_n'$, then $\bigcup A_n = F \cap (\bigcup A_n')$, so $\Phi(\bigcup A_n) = \bigcup \Phi(A_n)$. By Lemma 3.4, there is an $\mathscr{F}$ to $\mathscr{B}(F)$ measurable function v: $S \to F$ such that $v^{-1}(A) \in \Phi(A)$ for all $A \in \mathscr{B}(F)$. Thus $v^{-1}(A) \equiv u^{-1}(A)\ [\mu]$ for all $A \in \mathscr{B}(X)$, so $v \equiv u\,[\mu]$. ∎

THEOREM 3.6. *Let X be a compact Hausdorff space, let $\langle S, \mathscr{F}, \mu\rangle$ be a complete probability space, let T be a thick subset of S, and let u: $T \to X$ be $T \cap F$ to $\mathscr{B}(X)$ measurable. Then there is an $\mathscr{F}$ to $\mathscr{B}(X)$ measurable function v: $S \to X$ such that $u \equiv v|_T\ [\mu_T]$.*

Proof. Define Φ: $\mathscr{B}(X) \to \mathscr{F}/\mu$ as follows. For $A \in \mathscr{B}(X)$, choose $B \in \mathscr{F}$ such that $u^{-1}(A) = T \cap B$, and define $\Phi(A) = [B]_\mu$. Then Φ is well defined since T is thick, and Φ is a σ-homomorphism as in the previous proof. By Proposition 3.4 there is an $\mathscr{F}$ to $\mathscr{B}(X)$ measurable function v: $S \to X$ such that $v^{-1}(A) \in \Phi(A)$ for all $A \in \mathscr{B}(X)$. Thus

$$(v|_T)^{-1}(A) = T \cap v^{-1}(A) \equiv u^{-1}(A)\ [\mu_T] \quad \text{for all } A \in \mathscr{B}(X),$$

so $u \equiv v|_T\ [\mu_T]$. ∎

If X is a completely regular space and $x \in X$, let ε_x denote the unit mass at x. Define $\mathfrak{E} = \mathfrak{E}_X = \{\varepsilon_x : x \in X\} \subseteq \mathfrak{P}(X)$. The map $x \mapsto \varepsilon_x$ is a homeomorphism of X onto $\mathfrak{E}$ [26, Theorem 9, p. 187]. If $A \in \mathscr{B}(X)$, we will write $\mathfrak{N}_A = \{\mu \in \mathfrak{P}(X) : 0 < \mu(A) < 1\}$. Note $\mathfrak{N}_A \in \mathscr{B}(\mathfrak{P}(X))$ by Lemma 3.1.

PROPOSITION 3.7. *Let X be a compact Hausdorff space and let $\gamma \in \mathfrak{P}(\mathfrak{P}(X))$. If $\gamma(\mathfrak{N}_A) = 0$ for all $A \in \mathscr{B}(X)$, then $\mathfrak{E}$ is γ-thick.*

Proof. We first prove the proposition in the special case that X is totally disconnected (hence zero-dimensional [9, Theorem 3.5]). The algebra of open and closed sets in X generates $\mathscr{B}(X)$. If A is open and closed, then χ_A is continuous, so the map $\mu \mapsto \mu(A)$ is continuous, and therefore

$$\mathfrak{N}_A = \{\mu \in \mathfrak{P}(X) : 0 < \mu(A) < 1\}$$

is an open Baire set. Note

$$\mathfrak{E} = \mathfrak{P}(X) \setminus \bigcup \{\mathfrak{N}_A : A \text{ is open and closed}\}.$$

By the regularity of the regular Borel extension $\gamma^\wedge$ of γ, we have $\gamma^\wedge(E) = 1$ since $\gamma(\mathfrak{N}_A) = 0$ for all open and closed A.

Now return to the general case. Let $Y = \{0, 1\}^{\mathscr{B}(X)}$ have the product topology, so that Y is compact and totally disconnected. For $x \in X$, $A \in \mathscr{B}(X)$, define $u: X \to Y$ by

$$u(x)(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then u is measurable from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$. By Proposition 3.2, $u_*: \mathfrak{P}(X) \to \mathfrak{P}(Y)$ is $\mathscr{B}(\mathfrak{P}(X))$ to $\mathscr{B}(\mathfrak{P}(Y))$ measurable. Let $\beta = u_{**}(\gamma) \in \mathfrak{P}(\mathfrak{P}(Y))$. For $A \in \mathscr{B}(X)$, define

$$M_A = \{y \in Y : y(A) = 1\}.$$

Thus M_A is open and closed in Y and $A = u^{-1}(M_A)$. It follows that $\mathfrak{N}_A = u_*^{-1}(\mathfrak{N}_{M_A})$. Also, $\mathfrak{E}_X = u_*^{-1}(\mathfrak{E}_Y)$. By assumption, $\gamma(\mathfrak{N}_A) = 0$ for all $A \in \mathscr{B}(X)$, so $\beta(\mathfrak{N}_{A'}) = 0$ for all $A' \in \mathscr{B}(Y)$. By the first part of the proof, $\beta^\wedge(\mathfrak{E}_Y) = 1$. We claim that $\gamma^*(\mathfrak{E}_X) = 1$. Let $\mathfrak{A} \in \mathscr{B}(\mathfrak{P}(X))$ with $\mathfrak{A} \supseteq \mathfrak{E}_X$. By Lemma 3.3 there is $\mathfrak{A}' \in \mathscr{B}(\mathfrak{P}(Y))$ with $u_*^{-1}(\mathfrak{A}') = \mathfrak{A}$. It follows that $\mathfrak{E}_Y \cap u_*(\mathfrak{P}(X)) \subseteq \mathfrak{A}'$. Thus $\beta(\mathfrak{A}') = 1$, so $\gamma(\mathfrak{A}) = 1$. This shows $\gamma^*(\mathfrak{E}_X) = 1$, i.e., $\mathfrak{E}_X$ is thick. ■

We next prove a special case of Theorem 1.3.

THEOREM 3.8. *Let X and Y be compact Hausdorff spaces and let $p: X \to Y$ be continuous and surjective. Then for every Baire probability measure μ on Y there is a $\mathscr{B}(Y)^\mu$ to $\mathscr{B}(X)$ measurable weak section of p.*

Proof. (i) Let $\mathfrak{K} = \{\lambda \in \mathfrak{P}(X) : p_*(\lambda) = \mu\}$. By Proposition 2.1, $\mathfrak{K}$ is convex and closed (hence compact). By Theorem 2.2, $\mathfrak{K}$ is nonvoid. The

Krein-Milman theorem asserts that $\mathfrak{K}$ has an extreme point λ. Now $p_*(\lambda) = \mu$, so [5, Theorem 3.3] λ has a disintegration with respect to p, i.e., there is a function $u\colon Y \to \mathfrak{P}(X)$ such that for all $A \in \mathscr{B}(X)$ and $B \in \mathscr{B}(Y)$, the map $y \mapsto u(y)(A)$ is $\mathscr{B}(Y)^\mu$-measurable and

$$\lambda(A \cap p^{-1}(B)) = \int_B u(y)(A)\, d\mu(y).$$

By Proposition 3.2, the map u is $\mathscr{B}(Y)^\mu$ to $\mathscr{B}(\mathfrak{P}(X))$ measurable.

(ii) We will show that the compact set $\mathfrak{E} = \{\varepsilon_x\colon x \in X\}$ is $u_*(\mu)$-thick. By Proposition 3.7, it suffices to show that $u_*(\mu)(\mathfrak{N}_A) = 0$ for all $A \in \mathscr{B}(X)$, where $\mathfrak{N}_A = \{\nu \in \mathfrak{P}(X)\colon 0 < \nu(A) < 1\}$. Suppose (for purposes of contradiction) that $u_*(\mu)(\mathfrak{N}_A) > 0$ for some $A \in \mathscr{B}(X)$. Define $h\colon Y \to \mathbf{R}$ by $h(y) = u(y)(A)$; then h is $\mathscr{B}(Y)^\mu$ to $\mathscr{B}(\mathbf{R})$ measurable. Define $u_1, u_2\colon Y \to \mathfrak{P}(X)$ by

$$u_1(y) = \begin{cases} \dfrac{1}{h(y)}\, u(y)_A & \text{if } 0 < h(y) < 1, \\ u(y) & \text{otherwise,} \end{cases}$$

$$u_2(y) = \begin{cases} \dfrac{1}{1 - h(y)}\, u(y)_{X\setminus A} & \text{if } 0 < h(y) < 1, \\ u(y) & \text{otherwise.} \end{cases}$$

Then u_1, u_2 are $\mathscr{B}(Y)^\mu$ to $\mathscr{B}(\mathfrak{P}(X))$ measurable, $0 \leq h(y) \leq 1$,

$$\mu(\{y\colon 0 < h(y) < 1\}) > 0,$$

and $u(y) = h(y)u_1(y) + (1 - h(y))u_2(y)$. Let r be such that $0 < r < 1$ and

$$\mu(\{y\colon 0 < h(y) < r\}) > 0.$$

Define $u_3, u_4\colon Y \to \mathfrak{P}(X)$ by

$$u_3(y) = \begin{cases} u_1(y) & \text{if } h(y) \geq r, \\ \dfrac{h(y)}{r}\, u_1(y) + \dfrac{r - h(y)}{r}\, u_2(y) & \text{otherwise,} \end{cases}$$

$$u_4(y) = \begin{cases} \dfrac{h(y) - r}{1 - r}\, u_1(y) + \dfrac{1 - h(y)}{1 - r}\, u_2(y) & \text{if } h(y) \geq r, \\ u_2(y) & \text{otherwise,} \end{cases}$$

so that $ru_3(y) + (1 - r)u_4(y) = u(y)$. Let

$$\lambda_3 = \int u_3(y)\, d\mu(y), \quad \lambda_4 = \int u_4(y)\, d\mu(y)$$

(Pettis integrals). Then $\lambda = r\lambda_3 + (1 - r)\lambda_4$.

I claim that $p_*(\lambda_3) = \mu$. Let $B \in \mathscr{B}(Y)$. For all $B' \in \mathscr{B}(Y)$ we have

$$\begin{aligned}\int_{B'} \chi_B(y)\, d\mu(y) &= \int_{B' \cap B} 1\, d\mu(y) \\ &= \int_{B' \cap B} u(y)(X)\, d\mu(y) \\ &= \lambda(p^{-1}(B' \cap B)) \\ &= \lambda(p^{-1}(B) \cap p^{-1}(B')) \\ &= \int_{B'} u(y)(p^{-1}(B))\, d\mu(y),\end{aligned}$$

so $u(y)(p^{-1}(B)) = \chi_B(y)$ for μ-almost all y. Now

$$0 \leq u_3(y)(p^{-1}(B)) \leq 1, \quad 0 \leq u_4(y)(p^{-1}(B)) \leq 1,$$

and $u = ru_3 + (1 - r)u_4$, so

$$u_3(y)(p^{-1}(B)) = u_4(y)(p^{-1}(B)) = \chi_B(y) \quad \text{for } \mu\text{-almost all } y.$$

Then $p_*(\lambda_3)(B) = \int p_*(u_3(y))(B)\, d\mu(y) = \int \chi_B(y)\, d\mu(y) = \mu(B)$, so that $p_*(\lambda_3) = \mu$. Similarly $p_*(\lambda_4) = \mu$. Thus $\lambda_3, \lambda_4 \in \mathfrak{K}$.

We next claim that $\langle u_3(y)\rangle_{y \in Y}$ disintegrates λ_3 and $\langle u_4(y)\rangle_{y \in Y}$ disintegrates λ_4. Indeed, if $A' \in \mathscr{B}(X)$, $B \in \mathscr{B}(Y)$, then (since $u_3(y)(p^{-1}(B)) = 0$ for almost all $y \notin B$)

$$\begin{aligned}\lambda_3(A' \cap p^{-1}(B)) &= \int u_3(y)(A' \cap p^{-1}(B))\, d\mu(y) \\ &= \int_B u_3(y)(A' \cap p^{-1}(B))\, d\mu(y) \\ &\leq \int_B u_3(y)(A')\, d\mu(y).\end{aligned}$$

Similarly $\lambda_4(A' \cap p^{-1}(B)) \leq \int_B u_4(y)(A')\, d\mu(y)$. But $u = ru_3 + (1 - r)u_4$ and

$$\lambda(A' \cap p^{-1}(B)) = \int_B u(y)(A')\, d\mu(y),$$

so

$$\lambda_3(A' \cap p^{-1}(B)) = \int_B u_3(y)(A')\, d\mu(y)$$

and

$$\lambda_4(A' \cap p^{-1}(B)) = \int_B u_4(y)(A')\, d\mu(y).$$

Finally, take $B = \{y \in Y: 0 < h(y) < r\}$. Then $\mu(B) > 0$ and for all $y \in B$ we have $u_3(y)(A) = h(y)/r > 0$, $u_4(y)(A) = 0$. Thus

$$\lambda_3(A \cap p^{-1}(B)) = \int_B u_3(y)(A)\, d\mu(y) > 0$$

and

$$\lambda_4(A \cap p^{-1}(B)) = \int_B u_4(y)(A)\, d\mu(y) = 0.$$

Thus $\lambda_3 \neq \lambda_4$, which contradicts the choice of λ as an extreme point of $\mathfrak{K}$. This contradiction shows that $u_*(\mu)(\mathfrak{N}_A) = 0$, hence that $\mathfrak{E}$ is $u_*(\mu)$-thick.

(iii) By Theorem 3.5, there is $u'\colon Y \to \mathfrak{P}(X)$ with $u' \equiv u\,[\mu]$ and $u'(y) \in \mathfrak{E}$ for all $y \in Y$. But $\mathfrak{E}$ is homeomorphic to X and $\mathscr{B}(\mathfrak{E}) = \mathfrak{E} \cap \mathscr{B}(\mathfrak{P}(X))$, so there is $v\colon Y \to X$ which is $\mathscr{B}(Y)^\mu$ to $\mathscr{B}(X)$ measurable with $u'(y) = \varepsilon_{v(y)}$ for all $y \in Y$. If $A \in \mathscr{B}(X)$, then $u(y)(A) = u'(y)(A)$ [a.e. μ] by Proposition 1.1, so

$$\begin{aligned}
\lambda_{p^{-1}(B)}(A) &= \lambda(A \cap p^{-1}(B)) \\
&= \int_B u(y)(A)\, d\mu(y) \\
&= \int_B u'(y)(A)\, d\mu(y) \\
&= \int_B \varepsilon_{v(y)}(A)\, d\mu(y) \\
&= \mu(B \cap v^{-1}(A)) \\
&= v_*(\mu_B)(A),
\end{aligned}$$

so $v_*(\mu_B) = \lambda_{p^{-1}(B)}$. Finally, $(p \circ v)_*(\mu_B) = p_*(v_*(\mu_B)) = p_*(\lambda_{p^{-1}(B)}) = \mu_B$, so v is a weak section of p. ■

Theorem 3.8 constitutes the main step in the proof of Theorem 1.3, which we now complete.

Proof of Theorem 1.3. Recall that X and Y are completely regular spaces, that $p\colon X \to Y$ is continuous, $\lambda \in \mathfrak{P}(X)$, $\mu \in \mathfrak{P}(Y)$, and $p_*(\lambda) = \mu$. By Proposition 1.4, Y is (according to μ) locally somewhere the image (under p) of a compact set in X. By Lemma 2.5, there exist compact sets $K_1, K_2, \ldots,$ in X and disjoint Baire sets $B_1, B_2, \ldots,$ in Y such that

$$\bigcup_{n=1}^{\infty} B_n = Y, \quad p(K_n) \subseteq B_n \quad \text{and} \quad \mu^*\left(\bigcup_{n=1}^{\infty} p(K_n)\right) = 1.$$

For each n, there is by Theorem 3.8 a function $u_n\colon p(K_n) \to K_n$ such that

$$p \circ u_n \equiv 1_{p(K_n)}\,[\mu_{p(K_n)}].$$

By Theorem 3.6, there is $v_n\colon B_n \to K_n$ with $v_n|_{p(K_n)} \equiv u_n\ [\mu_{p(K_n)}]$, so

$$p \circ v_n \equiv 1_{B_n}\ [\mu_{B_n}].$$

Define $v\colon Y \to X$ by $v(y) = v_n(y)$ for $y \in B_n$. Then $p \circ v \equiv 1_Y\ [\mu]$. ■

4. Concluding remarks

We display a counterexample to a possible strengthening of the main theorem. Weak sections do not always exist.

Let T be the unit circle in the complex plane; T is a compact group under multiplication. Let $\mathscr{F} = \mathscr{B}(T)$. Let $D \subseteq T$ be a countable dense subgroup, and let $S = T/D$ have the quotient sigma-algebra $\mathscr{G}$; i.e., for $B \subseteq S$, define $B \in \mathscr{G}$ iff $p^{-1}(B) \in \mathscr{F}$, where $p\colon T \to T/D$ is the canonical projection. Let λ be normalized Haar measure on T, and let $\mu = p_*(\lambda)$. Now D acts ergodically on T, i.e., $\mu(B) = 0$ or 1 for all $B \in \mathscr{G}$. Note that $\langle \lambda \rangle_{s \in S}$ is a disintegration of λ with respect to p (but not a strict disintegration).

We claim that p has no weak section. Suppose $q\colon S \to T$ were a weak section. Then $q_*(\mu)$ would be a Borel measure on the compact set T; also $q_*(\mu)(A) = 0$ or 1 for all $A \in \mathscr{F}$. Hence $q_*(\mu) = \varepsilon_x$ for some $x \in T$. Finally we have

$$\begin{aligned} 1 &= \varepsilon_x(p^{-1}(p(x))) = p_*(\varepsilon_x)(\{p(x)\}) \\ &= p_*(q_*(\mu))(\{p(x)\}) = \mu(\{p(x)\}) \\ &= \lambda(xD) = \sum_{d \in D} \lambda(\{xd\}) = 0, \end{aligned}$$

a contradiction.

In this case there is a measure $\lambda \in \mathfrak{P}(T)$ with $p_*(\lambda) = \mu$, but the set

$$\mathfrak{K} = \{\nu \in \mathfrak{P}(T)\colon p_*(\nu) = \mu\}$$

is not closed and has no extreme points; if there were an extreme point, the argument of Theorem 3.8 could be applied to produce a weak section.

Here are a few remarks concerning the choice of definitions for this paper.

The Baire sets have been chosen rather than the Borel sets for use in "large" spaces; one reason is the following example showing that the pointwise limit of a sequence of Borel measurable functions $f_n\colon T \to S$ need not be Borel measurable. Let $T = [0, 1]$ and $S = [0, 1]^{[0,1]}$ with the product topology. For $n = 1, 2, 3, \ldots$, define $f_n\colon T \to S$ as follows:

$$f_n(t)(x) = 1 \wedge n|t - x|.$$

The f_n are continuous, hence Borel measurable. But f_n converges pointwise to the function $f\colon T \to S$ given by

$$f(t)(x) = \begin{cases} 0 & \text{if } t = x, \\ 1 & \text{if } t \neq x. \end{cases}$$

If $A \subseteq [0, 1]$ is nonmeasurable, then $K = \{s \in S: s(x) = 1 \text{ for all } x \notin A\}$ is closed, hence Borel, but $f^{-1}(K) = A$ is not measurable. On the other hand, it is easy to show that the pointwise limit of a sequence of Baire functions is Baire.

For real-valued functions on a measure space, equality almost everywhere is an extremely fruitful equivalence relation. For functions with values in a "large" space, it is not such a useful relation; the "weak equivalence" used here seems to be better for some purposes (the name comes from the case of functions with values in a locally convex space, where the notion is well known). Here is a standard example which shows that the two notions are not the same. Let T, S, and $f: T \to S$ be as in the previous paragraph. Define $g: T \to S$ by $g(t)(x) = 0$. Then f and g are Baire measurable functions. Now $f(t) \neq g(t)$ for all t; but if h is a bounded real-valued Baire function on S, then $h(f(t)) = h(g(t))$ for all but countably many t, so $f \equiv g$ [Lebesgue].

One cannot hope to prove uniformization-type theorems in the situation envisioned in this paper. If T is compact and has no G_δ-points, then $S \times T$ contains no Baire set which is uniform. Neither can one hope to obtain section theorems from measurable selection theorems: If T is compact with cardinal greater than the continuum and $p: T \to T$ is the identity, then its graph in $T \times T$ is not a Baire set, so a measurable selection theorem for Baire sets will not yield a measurable section for p.

We mention without details some extensions of results proved above.

One could proceed further (in the compact case) using the results in the proof of Theorem 3.8 to establish a representation for any $\lambda \in \mathfrak{P}(X)$ with $p_*(\lambda) = \mu$. The extreme points of $\mathfrak{K}$ are of the form $v_*(\mu)$, where v is a weak section of p. Further, if v and u are weak sections of p, then $v_*(\mu) = u_*(\mu)$ if and only if $v \equiv u\ [\mu]$, so the set Γ of weak equivalence classes of weak sections of p can be identified with the set of extreme points of $\mathfrak{K}$. According to the Choquet-Bishop-deLeeuw theorem [21, p. 24], there is for each $\lambda \in \mathfrak{K}$ a probability measure γ on Γ such that $\lambda = \int_\Gamma v_*(\mu)\, d\gamma(v)$. This generalizes Proposition 7 of [14].

There is known to be a connection between sections and liftings (see [11, p. 169]). Let Y be a compact Hausdorff space, and let μ be a Baire measure on Y with support all of Y. Let X be the Stone space for the measure algebra of $\langle Y, \mu \rangle$; it will be realized below as the set of multiplicative linear functionals on $L^\infty(Y, \mu)$. There is a canonical continuous surjective map $p: X \to Y$ (p is essentially restriction of multiplicative linear functionals on the algebra $L^\infty(Y, \mu)$ to the subalgebra $\mathfrak{C}(Y)$). If $q: Y \to X$ is a $\mathscr{B}(Y)^\mu$-measurable weak section of p, then a lifting θ for $L^\infty(Y, \mu)$ is defined by $\theta(f)(y) = q(y)(f)$, for $f \in L^\infty(Y, \mu)$, $y \in Y$. Of more interest, if q is a measurable *section* of p, then θ is a *strong lifting* (i.e., $\theta(f) = f$ for $f \in \mathfrak{C}(Y)$) and if $p(q(y)) = y$ a.e. then θ is an *almost strong* lifting [11, p. 127]. Whether strong liftings exist in general is not known. If there do exist Y, μ such that $L^\infty(Y, \mu)$ has no strong lifting (or, equivalently [11, p. 127], has no almost strong lifting), then it would follow that Theorems 1.3 and 3.8 cannot be improved by replacing "weak section" by "section" or by "$p \circ q = 1$ a.e." (This paragraph resulted from a comment made by the referee, and a discussion with J. Rosenblatt.)

It can be shown that Theorems 3.5 and 3.6 are true for completely regular spaces X when the hypothesis that $u_*(\mu)$ be tight is added. This is done by applying the respective theorems together with Lemma 2.5.

It would be interesting to generalize Theorem 1.3 to allow certain discontinuous maps p and/or nontight measures λ. It would be interesting to know whether Proposition 3.4 is true for more general spaces X (such as realcompact spaces or measure-compact spaces).

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NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO