θ-PERFECT AND θ-ABSOLUTELY CLOSED FUNCTIONS

BY

R. F. DICKMAN, JR., AND JACK R. PORTER[1]

1. Introduction

Given a function f: $X \to Y$, a class $\mathscr{A}$ of topological spaces, and a class $\mathscr{C}$ of functions, the extension function problem is to find a topological extension Z of X and extension function F: $Z \to Y$ of f such that $Z \in \mathscr{A}$ and $F \in \mathscr{C}$. The first author [5] proved that if f is continuous, then it is possible to construct a continuous perfect extension F on some topological extension Z of X and noted that if X and Y are Hausdorff spaces, then Z is not necessarily Hausdorff. Viglino [21] started with a continuous function f and Hausdorff spaces X and Y and required that F be continuous and Z be Hausdorff; he showed it was possible to obtain a maximal continuous extension function F (maximal in the sense that there is no proper continuous extension of F defined on a Hausdorff topological extension). Such maximal continuous extensions are called absolutely closed and are characterized in [6] in terms of a closedness-like property of F and a compactness-like property of point-inverse of F—analogous to the perfect function setting.

The notion of "θ-continuity" between Hausdorff spaces is more useful in certain cases (cf., [8], [11], [13], [16]) than "continuity." In this paper, we start with a θ-continuous function f and Hausdorff spaces X and Y and require that F be θ-continuous and Z be Hausdorff. Maximal θ-continuous extension functions are called θ-absolutely closed and are investigated in Section 4; in particular, a θ-continuous function between H-closed spaces is θ-absolutely closed. We are able to show that if Y is regular or H-closed, Urysohn, then f has a θ-absolutely closed extension F. A concept stronger than θ-absolutely closure, called θ-perfect, is developed in Section 3 and characterized in terms of a closedness-like property and compactness-like property of point-inverses.

A θ-continuous function between Hausdorff spaces that has a θ-continuous extension between their Katětov extensions is called a θ-p-map and is studied in Section 5. Now θ-p-maps are related to θ-absolutely closed functions as every θ-p-map from a Hausdorff space into an H-closed space has a θ-absolutely closed extension. In Section 6, the compactness-like properties of point-inverses of θ-perfect and θ-absolutely closed functions are investigated and related. A filter concept, called almost convergence, is defined in this section, developed in Section 2, and used to obtain many of the results in the rest of the paper.

The reader is referred to [3] for the definitions not given here. Here are a few additional definitions and results that are needed throughout the paper.

Received March 10, 1975.

[1] The research of the second author was partially supported by a University of Kansas research grant.

In a space X, the set of regular open subsets, i.e., $\{\text{int}(\text{cl } A): A \subseteq X\}$, forms an open basis for a topology on the underlying set of X; this new topological space is denoted as X_s and called the *semiregularization* of X. A space X is *semiregular* if $X = X_s$ and is *semiregular at a point* $p \in X$ if $\{\text{int}(\text{cl } A)$: $p \in \text{int}(\text{cl } A)$ and $A \subseteq X\}$ is a neighborhood basis of p in X. The space X_s is semiregular.

Let $\mathscr{F}$ be a filter base on a space X. If $x \in X$ is an adherent point of $\mathscr{F}$, we write $x \in \text{ad}_X \mathscr{F}$, and if $\mathscr{F}$ converges to $x \in X$, we write $\mathscr{F} \to x$. If $\mathscr{F}$ and $\mathscr{G}$ are filter bases on X, we say that $\mathscr{G}$ is *finer* than $\mathscr{F}$, written as $\mathscr{F} < \mathscr{G}$, if for each $F \in \mathscr{F}$, there is $G \in \mathscr{G}$ such that $G \subseteq F$ and that $\mathscr{F}$ *meets* $\mathscr{G}$ if $F \cap G \neq \emptyset$ for every $F \in \mathscr{F}$ and $G \in \mathscr{G}$. The neighborhood filter of a set $A \subseteq X$, denoted as $\mathscr{N}_A$, is the set of neighborhoods of A; $\bar{\mathscr{N}}_A$ is used to denote the filter base $\{\text{cl } U: U \in \mathscr{N}_A\}$. Usually, $\mathscr{N}_{\{x\}}$ is denoted as $\mathscr{N}_x$.

Let $\mathscr{F}$ be a filter base on a space X. We say $\mathscr{F}$ *almost converges* to a subset $A \subseteq X$, written as $\mathscr{F} \rightsquigarrow A$ or $\mathscr{F} \rightsquigarrow_X A$, if for each cover $\mathscr{A}$ of A by subsets open in X, there is a finite subfamily $\mathscr{B} \subseteq \mathscr{A}$ and $F \in \mathscr{F}$ such that $F \subseteq \bigcup \{\text{cl } V: V \in \mathscr{B}\}$. We say $\mathscr{F}$ *almost converges to a point* $x \in X$, written as $\mathscr{F} \rightsquigarrow x$, if $\mathscr{F} \rightsquigarrow \{x\}$. Now, $\mathscr{N}_x \to x$, whereas, $\bar{\mathscr{N}}_x \rightsquigarrow x$. A point $x \in X$ is an *almost adherent point* of $\mathscr{F}$, written as $x \in \text{al } \mathscr{F}$ or $x \in \text{al}_X \mathscr{F}$, if $\mathscr{F}$ meets $\bar{\mathscr{N}}_x$.

For a set $A \subset X$, the *almost closure of* A or θ*-closure of* A, denoted as $\text{cl}_\theta A$, is $\text{al}_X \{A\}$ if $A \neq \emptyset$ and is $\emptyset$ if $A = \emptyset$; A is θ*-closed* if $A = \text{cl}_\theta A$. Correspondingly, the *almost interior* of A or θ*-interior of* A, denoted as $\text{int}_\theta A$, is $\{x \in X: \text{cl } U \subseteq A$ for some open set U containing $x\}$. The concepts of almost convergence, almost adherence, and almost closure were introduced by Veličko to study H-closed spaces [19] and to obtain a generalization of Taimanov's extension theorem [20]. Almost closure is used by FitzGerald and Swingle in [9], and a number of the results in this paper have been used by the authors in [7].

The following example is frequently used throughout the sequel.

(1.1) *Example.* [4, Example 3.14]. Let Y denote the well-known example of a noncompact, minimal Hausdorff space due to Urysohn. That is, for each pair of positive integers i, j, let $a_{i,j} = (1/i, 1/j)$, $b_{i,j} = (1/i, -1/j)$, and $c_i = (1/i, 0)$, let

$$W = \{a_{i,j}: (i,j) \in \mathbf{N} \times \mathbf{N}\} \cup \{b_{i,j}: (i,j) \in \mathbf{N} \times \mathbf{N}\} \cup \{c_i: i \in \mathbf{N}\},$$

and let $a_0 = (0, 1)$ and $b_0 = (0, -1)$. The topology for W is the topology inherited from the plane and a basic set containing a_0 (resp. b_0) is of the form

$$U_n(a_0) = \{a_0\} \cup \{a_{i,j}: i \geq n, j \in \mathbf{N}\} \text{ (resp. } V_n(b_0) = \{b_0\} \cup \{b_{i,j}: i \geq n, j \in \mathbf{N}\}.$$

Let X be the subspace consisting of a_0 and all of the $a_{i,j}$'s and c_i's; the subspace X is H-closed, Urysohn and not compact.

We are indebted to the referee for his useful comments and suggestions.

2. Basic results

In this section a number of useful results about almost convergence, almost adherence, and almost closure are derived and used to obtain characterizations of topological concepts ranging from Urysohn to θ-continuity.

(2.1) *Let $\mathscr{F}$ and $\mathscr{G}$ be filter bases on a space X, $A \subseteq X$, and $x \in X$.*

(a) *If $\mathscr{F} \rightsquigarrow A$, then $\mathscr{N}_A < \mathscr{F}$.*
(b) *$\mathscr{F} \rightsquigarrow x$ if and only if $\mathscr{N}_x < \mathscr{F}$.*
(c) *If $\mathscr{F} < \mathscr{G}$, then $\mathrm{al}_X\, \mathscr{G} \subseteq \mathrm{al}_X\, \mathscr{F}$.*
(d) *If $\mathscr{F} < \mathscr{G}$ and $\mathscr{F} \rightsquigarrow A$, then $\mathscr{G} \rightsquigarrow A$.*
(e) $\mathrm{al}_X\, \mathscr{F} = \bigcap \{\mathrm{cl}_\theta\, F \colon F \in \mathscr{F}\}$.
(f) *If $\mathscr{F} \rightsquigarrow x$ and $x \in A$, then $\mathscr{F} \rightsquigarrow A$.*
(g) *$\mathscr{F} \rightsquigarrow A$ if and only if $\mathscr{F} \rightsquigarrow A \cap \mathrm{al}_X\, \mathscr{F}$.*
(h) *If $\mathscr{F} \rightsquigarrow A$, then $A \cap \mathrm{al}_X\, \mathscr{F} \neq \emptyset$.*
(i) [19] *If $U \subseteq X$ is open, then $\mathrm{cl}_\theta\, U = \mathrm{cl}\, U$.*
(j) *If $\mathscr{F}$ is an open filter base, then $\mathrm{ad}_X\, \mathscr{F} = \mathrm{al}_X\, \mathscr{F}$.*
(k) *If $\mathscr{U}$ is an open ultrafilter on X, then $\mathscr{U} \to x$ if and only if $\mathscr{U} \rightsquigarrow x$.*

The converse of 2.1(a) is false as $\mathscr{F}$ does not almost converge to A, even though $\mathscr{N}_A < \mathscr{F}$, when $\mathscr{F} = \{A\}$ and A is a noncompact subspace of a regular Hausdorff space X. Also, the converse of 2.1(f) is false as $\mathscr{F}$ does not almost converge to x for any $x \in A$, even though $\mathscr{F} \rightsquigarrow A$, when $\mathscr{F} = \{A\}$ and A is an infinite compact subset of a Hausdorff space X. By 2.2(k), convergence and almost convergence are equivalent for open ultrafilters; however, for open filter bases, this equivalence implies semiregularity.

(2.2) *A space X is semiregular at a point $p \in X$ if and only if for every open filter base $\mathscr{G}$, $\mathscr{G} \rightsquigarrow p$ implies $\mathscr{G} \to p$.*

Proof. Suppose X is semiregular at a point $p \in X$ and $\mathscr{G}$ is an open filter base on X such that $\mathscr{G} \rightsquigarrow p$. Let B be a regular open set containing p. There is an open $U \in \mathscr{G}$ such that $U \subseteq \mathrm{cl}\, B$. Hence, $U = \mathrm{int}\, U \subseteq \mathrm{int}\,(\mathrm{cl}\, B) = B$. Thus, $\mathscr{G} \to p$. Conversely, suppose for every filter base $\mathscr{G}$, $\mathscr{G} \rightsquigarrow p$ implies $\mathscr{G} \to p$. Let

$$\mathscr{G} = \{\mathrm{int}\,(\mathrm{cl}\, U) \colon p \in U,\ U \text{ open}\}.$$

$\mathscr{G}$ is an open filter base and $\mathscr{G} \rightsquigarrow p$. So, $\mathscr{G} \to p$ implying X is semiregular at p.

In the space X described in Example 1.1, let

$$F_n = \{c_k, a_{k,l} \colon k, l \geq n\}.$$

$\mathscr{F} = \{F_n \colon n \in \mathbf{N}\}$ is an open filter base on X and $\mathscr{F} \rightsquigarrow a_0$, but $\mathscr{F}$ does not converge to any point of X.

(2.3) *A space X is Hausdorff if and only if for cach $p \in X$, $\mathrm{cl}_\theta\, \{p\} = \{p\}$.*

It is easy to prove for each $A \subseteq X$ that $\text{cl } A \subseteq \text{cl}_\theta A$; the opposite inclusion, $\text{cl } A \supseteq \text{cl}_\theta A$ for each $A \subseteq X$, is equivalent to the space X being regular. One direction of this equivalence was noted by Veličko [20].

(2.4) *A space X is regular if and only if for every $A \subseteq X$, $\text{cl}_\theta A = \text{cl } A$.*

Recall that a space X is *Urysohn* if every pair of distinct points are contained in disjoint closed neighborhoods.

(2.5) *A space X is Urysohn if and only if no filter base has more than one almost convergence point.*

Let X be a space. A subset $A \subseteq X$ is *quasi-H-closed relative to X* [15, p. 160] if every cover of A by open subsets of X contains a finite subfamily whose union is dense in A. If X is also Hausdorff, we say A is *H-closed relative to X.*

(2.6) *The following are equivalent for a subset $A \subseteq X$:*

(a) *A is quasi-H-closed relative to X.*
(b) *For every filter base $\mathscr{F}$ on A, $\mathscr{F} \rightsquigarrow_X A$.*
(c) *For every filter base $\mathscr{F}$ on A, $\text{al}_X \mathscr{F} \cap A \neq \emptyset$.*

Proof. Clearly (a) implies (b), and by 2.1(h), (b) implies (c). To show (c) implies (a), let $\mathscr{A}$ be a cover of A by open subsets of X such that the union of any finite subfamily of $\mathscr{A}$ is not dense in A. Then

$$\mathscr{F} = \{A \backslash \text{cl}_X (\textstyle\bigcup_S U) \colon S \text{ is finite subfamily of } \mathscr{A}\}$$

is a filter base on A and $\text{al}_X \mathscr{F} \cap A = \emptyset$. This contradiction yields that A is quasi-H-closed relative to X.

One consequence of 2.6 is that H-closedness in a Hausdorff space X is equivalent to every filter base on X almost converging to X and to every filter base on X having nonvoid almost adherence. The latter equivalence was obtained by Veličko [19]; he also established the next result.

(2.7) *A θ-closed subset of an H-closed space is H-closed.*

(2.8) *Let X be an H-closed, Urysohn space and $A \subseteq X$. The following are equivalent:*

(a) *A is H-closed relative to X.*
(b) [19] *$A = \text{cl}_\theta A$.*
(c) [15] *A is a compact subspace of X_s.*

In [22], Whyburn defined a filter base $\mathscr{F}$ to be *directed toward* a set $A \subseteq X$ provided for every filter base $\mathscr{G}$, $\mathscr{F} < \mathscr{G}$ implies $\text{ad}_X \mathscr{G} \cap A \neq \emptyset$ and used this concept to prove that a perfect (not necessarily continuous) function is compact. This concept and almost convergence are characterized and related in the next result.

(2.9) *Let $\mathcal{F}$ be a filter base on a space X and $A \subseteq X$. Then $\mathcal{F}$ is directed towards A (resp. for every filter base $\mathcal{G}$, $\mathcal{F} < \mathcal{G}$ implies $\mathrm{al}_X \mathcal{G} \cap A \neq \emptyset$) if and only if for every cover $\mathcal{A}$ of A by open subsets of X, there is a finite subfamily $\mathcal{B} \subseteq \mathcal{A}$ and an $F \in \mathcal{F}$ such that $F \subseteq \bigcup \mathcal{B}$ (resp. $\mathcal{F} \rightsquigarrow A$).*

Proof. The proof of the two facts are similar; so, we will only prove the fact in the parentheses. Suppose for every filter base $\mathcal{G}$, $\mathcal{F} < \mathcal{G}$ implies $\mathrm{al}_X \mathcal{G} \cap A \neq \emptyset$. If $\mathcal{F} \rightsquigarrow x$ for some $x \in A$, then by 2.1(f), $\mathcal{F} \rightsquigarrow A$. So, suppose for every $x \in A$, $\mathcal{F}$ does not $\rightsquigarrow x$. Let $\mathcal{A}$ be a cover of A by subsets open in X. For each $x \in A$, there is an open set U_x containing x and $V_x \in \mathcal{A}$ such that $U_x \subseteq V_x$ and $F \backslash \mathrm{cl}_X U_x \neq \emptyset$ for every $F \in \mathcal{F}$. Thus, $\mathcal{G}_x = \{F \backslash \mathrm{cl}_X U_x : F \in \mathcal{F}\}$ is a filter base on X and $\mathcal{F} < \mathcal{G}_x$. Now, $x \notin \mathrm{al}_X \mathcal{G}_x$. Assume that $\bigcup \{\mathcal{G}_x : x \in A\}$ forms a filter subbase with $\mathcal{G}$ denoting the generated filter. Then $\mathcal{F} < \mathcal{G}$ and $A \cap \mathrm{al}_X \mathcal{G} = \emptyset$. This contradiction implies there is a finite subset $B \subseteq A$ and $F_x \in \mathcal{F}$ for $x \in B$ such that

$$\emptyset = \bigcap \{F_x \backslash \mathrm{cl}_X U_x : x \in B\}.$$

There is $F \in \mathcal{F}$ such that $F \subset \bigcap \{F_x : x \in B\}$. It easily follows that

$$\emptyset = \bigcap \{F \backslash \mathrm{cl}_X U_x : x \in B\} \quad \text{and} \quad F \subseteq \bigcup \{\mathrm{cl}_X V_x : x \in B\}.$$

Thus $\mathcal{F} \rightsquigarrow A$. Conversely, suppose $\mathcal{F} \rightsquigarrow A$ and $\mathcal{G}$ is a filter base such that $\mathcal{F} < \mathcal{G}$. By 2.1(d), $\mathcal{G} \rightsquigarrow A$, and by 2.1(h), $A \cap \mathrm{al}_X \mathcal{G} \neq \emptyset$.

A function $f: X \to Y$ is *θ-continuous* (resp. *weakly θ-continuous* or *$w\theta$-continuous*) if for every $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(\mathrm{cl}\ U) \subseteq \mathrm{cl}\ V$ (resp. $f(U) \subseteq \mathrm{cl}\ V$). Clearly, every continuous function is θ-continuous. There are many advantages in using θ-continuity and $w\theta$-continuity in investigating Hausdorff spaces and, in particular, H-closed spaces. For example, every Hausdorff space is the irreducible, perfect, θ-continuous image of some extremally disconnected Tychonoff space (see [11]) and while not every continuous function from a Hausdorff space X into an H-closed, Urysohn space Y has a continuous extension to the Katětov extension κX of X (see [10]), every θ-continuous function from X to Y has a θ-continuous extension to κX. The notions of almost convergence and almost adherence can be used to characterize θ-continuity.

(2.10) *Let $f: X \to Y$ be a function. The following are equivalent:*

(a) *f is θ-continuous.*
(b) *For every filter base $\mathcal{F}$ on X, $\mathcal{F} \rightsquigarrow x$ implies $f(\mathcal{F}) \to f(x)$.*
(c) *For every filter base $\mathcal{F}$ on X, $f(\mathrm{al}\ \mathcal{F}) \subseteq \mathrm{al}\ f(\mathcal{F})$.*
(d) *For every open $U \subseteq Y$, $f^{-1}(U) \subseteq \mathrm{int}_\theta f^{-1}(\mathrm{cl}_\theta U)$.*

Proof. The proof of the equivalence of (a), (b), and (d) is straightforward.
(a) *implies* (c). Suppose $\mathcal{F}$ is a filter base on X, $x \in \mathrm{al}\ \mathcal{F}$, $F \in \mathcal{F}$, and U is a

neighborhood of $f(x)$. There is a neighborhood V of x such that $f(\text{cl } V) \subseteq \text{cl } U$. Since $\text{cl } V \cap F \neq \emptyset$, then $\text{cl } U \cap f(F) \neq \emptyset$. So, $f(x) \in \text{al } f(\mathscr{F})$. This shows that $f(\text{al } \mathscr{F}) \subseteq \text{al } f(\mathscr{F})$.

(c) *implies* (a). Let $\mathscr{U}$ be an ultrafilter containing $f(\mathscr{N}_x)$. Now, $f^{-1}(\mathscr{U})$ is a filter base since $f(X) \in \mathscr{U}$ and $f^{-1}(\mathscr{U})$ meets $\mathscr{N}_x$. So, $f^{-1}(\mathscr{U}) \cup \mathscr{N}_x$ is contained in some ultrafilter $\mathscr{V}$. Now $ff^{-1}(\mathscr{U})$ is an ultrafilter base that generates $\mathscr{U}$. Since $ff^{-1}(\mathscr{U}) < f(\mathscr{V})$, then $f(\mathscr{V})$ also generates $\mathscr{U}$; hence $\text{al } f(\mathscr{V}) = \text{al } \mathscr{U}$. Since $x \in \text{al } \mathscr{V}$, then $f(x) \in f(\text{al } \mathscr{V}) \subseteq \text{al } f(\mathscr{V}) = \text{al } \mathscr{U}$. So, $\mathscr{U}$ meets $\mathscr{N}_{f(x)}$ and

$$\mathscr{N}_{f(x)} \subseteq \bigcap \{\mathscr{U} : \mathscr{U} \text{ ultrafilter, } \mathscr{U} \supseteq f(\mathscr{N}_x)\}$$

(denote this intersection by $\mathscr{G}$). But $\mathscr{G}$ is the filter generated by $f(\mathscr{N}_x)$ (see [3, Proposition I.6.6]); so, $\mathscr{N}_{f(x)} < f(\mathscr{N}_x)$. Hence, f is θ-continuous.

(2.10.1) COROLLARY. *If $f: X \to Y$ is θ-continuous and $A \subseteq X$, then $f(\text{cl}_\theta A) \subseteq \text{cl}_\theta f(A)$.*

Since the composition of θ-continuous functions is a θ-continuous function and the identity function is θ-continuous, then θ-continuous functions can be used as maps in defining categories where the objects are topological spaces. However, the composition of $w\theta$-continuous functions is not necessarily $w\theta$-continuous; this defect is skirted in an interesting manner by Rudolf [18, Theorem III, 3.2]. Here are some similarly proven facts about $w\theta$-continuous functions.

(2.11) *Let $f: X \to Y$ be a function. The following are equivalent:*

(a) *f is $w\theta$-continuous.*
(b) *For every filter base $\mathscr{F}$ on X, $\mathscr{F} \to x$ implies $f(\mathscr{F}) \rightsquigarrow f(x)$.*
(c) *For every filter base $\mathscr{F}$ on X, $f(\text{ad } \mathscr{F}) \subseteq \text{al } f(\mathscr{F})$.*
(d) *For every open $U \subseteq Y$, $f^{-1}(U) \subseteq \text{int } f^{-1}(\text{cl } U)$.*

(2.12) *If $f: X \to Y$ is $w\theta$-continuous, then:*

(a) *For each $A \subseteq X$, $f(\text{cl } A) \subseteq \text{cl}_\theta f(A)$.*
(b) [18] *For each $B \subseteq Y$, $f(\text{cl}(\text{int}(\text{cl } f^{-1}(B)))) \subseteq \text{cl } B$.*
(c) [18] *For each open $U \subseteq X$, $f(\text{cl } U) \subseteq \text{cl } f(U)$.*

3. θ-perfect functions

In [22], Whyburn proved that a function is perfect (i.e., closed and point-inverses are compact) if and only if for every filter base $\mathscr{F}$ on $f(X)$, $\mathscr{F} \to y$, implies $f^{-1}(\mathscr{F})$ is directed towards $f^{-1}(y)$ and that a perfect function is compact (i.e., inverse image of compact sets are compact). In view of 2.9, we say that a function $f: X \to Y$ is *θ-perfect* if for every filter base $\mathscr{F}$ on $f(X)$, $\mathscr{F} \rightsquigarrow y$ implies $f^{-1}(\mathscr{F}) \rightsquigarrow f^{-1}(y)$.

(3.1) *Let $f: X \to Y$ be a function. The following are equivalent:*

(a) *f is θ-perfect.*
(b) *For every filter base $\mathscr{F}$ on X, $f(\mathrm{al}\,\mathscr{F}) \supseteq \mathrm{al}\, f(\mathscr{F})$.*
(c) *For every filter base $\mathscr{F}$ on $f(X)$, $\mathscr{F} \rightsquigarrow B \subseteq Y$ implies $f^{-1}(\mathscr{F}) \rightsquigarrow f^{-1}(B)$.*

Proof. Clearly (c) implies (a).

(a) *implies* (b). Suppose $\mathscr{F}$ is a filter base on X and $y \in \mathrm{al}\, f(\mathscr{F})$. Assume, by way of contradiction, that $f^{-1}(y) \cap \mathrm{al}\,\mathscr{F} = \emptyset$. For each $x \in f^{-1}(y)$, there is open U_x of x and $F_x \in \mathscr{F}$ such that $\mathrm{cl}\, U_x \cap F_x = \emptyset$. Since $f^{-1}(\overline{\mathscr{N}}_y) \rightsquigarrow f^{-1}(y)$ and $\{U_x: x \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$, there is a $V \in \mathscr{N}_y$ and a finite subset $B \subseteq f^{-1}(y)$ such that $f^{-1}(\mathrm{cl}\, V) \subseteq \bigcup \{\mathrm{cl}\, U_x: x \in B\}$. There is an $F \in \mathscr{F}$ such that $F \subseteq \bigcap \{F_x: x \in B\}$. Thus, $F \cap f^{-1}(\mathrm{cl}\, V) = \emptyset$ implying $\mathrm{cl}\, V \cap f(F) = \emptyset$, a contradiction as $y \in \mathrm{al}\, f(\mathscr{F})$. This shows that $y \in f(\mathrm{al}\,\mathscr{F})$.

(b) *implies* (c). Suppose $\mathscr{F}$ is a filter base on $f(X)$ and $\mathscr{F} \rightsquigarrow B \subseteq Y$. Let $\mathscr{G}$ be a filter base on X such that $f^{-1}(\mathscr{F}) < \mathscr{G}$. Then $\mathscr{F} < f(\mathscr{G})$ and $\mathrm{al}\, f(\mathscr{G}) \cap B \neq \emptyset$. Hence, $f(\mathrm{al}\,\mathscr{G}) \cap B \neq \emptyset$ and $\mathrm{al}\,\mathscr{G} \cap f^{-1}(B) \neq \emptyset$. By 2.9, $f^{-1}(\mathscr{F}) \rightsquigarrow f^{-1}(B)$.

(3.1.1) COROLLARY. *If $f: X \to Y$ is θ-perfect, then:*

(a) *For each $A \subseteq X$, $\mathrm{cl}_\theta f(A) \subseteq f(\mathrm{cl}_\theta A)$.*
(b) *For each θ-closed $A \subseteq X$, $f(A)$ is θ-closed.*
(c) *For each subset K quasi-H-closed relative to Y, $f^{-1}(K)$ is quasi-H-closed relative to X.*

Proof. (a) is an immediate consequence of 3.1, and (b) follows easily from (a). To prove (c), we will use 2.6. Let $\mathscr{G}$ be a filter base on $f^{-1}(K)$. Then $f(\mathscr{G})$ is a filter base on K. By 2.6, $\mathrm{al}\, f(\mathscr{G}) \cap K \neq \emptyset$ and by 3.1(b), $\mathrm{al}\,\mathscr{G} \cap f^{-1}(K) \neq \emptyset$. By 2.6, $f^{-1}(K)$ is quasi-H-closed relative to X.

A function $f: X \to Y$ is said to be *θ-compact* if f possesses property (c) of 3.1.1. In order to obtain a characterization of θ-perfect in terms of closedness-like and compactness-like properties as Whyburn did for perfect functions, stronger properties than (a) and (c) of 3.1.1 are needed. A subset A of a space X is θ-rigid provided whenever $\mathscr{F}$ is a filter base on X and $A \cap \mathrm{al}_X\,\mathscr{F} = \emptyset$, there is an open U containing A and $G \in \mathscr{F}$ such that $\mathrm{cl}\, U \cap G = \emptyset$. A function $f: X \to Y$ is *almost closed* if for any set $A \subseteq X$, $f(\mathrm{cl}_\theta A) = \mathrm{cl}_\theta f(A)$. Before characterizing θ-rigidity, we will show that a θ-continuous, θ-compact function into a Urysohn space with a certain property (the "θ-closure" and "quasi-H-closed relative" analogue of property α in [22]) is almost closed.

(3.2) *Suppose $F: X \to Y$ is θ-continuous and θ-compact and Y is Urysohn with this property: For each $A \subseteq Y$ and $p \in \mathrm{cl}_\theta A$, there is a subset H quasi-H-closed relative to Y such that $p \in \mathrm{cl}_\theta (H \cap A)$. Then f is almost closed.*

Proof. Let $B \subseteq X$. By 2.10.1, $f(\mathrm{cl}_\theta B) \subseteq \mathrm{cl}_\theta f(B)$. Suppose $p \in \mathrm{cl}_\theta f(B)$. There is a subset H quasi-H-closed relative to Y such that $p \in \mathrm{cl}_\theta (H \cap f(B))$. Then

$$\mathscr{F} = \{\mathrm{cl}\, U \cap H \cap f(B) \colon U \in \mathscr{N}_p\}$$

is a filter base on Y such that $\mathscr{F} \rightsquigarrow p$. Now, $\mathscr{G} = \{B \cap f^{-1}(F) \colon F \in \mathscr{F}\}$ is a filter base on $B \cap f^{-1}(H)$. Since $f^{-1}(H)$ is quasi-H-closed relative to X, then there is $x \in \mathrm{al}_X\, \mathscr{G} \cap f^{-1}(H)$. By 2.10, $f(x) \in \mathrm{al}_Y f(\mathscr{G}) \subseteq \mathrm{al}_Y\, \mathscr{F}$. Since $\mathscr{F} \rightsquigarrow p$ and Y is Urysohn, $\mathrm{al}_Y\, \mathscr{F} = \{p\}$. Thus, $p \in f(\mathrm{cl}_\theta B)$.

(3.3) *Let A be a subset of a space X. The following are equivalent:*

(a) *A is θ-rigid in X.*

(b) *For any filter base $\mathscr{F}$ on X, if $A \cap \mathrm{al}_X\, \mathscr{F} = \emptyset$, then for some $F \in \mathscr{F}$, $A \cap \mathrm{cl}_\theta F = \emptyset$.*

(c) *For each cover $\mathscr{A}$ of A by open subsets of X, there is a finite subfamily $\mathscr{B} \subseteq \mathscr{A}$ such that $A \subseteq \mathrm{int\,cl}\,(\bigcup \mathscr{B})$.*

Proof. The proof that (a) implies (b) is straightforward.

(b) *implies* (c). Let $\mathscr{A}$ be cover of A by open subsets of X and

$$\mathscr{F} = \left\{\bigcap_{U \in \mathscr{B}} X \backslash \mathrm{cl}\, U \colon \mathscr{B} \text{ is finite subset of } \mathscr{A}\right\}.$$

If $\mathscr{F}$ is not a filter base, then for some finite subfamily $\mathscr{B} \subseteq \mathscr{A}$, $X \subseteq \bigcup \{\mathrm{cl}\, U \colon U \in \mathscr{B}\}$; thus, $A \subseteq X \subseteq \mathrm{int\,cl}\,(\bigcup \mathscr{B})$ which completes the proof in the case that $\mathscr{F}$ is not a filter base. So, suppose $\mathscr{F}$ is a filter base. Then $A \cap \mathrm{al}\, \mathscr{F} = \emptyset$ and there is an $F \in \mathscr{F}$ such that $A \cap \mathrm{cl}_\theta F = \emptyset$. For each $x \in A$, there is open V_x of x such that $\mathrm{cl}\, V_x \cap F = \emptyset$. Let $V = \bigcup \{V_x \colon x \in A\}$. Now, $V \cap F = \emptyset$. Since $F \in \mathscr{F}$, then for some finite subfamily $\mathscr{B} \subseteq \mathscr{A}$, $F = \bigcap \{X \backslash \mathrm{cl}\, U \colon U \in \mathscr{B}\}$. It follows that $V \subseteq \mathrm{cl}\,(\bigcup \mathscr{B})$ and hence, $A \subseteq \mathrm{int\,cl}\,(\bigcup \mathscr{B})$.

(c) *implies* (a). Let $\mathscr{F}$ be a filter base on X such that $A \cap \mathrm{al}\, \mathscr{F} = \emptyset$. For each $x \in A$ there is open V_x of x and $F_x \in \mathscr{F}$ such that $\mathrm{cl}\, V_x \cap F_x = \emptyset$. Now $\{V_x \colon x \in A\}$ is a cover of A by open subsets of X; so, there is finite subset $B \subseteq A$ such that $A \subseteq \mathrm{int\,cl}\,(\bigcup \{V_x \colon x \in B\})$. Let $U = \mathrm{int\,cl}\,(\bigcup \{V_x \colon x \in B\})$. There is $F \in \mathscr{F}$ such that $F \subseteq \bigcap \{F_x \colon x \in B\}$. Since $\mathrm{cl}\,(U) = \bigcup \{\mathrm{cl}\, V_x \colon x \in B\}$, then $\mathrm{cl}\, U \cap F = \emptyset$. Thus, A is θ-rigid in X.

Recall that a subset $A \subseteq X$ is compact if and only if for every filter base $\mathscr{F}$ on X such that $A \cap \mathrm{ad}\,(\mathscr{F}) = \emptyset$, there is $F \in \mathscr{F}$ such that $A \cap \mathrm{cl}\, F = \emptyset$. Thus, by 3.3(b), θ-rigidity is a generalization of compactness with closure and adherence replaced by almost closure and almost adherence, respectively. The open cover characterization of θ-rigidity indicates the closeness of this property to compactness. This remark is emphasized by this easily proven fact: A subset

$A \subseteq X$ is a compact subspace of X_s if and only if for every cover $\mathscr{A}$ of A by open subsets of X, there is a finite subfamily $\mathscr{B} \subseteq \mathscr{A}$ such that

$$A \subseteq \bigcup \{\operatorname{int}_X \operatorname{cl}_X (U): U \in \mathscr{B}\}.$$

(3.4) *A θ-continuous function f: $X \to Y$ is θ-perfect if and only if*

(a) *f is almost closed, and*
(b) *point-inverses are θ-rigid.*

Proof. If f is θ-continuous and θ-perfect, then by 3.1.1 and 2.10.1, f is almost closed. To show $f^{-1}(y)$, for $y \in Y$, is rigid, let $\mathscr{F}$ be a filter base on X such that $f^{-1}(y) \cap \operatorname{al} \mathscr{F} = \emptyset$. So, $y \notin f(\operatorname{al} \mathscr{F})$ and by 3.1(b), $y \notin \operatorname{al} f(\mathscr{F})$. There is open U of y and $F \in \mathscr{F}$ such that $\operatorname{cl} U \cap f(F) = \emptyset$. Therefore, $f^{-1}(\operatorname{cl} U) \cap F = \emptyset$. Since f is θ-continuous, then for each $x \in f^{-1}(y)$, there is open V of x such that $\operatorname{cl} V \subseteq f^{-1}(\operatorname{cl} U)$. So, $f^{-1}(y) \cap \operatorname{cl}_\theta F = \emptyset$. Conversely, suppose a θ-continuous function f satisfies (a) and (b). Let $\mathscr{F}$ be a filter base on $f(X)$ such that $\mathscr{F} \rightsquigarrow y$. Let $\mathscr{G}$ be a filter base on X such that $f^{-1}(\mathscr{F}) < \mathscr{G}$. So, $\mathscr{F} < f(\mathscr{G})$ implying that $y \in \operatorname{al} f(\mathscr{G})$. So, for every $G \in \mathscr{G}$, $y \in \operatorname{cl}_\theta f(G) \subseteq f(\operatorname{cl}_\theta G)$. Hence, $f^{-1}(y) \cap \operatorname{cl}_\theta G \neq \emptyset$ for every $G \in \mathscr{G}$. By (b), $f^{-1}(y) \cap \operatorname{al} \mathscr{G} \neq \emptyset$. By 3.1, f is θ-perfect.

Actually, in the proof of the converse of 3.4, we have shown that property (a) of 3.4 can be reduced to this statement: For each $A \subseteq X$, $f(\operatorname{cl}_\theta A) \supseteq \operatorname{cl}_\theta f(A)$; in fact, we have shown the next result (the function is not necessarily θ-continuous).

(3.4.1) COROLLARY. *Let f: $X \to Y$. If*

(a) *for each $A \subseteq X$, $\operatorname{cl}_\theta f(A) \subseteq f(\operatorname{cl}_\theta A)$ and*
(b) *point-inverses are θ-rigid,*

then f is θ-perfect.

(3.4.2) COROLLARY. *If f: $X \to Y$ satisfies* (a) *and* (b) *of* 3.4, *then f^{-1} preserves θ-rigidity.*

Proof. Let $K \subseteq Y$ be θ-rigid and $\mathscr{F}$ be a filter base on X such that $\operatorname{al}_X \mathscr{F} \cap f^{-1}(K) = \emptyset$. By 3.4.1 and 3.1, $\operatorname{al} f(\mathscr{F}) \cap K = \emptyset$. So, there is $F \in \mathscr{F}$ such that $\operatorname{cl}_\theta f(F) \cap K = \emptyset$. But $\operatorname{cl}_\theta f(F) = f(\operatorname{cl}_\theta F)$. So, $\operatorname{cl}_\theta F \cap f^{-1}(K) = \emptyset$. So, by 3.3, $f^{-1}(K)$ is θ-rigid.

(3.4.3) COROLLARY. *The identity functions from X to X_s and from X_s to X are θ-perfect.*

Proof. 3.4 *can be applied since both identity functions are θ-continuous.* Property (b) of 3.4 follows directly from 3.3(b); property (a) of 3.4 follows the fact that for $A \subseteq X$, the θ-closure of A in X is the same as the θ-closure of A in X_s.

For the space X described in Example 1.1, the identity function from X to X_s is continuous and θ-perfect but is not closed and, hence, not perfect.

(3.5) *Suppose* $f\colon X \to Y$ *has θ-rigid point-inverses. Then:*

(a) *f is θ-continuous if and only if for each $y \in Y$ and open set V containing y, there is an open set U containing $f^{-1}(y)$ such that $f(\mathrm{cl}\ U) \subseteq \mathrm{cl}\ V$.*

(b) *If for each $y \in Y$ and open set U containing $f^{-1}(y)$, there is an open set V of y such that $f^{-1}(\mathrm{cl}\ V) \subseteq \mathrm{cl}\ U$, then for each $A \subseteq X$, $\mathrm{cl}_\theta f(A) \subseteq f(\mathrm{cl}_\theta A)$.*

Proof. The proof of one direction of (a) is obvious, and the proof of the other direction is straightforward using 3.3(c). To prove (b), let $\emptyset \neq A \subseteq X$ and $y \notin f(\mathrm{cl}_\theta A)$. Then $f^{-1}(y) \cap \mathrm{cl}_\theta A = \emptyset$. Now $\mathscr{F} = \{A\}$ is a filter base and $\mathrm{al}\ \mathscr{F} \cap f^{-1}(y) = \emptyset$. So, there is open set U containing $f^{-1}(y)$ such that $\mathrm{cl}\ U \cap A = \emptyset$. There is open V of y such that $f^{-1}(\mathrm{cl}\ V) \subseteq \mathrm{cl}\ U$. So, $\mathrm{cl}\ V \cap f(A) = \emptyset$. Hence, $y \notin \mathrm{cl}_\theta f(A)$.

The next result is closely related to 3.5(b); the proof is straightforward.

(3.6) *Let* $f\colon X \to Y$. *The following are equivalent:*

(a) *For every θ-closed $A \subseteq X$, $f(A)$ is θ-closed.*

(b) *For every $B \subseteq Y$ and θ-open U containing $f^{-1}(B)$, there is θ-open V containing B such that $f^{-1}(V) \subseteq U$.*

(3.7) *If $f\colon X \to Y$ is continuous (resp. θ-continuous) and Y is Hausdorff (resp. Urysohn), then f is perfect (resp. θ-perfect) if and only if for every filter base $\mathscr{F}$ on X, if $f(\mathscr{F}) \to y \in Y$ (resp. $f(\mathscr{F}) \rightsquigarrow y \in Y$), then $\mathrm{ad}_X\ \mathscr{F} \neq \emptyset$ (resp. $\mathrm{al}_X\ \mathscr{F} \neq \emptyset$).*

Proof. The proof of the "perfect" and "θ-perfect" parts are similar; so, we only present the proof of the "θ-perfect" part here. Suppose f is θ-perfect and $f(\mathscr{F}) \rightsquigarrow y$. So, $f^{-1}f(\mathscr{F}) \rightsquigarrow f^{-1}(y)$. Since $f^{-1}f(\mathscr{F}) < \mathscr{F}$, then by 2.1(d), $\mathscr{F} \rightsquigarrow f^{-1}(y)$; by 2.1(h), $\mathrm{al}\ \mathscr{F} \neq \emptyset$. Conversely, suppose for every filter base $\mathscr{F}$ on X, if $f(\mathscr{F}) \rightsquigarrow y \in Y$, then $\mathrm{al}_X\ \mathscr{F} \neq \emptyset$. Suppose $\mathscr{G}$ is a filter base on $f(X)$ such that $\mathscr{G} \rightsquigarrow y \in Y$, and suppose $\mathscr{H}$ is a filter base on X such that $f^{-1}(\mathscr{G}) < \mathscr{H}$. Then $\mathscr{G} = ff^{-1}(\mathscr{G}) < f(\mathscr{H})$. So, $f(\mathscr{H}) \rightsquigarrow y$. Hence, $\mathrm{al}_X\ \mathscr{H} \neq \emptyset$. Let $z \in Y \backslash \{y\}$. Since Y is Urysohn, there are open sets U_z of z and U_y of y such that $\mathrm{cl}\ U_z \cap \mathrm{cl}\ U_y = \emptyset$. There is $H \in \mathscr{H}$ such that $f(H) \subseteq \mathrm{cl}\ U_y$. For each $x \in f^{-1}(z)$, there is open V_x of x such that $f(\mathrm{cl}\ V_x) \subseteq \mathrm{cl}\ U_z$. So, $\mathrm{cl}\ V_x \cap H = \emptyset$. It follows that $f^{-1}(z) \cap \mathrm{al}_X\ \mathscr{H} = \emptyset$ for each $z \in Y \backslash \{y\}$. So, $\mathrm{al}_X\ \mathscr{H} \cap f^{-1}(y) \neq \emptyset$ and f is θ-perfect.

(3.7.1) COROLLARY. *If $f\colon X \to Y$ is θ-continuous, X is quasi-H-closed, and Y is Urysohn, then f is θ-perfect.*

Proof. Since X is quasi-H-closed, then every filter base on X has nonvoid almost adherence; now, the corollary follows directly from 3.4.

(3.7.2) COROLLARY. *If $f: X \to Y$ is θ-continuous, X is Hausdorff, and Y is H-closed, Urysohn, then f has an unique, θ-continuous, θ-perfect extension from κX to Y.*

Proof. By Corollary II.3.3 in [18], f has an unique θ-continuous extension from κX to Y; this extension is θ-perfect by 3.7.1.

4. Absolutely closed and θ-absolutely closed functions

In [21], Viglino defines a continuous function $f: X \to Y$, where X is Hausdorff, to be *absolutely closed* if there does not exist a Hausdorff space Z and a continuous function $f: Z \to Y$ such that X is proper dense subset of Z and $F \mid X = f$. Correspondingly, a θ-continuous (resp. $w\theta$-continuous) function $f: X \to Y$ is defined to be *θ-absolutely closed* (resp. *$w\theta$-absolutely closed*) if there does not exist a Hausdorff space Z and a θ-continuous (resp. *$w\theta$-continuous*) function $F: Z \to Y$ such that X is proper dense subset of Z and $F \mid X = f$. Without the Hausdorff restrictions on X and Z, then no continuous (resp. *θ-continuous, $w\theta$-continuous*) function with nonempty domain is absolutely closed (resp. θ-absolutely closed, $w\theta$-absolutely closed).

(4.1) *Let $f: X \to Y$ be θ-continuous (resp. $w\theta$-continuous) with X a Hausdorff space. The following are equivalent:*

(a) *f is θ-absolutely closed (resp. $w\theta$-absolutely closed).*

(b) *For each open filter base $\mathscr{F}$ on X, if $f(\overline{\mathscr{F}}) \rightsquigarrow y \in Y$ (resp. $f(\mathscr{F}) \rightsquigarrow y \in Y$), then* ad $\mathscr{F} \neq \emptyset$.

(c) *For each open ultrafilter base $\mathscr{U}$ on X, if $f(\overline{\mathscr{U}}) \rightsquigarrow y \in Y$ (resp. $f(\mathscr{U}) \rightsquigarrow y \in Y$), then* ad $\mathscr{U} \neq \emptyset$.

Proof. The proof for both parts are very similar; so, we only present the proof for the part not in the parentheses. Clearly, (b) implies (c).

(a) *implies* (b). Let $\mathscr{F}$ be open filter base on X. Suppose $f(\overline{\mathscr{F}}) \rightsquigarrow y \in Y$. Assume ad $\mathscr{F} = \emptyset$. Let $Z = X \cup \{\mathscr{F}\}$ with the simple extension topology (see [1]). Then Z is Hausdorff. Define $F: Z \to Y$ by $F \mid X = f$ and $F(\mathscr{F}) = y$. F is θ-continuous and f is not θ-absolutely closed. This contradiction implies that ad $\mathscr{F} \neq \emptyset$.

(c) *implies* (a). Suppose $F: Z \to Y$ is θ-continuous where Z is Hausdorff, X is a proper dense subspace of Z, and $F \mid X = f$. So, there is $z \in Z \backslash X$. Let $\mathscr{U}$ be an open ultrafilter on X that contains the trace on X of the neighborhood filter of z. Then $f(\overline{\mathscr{U}}) \rightsquigarrow f(z)$ since F is θ-continuous, but $\mathrm{ad}_X\, \mathscr{U} = \emptyset$ since Z is Hausdorff.

(4.1.1) COROLLARY. *Let $f: X \to Y$ be θ-perfect with X a Hausdorff space. If f is continuous (resp. θ-continuous, $w\theta$-continuous), then f is absolutely closed (resp. θ-absolutely closed, $w\theta$-absolutely closed).*

We now obtain two variations of Corollary 1.1 in [21].

(4.2) *Suppose $f\colon X \to Y$ is θ-continuous where X is Hausdorff and Y is H-closed and Urysohn. The following are equivalent:*

(a) *X is H-closed.*
(b) *f is θ-absolutely closed.*
(c) *f is θ-perfect.*

Proof. Clearly (a) implies (b). By 3.7.2, (b) implies (c) and, by 3.1.1(c), (c) implies (a).

Without Urysohn in the hypothesis of 4.2, it follows that (a) implies (b) and by 3.1.1(c), (c) implies (a). However, for the spaces X and Y described in Example 1.1, the identity function from X into Y is an example of a continuous, absolutely closed, and θ-absolutely closed function from an H-closed space into an H-closed space that is not θ-perfect since $f(\mathrm{cl}_\theta X) = f(X) \neq \mathrm{cl}_\theta f(X)$.

(4.3) *If $f\colon X \to Y$ is continuous, X is Hausdorff, and Y is regular Hausdorff, then the following are equivalent:*

(a) *f is θ-perfect.*
(b) *f is absolutely closed.*
(c) *f is θ-absolutely closed.*

Proof. Clearly (c) implies (b), and by 4.1.1, (a) implies (b) and (c). We will use 3.7 to show (b) implies (a). Let $\mathscr{F}$ be a filter base on X such that $f(\mathscr{F}) \rightsquigarrow y \in Y$. Since Y is regular, $f(\mathscr{F}) \to y$. Now

$$O(\mathscr{F}) = \{U \subseteq X\colon U \text{ open and } U \supseteq F \text{ for some } F \in \mathscr{F}\}$$

is an open filter on X. Since f is continuous, then $f^{-1}(\mathscr{N}_y)$ is an open filter base on X and $f^{-1}(\mathscr{N}_y) \subseteq O(\mathscr{F})$. Thus, $f(O(\mathscr{F})) \to y$. By (b) and Theorem 1.2 in [21], $\mathrm{ad}_X\, O(\mathscr{F}) \neq \emptyset$. It easily follows that $\mathrm{al}_X\, \mathscr{F} \neq \emptyset$.

It seems natural to ask if the list of equivalent conditions of 4.3 can be extended to: (d) f is perfect. A negative answer follows by considering a function $f\colon X \to Y$ where Y is a one-point space and X is an H-closed, noncompact space. Another natural question is whether (a), (b), and (c) of 4.3 are equivalent if f is continuous, X is Hausdorff, and Y is Urysohn? We know by the proof of 4.3, that (c) implies (b), (a) implies (b), and (a) implies (c) with only Y Hausdorff. Let X and Y be the spaces described in Example 1.1, and define the function $f\colon Y\backslash\{b_0\} \to X$ by $f|_X$ is the identity function and $f(b_{ij}) = c_i$. This shows that (b) implies (c) and (b) implies (a) are false, in general, even if f is continuous and closed and the range is Urysohn. We have not been able to resolve if (c) implies (a) when Y is Urysohn.

Problem. Prove or disprove that a continuous, θ-absolutely closed function from a Hausdorff space into a Urysohn space is θ-perfect.

Since every continuous function from a Hausdorff space to a Hausdorff space has an absolutely closed continuous extension by Theorems 2.1 and 2.2 in [21], then the next fact follows directly from 4.3.

(4.3.1) COROLLARY. *Every continuous function from a Hausdorff space into a regular Hausdorff space has a θ-perfect continuous extension.*

Recall by 3.7.2 that a θ-continuous function from a Hausdorff space into an H-closed Urysohn space has a θ-continuous, θ-perfect extension. The common denominator of this fact and Corollary 4.3.1 gives rise to the following problem.

Problem. Prove or disprove that a θ-continuous function from a Hausdorff space into an Urysohn space has a θ-continuous, θ-absolutely closed extension.

For a Hausdorff space X, let κX denote the Katětov H-closed extension (see [12]). A continuous function $f: X \to Y$, where X and Y are Hausdorff spaces, is a *p-map* [10] or said to be *proper* [13] if there is a continuous extension $\kappa f: \kappa X \to \kappa Y$. A p-map f is τ-perfect if $\kappa f(\kappa X\backslash X) \subseteq \kappa Y\backslash Y$.

In [2], a subset $A \subseteq X$ is said to be *far from the remainder* (f.f.r.) if each free open ultrafilter on X contains an open set whose closure is disjoint from A. A function is *regular-closed* [6] if the image of every regularly closed subset is closed. Blaszezyk and Mioduszewski [2] proved that a p-map is τ-perfect if and only if it is regular-closed and point-inverses are f.f.r. Dickman [6] noted that a function is τ-perfect if and only if it is absolutely closed and a p-map. By 4.1.1, a continuous, θ-perfect function is regular-closed and point-inverses are f.f.r.

(4.4) *Suppose $f: X \to Y$ is a continuous, θ-absolutely closed function where X and Y are Hausdorff. Then:*

(a) *Point-inverses are θ-rigid.*
(b) *For each $A \subseteq X, f(\mathrm{cl}_\theta A)$ is closed (in particular, f is regular-closed).*

Proof. Ad(a). Suppose $y \in Y$ and $\mathscr{F}$ is a filter base on X such that $\mathrm{al}_X \mathscr{F} \cap f^{-1}(y) = \emptyset$. Assume, by way of contradiction, that

$$\mathscr{S} = \{U \cap V: U \text{ open}, U \supseteq f^{-1}(y), V \text{ open}, V \supseteq F \text{ for some } F \in \mathscr{F}\}$$

is an open filter base. Now, by the continuity of $f, f(\overline{\mathscr{S}}) \rightsquigarrow y$. By 4.1, ad $\mathscr{S} \neq \emptyset$. Since f is continuous, then $f(\text{ad } \mathscr{S}) \subseteq \text{ad } f(\mathscr{S}) \subseteq \{y\}$; so ad $\mathscr{S} \subseteq f^{-1}(y)$. Thus, $\emptyset \neq \text{ad } \mathscr{S} \subseteq f^{-1}(y) \cap \mathrm{al}_X \mathscr{F}$, a contradiction. So, there is an open set $U \supseteq f^{-1}(y)$ and open set $V \supseteq F$ for some $F \in \mathscr{F}$ such that $U \cap V = \emptyset$. Thus $\mathrm{cl}\, U \cap F = \emptyset$.

Ad(b). We first prove f is regular-closed and use this fact to prove that $f(\mathrm{cl}_\theta A)$ is closed for every $A \subseteq X$. Suppose $p \in \mathrm{cl}\, f(\mathrm{cl}\, U)$ where U is open. Then $\mathscr{S} = \{U \cap f^{-1}(V): V \in \mathscr{N}_p\}$ is open filter base and $f(\mathscr{S}) \to p$. In particular, $f(\overline{\mathscr{S}}) \rightsquigarrow p$ and since f is θ-absolutely closed, ad $\mathscr{S} \neq \emptyset$. Since f is continuous, $\emptyset \neq f(\operatorname{ad} \mathscr{S}) \subseteq \operatorname{ad} f(\mathscr{S}) \subseteq \{p\}$. Thus, $\operatorname{ad} \mathscr{S} \subseteq f^{-1}(p)$. But, $\operatorname{ad} \mathscr{S} \subseteq \operatorname{cl} U$. So, $f^{-1}(p) \cap \operatorname{cl} U \neq \emptyset$ implying $p \in f(\operatorname{cl} U)$. This completes the proof that f is regular-closed. Suppose $A \subseteq X$. Clearly, $f(\operatorname{cl}_\theta A)$ is closed if $A = \emptyset$. Suppose $A \neq \emptyset$ and $p \in \operatorname{cl} f(\operatorname{cl}_\theta A)$. Since $\operatorname{al} \mathscr{N}_A = \operatorname{ad} \mathscr{N}_A = \operatorname{cl}_\theta A$, then $f^{-1}(p) \cap \operatorname{al} \mathscr{N}_A \neq \emptyset$ would imply that $f(p) \in f(\operatorname{cl}_\theta A)$ and complete the proof of the closure of $f(\operatorname{cl}_\theta A)$. Assume, by way of contradiction, that $f^{-1}(p) \cap \operatorname{al} \mathscr{N}_A = \emptyset$. By 4.4(a), there is an open set $U \supseteq f^{-1}(p)$ and an open set $W \in \mathscr{N}_A$ such that $\operatorname{cl} U \cap W = \emptyset$. So, $U \cap \operatorname{cl} W = \emptyset$ implying $f(p) \notin f(\operatorname{cl} W)$. But $f(\operatorname{cl} W)$ is closed; so, $p \in \operatorname{cl} f(\operatorname{cl}_\theta A) \subseteq \operatorname{cl} f(\operatorname{cl} W) = f(\operatorname{cl} W)$ is a contradiction.

The results of 4.4 represent our furthermost position at characterizing θ-absolutely closure in terms of closedness-like and compactness-like properties as Whyburn [22] did for perfect functions, Dickman [6] for absolutely closed functions, and we did for θ-perfect functions in 3.4. We have had little success in determining the compactness-like properties of point-inverses of θ-absolutely closed functions. Also, we have not been able to prove or disprove the converse of 4.4.

Problem. If $f: X \to Y$ is a continuous surjection where X is Hausdorff and Y is Urysohn and if f satisfies (a) and (b) of 4.4, prove or disprove that f is θ-absolutely closed (such a function is necessarily absolutely closed by Theorem 2 in [6] and 6.2.1 in Section 6.)

We complete this section by giving a characterization of regular-closed continuous functions.

(4.5) *A continuous function $f: X \to Y$ is regular-closed if and only if for each $y \in Y$ and open set $U \supseteq f^{-1}(y)$ there is an open set V of y such that* $\operatorname{int} \operatorname{cl} U \supseteq f^{-1}(V)$.

Proof. Suppose f is regular-closed, $y \in Y$, and $f^{-1}(y) \subseteq U$ for some open set U. Assume, by way of contradiction, for each open set W containing y,

$$f^{-1}(W) \cap \operatorname{cl}(X \backslash \operatorname{cl} U) \neq \emptyset.$$

Then $y \in \operatorname{cl} f(\operatorname{cl}(X \backslash \operatorname{cl} U)) = f(\operatorname{cl}(X \backslash \operatorname{cl} U)) = f(X \backslash \operatorname{int} \operatorname{cl} U)$ implying

$$\emptyset \neq f^{-1}(y) \backslash \operatorname{int} \operatorname{cl} U \subseteq f^{-1}(y) \backslash U,$$

a contradiction. So, for some open set W of y, $f^{-1}(W) \cap \operatorname{cl}(X \backslash \operatorname{cl} U) = \emptyset$ implying $f^{-1}(W) \subseteq \operatorname{int} \operatorname{cl} U$. Conversely, let V be an open set in X. Assume, by way of contradiction, there is $y \in \operatorname{cl} f(\operatorname{cl} V) \backslash f(\operatorname{cl} V)$. Since $f^{-1}(y) \subseteq X \backslash \operatorname{cl} V$, there is an open set W of y such that $f^{-1}(W) \subseteq \operatorname{int} \operatorname{cl}(X \backslash \operatorname{cl} V) = X \backslash \operatorname{cl} V$. Since $y \in \operatorname{cl} f(\operatorname{cl} V)$, then $W \cap f(\operatorname{cl} V) \neq \emptyset$ implying $f^{-1}(W) \cap \operatorname{cl} V \neq \emptyset$, a contradiction as $f^{-1}(W) \subseteq X \backslash \operatorname{cl} V$. This completes the proof that f is regular-closed.

5. $w\theta$-p-maps and θ-p-maps

Corresponding to the definition of p-maps, we define a θ-continuous (resp. $w\theta$-continuous) function $f: X \to Y$ where X and Y are Hausdorff to be a *θ-p-map* (resp. $w\theta$-p-map) if f has a θ-continuous (resp. $w\theta$-continuous) extension $F: \kappa X \to \kappa Y$. D. Harris [10] defines an open cover to be a *p-cover* if there is a finite subfamily whose union is dense, and he proves that a continuous function between Hausdorff spaces is a p-map if and only if the inverse image of a p-cover is a p-cover. By a similar proof, one can prove the next result.

(5.1) *A $w\theta$-continuous function $f: X \to Y$ where X and Y are Hausdorff spaces is a $w\theta$-p-map if and only if for every p-cover $\mathscr{A}$ of Y, $\{\operatorname{int} f^{-1}(\operatorname{cl}(U)): U \in \mathscr{A}\}$ is a p-cover of X.*

Using the concepts of p-maps, θ-p-maps, and $w\theta$-p-maps, it is straightforward to obtain the following extension of Corollary 1.1 in [21].

(5.2) *Suppose $f: X \to Y$ is continuous (resp. θ-continuous, $w\theta$-continuous) where X is Hausdorff and Y is H-closed. Then X is H-closed if and only if f is absolutely closed (resp. θ-absolutely closed, $w\theta$-absolutely closed) and a p-map (resp. θ-p-map, $w\theta$-p-map).*

In the space Y described in Example 1.1, let $C = \{c_i: i \in \mathbf{N}\}$; let h be the identity function from C into $Y\backslash\{a_0\}$. h is a θ-perfect, perfect, continuous function that is not a p-map (and hence, not τ-perfect). We now give an example of a p-cover $\mathscr{A}$ of $Y\backslash\{a_0\}$ such that $\{h^{-1}(A): A \in \mathscr{A}\}$ is not a p-cover of C. Let

$$W_n = \{c_n\} \cup \{a_{n,i}: i \in \mathbf{N}\} \cup \{b_{n,i}: i \in \mathbf{N}\}$$

and

$$\mathscr{A} = \{W_n: n \in \mathbf{N}\} \cup \{U_1(a_0)\backslash\{a_0\}\} \cup \{V_1(b_0)\}$$

where $U_1(a_0)$ and $V_1(b_0)$ are defined in Example 1.1.

For 5.2 to be an extension of Corollary 1.1 in [21], we must show that a continuous (resp. θ-continuous, $w\theta$-continuous), absolutely closed (resp. θ-absolutely closed, $w\theta$-absolutely closed) function into a compact Hausdorff space is a p-map (resp. θ-p-map, $w\theta$-p-map). The continuous and $w\theta$-continuous cases follow easily from the covering characterizations. The θ-continuous case follows from 3.7.2. This illustrates the need for a covering characterization of θ-p-maps. We have been able to obtain a covering condition that is necessary but have not been able to prove or disprove its sufficiency.

Before introducing the covering condition, we need to recall that a cover $\mathscr{A}$ of a space is a *regular refinement* of a cover $\mathscr{B}$ if for each $A \in \mathscr{A}$, there is a $B \in \mathscr{B}$ such that $\operatorname{cl} A \subseteq B$.

(5.3) *A θ-p-map $f: X \to Y$ where X and Y are Hausdorff has this property:*
(*) *if $\mathscr{A}$ is a p-cover of Y, then $\{f^{-1}(\operatorname{cl} V): V \in \mathscr{A}\}$ has a regular open refinement which is a p-cover.*

Proof. The proof of (5.3) is very similar to the proof of the corresponding result in [10].

The degree that 5.3(*) is a sufficient condition for a θ-p-map is indicated in the result following this lemma.

(5.4) LEMMA. *If $f: X \to Y$ is a θ-continuous function where X is Hausdorff and Y is H-closed and f satisfies* 5.3(*), *then for every open ultrafilter $\mathscr{U}$ on X, $f(\overline{\mathscr{U}}) \rightsquigarrow p$ for $p \in Y$.*

Proof. Assume $f(\overline{\mathscr{U}})$ does not almost converge to any point $p \in Y$. Then for each $p \in Y$, there is an open set V_p containing p with the property that cl V_p does not contain any member of $f(\overline{\mathscr{U}})$. Since $\{V_p: p \in Y\}$ is a p-cover (Y is H-closed), then $\{f^{-1}(\text{cl } V_p): p \in Y\}$ has a regular open refinement $\mathscr{A}$ which is a p-cover. By a lemma in [10], there is $A \in \mathscr{A} \cap \mathscr{U}$. So, there is some $p \in Y$ such that cl $A \subseteq f^{-1}(\text{cl } V_p)$. Thus cl V_p contains an element of $f(\overline{\mathscr{U}})$, a contradiction.

As a consequence of 5.4 and 4.1, it follows that a θ-continuous, θ-absolutely closed function is θ-p-map if and only if it satisfies 5.3(*). The best that we can do toward proving that a θ-continuous function satisfying 5.3(*) is a θ-p-map is presented in the next result.

(5.5) *If $f: X \to Y$ is θ-continuous and satisfies* 5.3(*) *where X and Y are Hausdorff, then there is a wθ-continuous extension $F: \kappa X \to \kappa Y$ with this property: for each $z \in \kappa X$ and open set U of $F(z)$ in κY, there is an open set V containing z in κX such that $F(\text{cl}_{\kappa X} V) \subseteq \text{cl}_\theta \text{cl}_{\kappa Y} U$ where θ-closure is relative to κY.*

Proof. It is easy to show that $f: X \to \kappa Y$ is θ-continuous and satisfies 5.3(*). By 5.4, for each free open ultrafilter $\mathscr{U}$ on $X, f(\overline{\mathscr{U}})$ almost converges to some point in κY; we denote one of these points as $F(\mathscr{U})$. Extend the definition of F to X by defining $F(x) = f(x)$ for $x \in X$. Thus, F is an extension of f. Let $z \in \kappa X$ and U be an open set of $F(z)$ in κY. There is an open V containing z in κX such that

$$F(\text{cl}_X(V \cap X)) \subseteq \text{cl}_{\kappa X} U.$$

Since

$$\text{cl}_{\kappa X} V = \text{cl}_X(V \cap X) \cup \text{cl}_{\kappa X} V\backslash X,$$

then

$$F(\text{cl}_{\kappa X} V) \subseteq \text{cl}_{\kappa Y} U \cup F(\text{cl}_{\kappa X} V\backslash X).$$

Let $\mathscr{U} \in \text{cl}_{\kappa X} V\backslash X$. Then $V \cap X \in \mathscr{U}$. Since $f(\overline{\mathscr{U}})$ meets $\overline{\mathscr{N}_{F(\mathscr{U})}}^{\kappa Y}$, then

$$F(\mathscr{U}) \in \text{cl}_\theta(f(\text{cl}_X(V \cap X))) \subseteq \text{cl}_\theta(\text{cl}_{\kappa Y} U).$$

Hence, $F(\text{cl}_{\kappa X} V) \subseteq \text{cl}_\theta(\text{cl}_{\kappa Y} U)$.

In general, the function F constructed in the proof of 5.5 will not be θ-continuous; that is, the random selection process of obtaining F may not yield a θ-continuous extension even though some selection process might yield a θ-continuous extension.

Problem. Prove or disprove if $f: X \to Y$ is θ-continuous and satisfies 5.3(*) where X and Y are Hausdorff, then f is a θ-p-map.

6. θ-rigid and f.f.r. sets

In this section, we present some properties of θ-rigid subsets (defined in Section 3) and f.f.r. subsets (defined in Section 4) and compare these concepts to compactness and quasi-H-closure.

(6.1) *Disjoint θ-rigid subsets A and B of a Hausdorff space X can be separated by disjoint open sets.*

Proof. Let $a \in A$. For each $x \in B$, there are disjoint open sets U_x of a and V_x of x. There is a finite subset $C \subseteq B$ such that $B \subseteq \operatorname{int}(\operatorname{cl}(\bigcup \{V_x: x \in C\}))$. Let

$$U_a = \bigcap \{U_x: x \in C\} \quad \text{and} \quad V_a = \operatorname{int}(\operatorname{cl}(\bigcup \{V_x: x \in C\})).$$

Now $a \in U_a$, $B \subseteq V_a$, and $U_a \cap V_a = \emptyset$. Repeating the argument for each $a \in A$, we obtain disjoint open sets containing A and B.

(6.1.1) COROLLARY. *If every closed subset of a Hausdorff space X is θ-rigid in X, then X is compact.*

Proof. Since X is θ-rigid in X, then X is H-closed. By 6.1, X is normal. Hence, X is compact.

(6.1.2) COROLLARY. *If A is θ-rigid in a Hausdorff space X, then every point of $X\backslash A$ is contained in a closed neighborhood that misses A.*

Our next result characterizes θ-rigidity in terms of open filters yielding a means of comparison of the definition of f.f.r.

(6.2) *A is θ-rigid in X if and only if for every open filter $\mathscr{G}$ on X if $A \cap \operatorname{ad}_X \mathscr{G} = \emptyset$, there is an open $U \in \mathscr{G}$ such that $A \cap \operatorname{cl} U = \emptyset$.*

Proof. The proof follows from 3.3(b) and the fact that if $\mathscr{F}$ is a filter base on X and $O(\mathscr{F}) = \{U \subseteq X: U \text{ open and } U \supseteq F \text{ for some } F \in \mathscr{F}\}$, then $\operatorname{al}_X \mathscr{F} = \operatorname{ad}_X O(\mathscr{F})$.

(6.2.1) COROLLARY. *If A is θ-rigid in X, then A is f.f.r. in X.*

It is straightforward to show that a closed subset A of a Hausdorff space X is f.f.r. in X if and only if $\operatorname{cl}_\theta^{\kappa X} A = \operatorname{cl}_\theta^X A$ where $\operatorname{cl}_\theta^{\kappa X}$ (resp. $\operatorname{cl}_\theta^X$) denotes the θ-closure in κX (resp. X). We now obtain a characterization of a concept weaker than f.f.r.

(6.3) *Suppose X is Hausdorff and $A \subseteq X$. A is contained in the union of an H-closed subspace and a nowhere dense set if and only if for each free open ultra-*

filter $\mathscr{U}$ on X, there is $U \in \mathscr{U}$ such that $U \cap A = \emptyset$, In particular, the latter condition is equivalent to $\mathrm{cl}_X A = \mathrm{cl}_{\kappa X} A$.

Proof. Suppose for each free open ultrafilter $\mathscr{U}$ on X, there is $U \in \mathscr{U}$ such that $U \cap A = \emptyset$. Then $U \subseteq X \backslash \mathrm{cl}\, A$ implying $X \backslash \mathrm{cl}\, A \in \mathscr{U}$. By Theorem 2.4 in [14], $X \backslash \mathrm{int}\, (\mathrm{cl}\, (X \backslash \mathrm{cl}\, A))$ is H-closed. Hence, $\mathrm{cl}\, (\mathrm{int}\, (\mathrm{cl}\, A))$ is H-closed. Let $B = A \backslash \mathrm{cl}\, (\mathrm{int}\, (\mathrm{cl}\, A))$. Since $\mathrm{cl}\, B \cap \mathrm{int}\, (\mathrm{cl}\, A)$ is empty and $B \subset \mathrm{cl}\, A$, then $\mathrm{int}\, (\mathrm{cl}\, B) = \emptyset$ and B is nowhere dense. The proof of the converse is straightforward.

(6.3.1) COROLLARY. *If A is f.f.r. in a Hausdorff space X, then* $\mathrm{cl}\, (\mathrm{int}\, A)$ *is H-closed and A is the union of an H-closed subspace and a closed nowhere dense subset.*

These concepts are related with compactness and quasi-H-closure in the following diagram:

$$\begin{tikzcd}
 & A \text{ compact} \arrow[dl] \arrow[d] \arrow[dddr] & \\
A \text{ quasi-}H\text{-closed} \arrow[ddr] & A \text{ compact in } X_s \arrow[d] & \\
 & A\ \theta\text{-rigid in } X \arrow[d] & \\
 & A \text{ quasi-}H\text{-closed rel. to } X & A \text{ f.f.r. in } X
\end{tikzcd}$$

Clearly, if A is a compact subspace of X, then A is quasi-H-closed and A is compact subspace of X_s (see remark after 3.3). By 3.3(c) and the remark following 3.3, if A is a compact subspace of X_s, then A is θ-rigid in X. By 3.3(c), if A is θ-rigid in X, then A is H-closed relative to X (noted in the paragraph preceeding 2.1 in [15]). By 2.5 in [15], if X is Hausdorff and A is H-closed relative to X, then A is a closed subset of X; hence, for a Hausdorff space X, in all of the concepts, A is closed in X. In general, the converse of each of the implications is false even if X is Hausdorff. If X is regular and A is quasi-H-closed relative to X, then A is compact, and if X is H-closed, Urysohn and A is H-closed relative to X, then by 2.8, A is a compact subspace of X_s. If $A = X$, then the concepts A is θ-rigid in X, A is quasi-H-closed relative to X, A is quasi-H-closed, and A is f.f.r. in X are equivalent. In fact, for the space Y described in Example 1.1, Y is rigid in Y but Y is not a compact subspace of $Y = Y_s$. In [2] it is noted that the subspace X described in Example 1.1 is H-closed but X is not f.r.r. in $Y \backslash \{b_0\}$. The subspace $X = \{c_i \colon i \in \mathrm{N}\}$ is f.f.r. in Y but C is not H-closed relative to Y.

(6.4) *Suppose for each $B \subseteq X$, $\mathrm{cl}_\theta\, \mathrm{cl}_\theta\, B = \mathrm{cl}_\theta\, B$. If A is quasi-H-closed relative to X, then A is θ-rigid in X.*

Proof. Let $\mathscr{F}$ be a filter base on X such that $\mathrm{al}_X\, \mathscr{F} \cap A = \emptyset$. For each $a \in A$, there is $F_a \in \mathscr{F}$ such that $a \notin \mathrm{cl}_\theta\, F_a$. Hence, $a \notin \mathrm{cl}_\theta\, \mathrm{cl}_\theta\, F_a$ and there is an open set U_a containing a such that $\mathrm{cl}\, U_a \cap \mathrm{cl}_\theta\, F_a = \emptyset$. There is a finite subfamily $C \subseteq A$ such that $A \subseteq \bigcup \{\mathrm{cl}\, U_a: a \in C\}$, and there is $F \in \mathscr{F}$ such that $F \subseteq \bigcap \{F_a: a \in C\}$. It follows that $A \cap \mathrm{cl}_\theta\, F = \emptyset$. By 3.3(b), A is θ-rigid in X.

(6.5) *Remark.* Note that if X is a Hausdorff space such that for each open subset $B \subseteq X$, $\mathrm{cl}_\theta\, \mathrm{cl}_\theta\, B = \mathrm{cl}_\theta\, B$, then X is a Urysohn space.

References

1. B. Banaschewski, *Extensions of topological spaces*, Canad. Math. Bull., vol. 7 (1964), pp. 1–22.
2. A. Blaszczyk and J. Mioduszewski, *On factorization of maps through τX*, Coll. Math., vol. 23 (1971), pp. 45–52.
3. N. Bourbaki, *General topology*, Part 1, Addison-Wesley, Reading, Mass., 1966.
4. M. Berri, J. Porter, and R. Stephenson, *A survey of minimal topological spaces*, Proc. Kanpur Topology Conference, 1968, Academic Press, 1970, pp. 93–114.
5. R. F. Dickman, Jr., *On closed extensions of functions*, Proc. Nat. Acad. Sci. U.S.A., vol. 62 (1969), pp. 326–332.
6. ———, *Regular closed maps*, Proc. Amer. Math. Soc., vol. 39 (1973), pp. 414–416.
7. R. F. Dickman, Jr., and J. R. Porter, *θ-closed subsets of Hausdorff spaces*, Pacific J. Math., vol. 59 (1975), pp. 407–415.
8. S. Fomin, *Extensions of topological spaces*, Ann. Math., vol. 44 (1943), pp. 471–480.
9. R. W. FitzGerald and P. M. Swingle, *Core decompositions of continua*, Fund. Math., vol. 61 (1967), pp. 33–50.
10. D. Harris, *Katětov extension as a functor*, Math. Ann., vol. 193 (1971), pp. 171–175.
11. S. Iliadis and S. Fomin, *The method of centred systems in the theory of topological spaces*, Uspekhi Mat. Nauk 21 (1966), pp. 47–76 = Russian Math. Surveys, vol. 21 (1966), pp. 37–62.
12. M. Katětov, *Über H-abgeschlossene und bikompakte Räume*, Časopis Pěst. Mat. Fys., vol. 69 (1940), pp. 36–49.
13. J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Dissertationes Mathematicae, vol. 66 (1969).
14. J. Porter, *On locally H-closed spaces*, Proc. Lond. Math. Soc. (3), vol. 20 (1970), pp. 193–204.
15. J. R. Porter and J. D. Thomas, *On H-closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc., vol. 138 (1969), pp. 159–170.
16. J. R. Porter and C. Votaw, *H-closed extensions I*, General Topology and Appl., vol. 3 (1973), pp. 211–224.
17. L. Rudolf, *θ-continuous extensions of maps on τX*, Fund. Math., vol. 77 (1972), pp. 111–131.
18. ———, *Extending maps from dense subspaces*, Fund. Math., vol. 77 (1972), pp. 171–190.
19. N. V. Veličko, *H-closed topological spaces*, Mat. Sb., vol. 70 (1966), pp. 98–112 = Amer. Math. Soc. Transl., vol. 78 (1968), pp. 103–118.
20. ———, *On extension of mappings of topological spaces*, Sibirsk. Mat. Ž., vol. 6 (1965), pp. 64–69 = Amer. Math. Soc. Transl., vol. 98 (1970), pp. 41–47
21. G. Viglino, *Extensions of functions and spaces*, Trans. Amer. Math. Soc., vol. 179 (1973), pp. 61–69.
22. G. T. Whyburn, *Directed families of sets and closedness of functions*, Proc. Nat. Acad. Sci. U.S.A., vol. 54 (1965), pp. 688–692.

Virginia Polytechnic Institute and State University
Blacksburg, Virginia
University of Kansas
Lawrence, Kansas