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REPRESENTING CODIMENSION-ONE HOMOLOGY CLASSES ON CLOSED NONORIENTABLE MANIFOLDS BY SUBMANIFOLDS

BY

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In [5], Julie Patrusky and the author proved that a codimension-one homology class on a closed orientable connected piecewise linear manifold can be represented by a closed connected orientable submanifold precisely when the class is primitive. If M is a closed n-dimensional manifold, we will call a class in $H_{n-1}(M, Z)$ primitive if the induced class in $H_{n-1}(M, Z)/\text{torsion}$ is the zero class or is not a nontrivial multiple of any other class.

The representation theorem we prove here is for closed connected nonorientable P.L. manifolds and its proof is much more involved than is the proof of the orientable case. In dimension two our theorem implies that an integer homology class on a connected closed nonorientable surface can be represented by an embedded circle if and only if the class is primitive or twice a primitive class.

Recall that the Universal Coefficient Theorem implies that if M is a closed, connected, n-dimensional P.L. manifold, then $H_{n-1}(M, Z) = Z_2 \oplus F$ where F is a free abelian group. After triangulation, an orientable k-dimensional P.L. submanifold naturally represents a class in $H_k(M, Z)$. (See [8].) We will call a closed oriented $(n-1)$-dimensional submanifold $N \subset M$ representing a class $\delta \in H_{n-1}(M, Z)$ a minimal representative for δ if there is no other submanifold representative for δ having fewer components. Let $|N|$ denote the number of components of N.

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THEOREM 1. *Suppose M is a closed connected nonorientable n-dimensional P.L. manifold. Let σ denote the order two class in $H_{n-1}(M, Z)$. If N is a minimal representative for a nonzero $\delta \in H_{n-1}(M, Z)$, then:*

(1) *If $M-N$ is not connected, then each nonorientable component of $M-N$ has one end.*

(2) *Every component of $M-N$ with three ends comes from cuts along two components of N.*

(3) *Each orientable component of $M-N$ has at most four ends. If there is a component of $M-N$ with four ends, then $M-N$ is connected and $|N|=2$.*

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(4) *$M-N$ has one nonorientable component if and only if for some γ, a primitive class of infinite order, $\delta=(2r+1)\gamma$, where $r\geq 0$, and then $|N|=r+1$.*

(5) *$M-N$ has two nonorientable components if and only if for some γ, a primitive class of infinite order, $\delta=2r\gamma$, where $r\geq 0$, and then $|N|=r$.*

(6) *$M-N$ is orientable if and only if for some γ, a primitive class of infinite order, $\delta=2r\gamma+\sigma$, and then $|N|=r+1$.*

Unless otherwise stated, M will denote a connected compact P.L. manifold, possibly with boundary, and N will denote a codimension-one oriented P.L. submanifold of M.

The main technique of proof of parts 1, 2 and 3 of Theorem 1 is to show that we can take the internal connected sums of components of N when the conditions on the various parts of the theorem are not met, thereby reducing the number of path components of N.

One can take the interval oriented connected sum of codimension-one closed oriented submanifolds N_1 and N_2 if there exists a P.L. path $\alpha: [0, 1]\rightarrow M$ with the following properties:

(1) α is an embedding.

(2) $\alpha\cap N_1=\alpha(0)$, $\alpha\cap N_2=\alpha(1)$.

(3) After orienting the normal bundle to α, the intersection numbers of α with N_1 and N_2 have opposite sign.

The intersection sign of α with N_i is defined to be positive if the orientation of the normal bundle of α "agrees" with the orientation of N_i, and is defined to be negative otherwise. This definition of intersection sign makes perfect sense if everything is smooth, M is Riemannian and $\alpha(t)$ is orthogonal to N_1 and N_2 at the points $\alpha(0)$ and $\alpha(1)$ respectively. In general, the normal bundle of α can be considered to be an oriented disk bundle B over α embedded in M along α with the fiber disk D_0 above $\alpha(0)$ contained in N_1 and the fiber disk D_1 above $\alpha(1)$ contained in N_2. In this more general situation we can again define the intersection sign of α with N_1 and N_2 according to whether the orientation of D_i in the disk bundle agrees or disagrees with orientation induced by N_{i+1}.

Paths α with the above properties will be called *special connecting paths.* If instead of opposite intersection signs, α has the same sign of intersection with both N_1 and N_2, then we will call α a *non-special connecting path.*

If α is a special connecting path joining N_1 to N_2 and $B\subset M$ is the embedded normal disk bundle as given above, then one can form the connected sum of N_1 and N_2 along α by joining (N_1-D_1) and (N_1-D_2) along the associated boundary sphere bundle to B in M. It is straightforward to verify that the resulting oriented P.L. submanifold represents the homology class $[N_1]+[N_2]\in H_{n-1}(M, Z)$.

LEMMA 1. *If M is a compact connected nonorientable P.L. manifold with* *oriented boundary components $N_1, N_2, \ldots, N_k$, then there exist special and non-special connecting paths joining any $p \in N_i$ to any $q \in N_j$ when $i \neq j$.*

Proof. Case 1. The dimension of M is greater than two. Let α be an embedded loop in the interior of M with nonorientable normal bundle. Let γ_1 be an embedded path disjoint from α joining p to q, and let γ_2 be an internal connected sum of γ_1 and α. By construction, either γ_1 or γ_2 is a special connecting path and the other path is a non-special connecting path.

Case 2. The dimension of M is two. By the classification theorem for surfaces with boundary, M is a punctured sphere with cross caps. Let α be a circle embedded in a cross cap with a Möbius strip neighborhood. By our choice of α, it is clear that there is a path γ_1 disjoint from α joining p to q. Let γ_2 be the connected sum of α and γ_1. As above, either γ_1 or γ_2 is special and the other one is non-special.

LEMMA 2. *Suppose M is a connected compact P.L. manifold with oriented boundary components $N_1, N_2, \ldots, N_k$. If there are no special connecting paths joining N_i to N_j and no special connecting paths from N_j to N_s, then there is a special connecting path joining N_i to N_s.*

Proof. Let γ_1 be a non-special connecting path joining $p \in N_i$ to $q \in N_j$ and let γ_2 be a non-special connecting path joining q to $n \in N_s$ with $\gamma_1 \cap \gamma_2 = \{q\}$. If γ_3 is the composite path $\gamma_1\gamma_2$ pushed off on N_j, then it is straightforward to verify that γ_3 is a special connecting path.

LEMMA 3. *Let M be a closed connected nonorientable P.L. manifold. If $N \subset M$ is connected, then $[N] \in H_{n-1}(M, Z)$ is a primitive or twice a primitive homology class.*

Proof. If $[N] = \sigma$ or $[N] = 0$, then we are finished. Hence, from now on assume that $[N]$ has infinite order.

Let $H^*_{DR}(M, Q)$ denote the rational P.L. De Rham cohomology algebra as defined by D. Sullivan in [2], and let $H^k_{DR}(M, Z) = \{[\omega] \in H^k_{DR}(M, Q) \mid \int_c \omega \in Z$ for all integer cycles c on $M\}$. Note that $H^1_{DR}(M, Z)$ is the image of

$$H^1_{\mathrm{Sing}}(M, Z) \subset H^1_{\mathrm{Sing}}(M, Q) \xrightarrow{\int} H^1_{DR}(M, Q)$$

where $\int$ is the P.L. De Rham isomorphism. If M is smooth, then one can use the usual De Rham cohomology algebra arising from smooth differential forms.

Let $p: \tilde{M} \to M$ denote the oriented two sheeted cover of M and $g: \tilde{M} \to \tilde{M}$ be the order two deck transformation.

Case 1. $M - N$ is not connected. Suppose $[N] = k[Q]$ or $k[Q] + \sigma$ where Q is a P.L. integer chain representing a primitive class of infinite order.

Note that we may assume that both components of $M-N$ are nonorientable since otherwise $[N]=0$.

Since N disconnects M, the normal bundle to N is trivial and a neighborhood of N in M is orientable. Therefore the inclusion map $i: N \to M$ lifts to $\tilde{M}$ giving two oriented submanifolds N_1 and N_2 of $\tilde{M}$ which disconnect $\tilde{M}$ into two components C_1 and C_2. The components C_1 and C_2 are the inverse image under p of the two components of $M-N$. Elementary covering space theory shows that C_1 and C_2 are invariant under the deck transformation g and $g \mid N_1: N_1 \to N_2$ is orientation preserving. Since g is orientation reversing on C_1, it is straightforward to verify that, after picking a proper orientation on C_1, the closure of C_1 in $\tilde{M}$ gives a homology between N_1 and N_2. Hence, we have $[N_1]=[N_2]$.

By Theorem 1 in [5], $[N_1]$ is primitive. Therefore, there is an

$$[w] \in H_{DR}^{n-1}(\tilde{M}, Z) \quad \text{with} \quad \int_{N_1} w = 1 = \int_{N_2} w.$$

Since $\eta = w + g^* w$ is g-invariant, $\eta = p^*(\alpha)$ for some $[\alpha] \in H_{DR}^{n-1}(M, Z)$. Now, $k \int_Q \alpha = \int_N \alpha = \int_{N_1} p^*(\alpha) = \int_{N_1} \eta = 2$. Hence $k=1$ or $k=2$. If $[N]=2[Q]+\sigma$ or if $[N]$ is primitive, then $0 \neq [N] \in H_{n-1}(M, Z_2)$. Since N disconnects M, these cases cannot occur. Therefore, $[N]=2[Q]$.

Case 2. The normal bundle of N is nontrivial and $M-N$ is connected. In this case, $p^{-1}(N)=N_1 \subset \tilde{M}$ is connected. As above, we may assume there is a g-invariant closed $(n-1)$-form $\eta = p^*(\alpha)$ on $\tilde{M}$, where α is a closed integral $(n-1)$-form on M, and $\int_{N_1} \eta = 2$. This implies $\int_N \alpha = 1$, and hence $[N]$ is primitive.

Case 3. The normal bundle of N is trivial and $M-N$ is connected and orientable. In this case, the cycle $2N$ is an oriented boundary. Since $[N]$ has infinite order, this case does not occur.

DEFINITION. The *end closure* T of a path component U of $M-N$ is obtained by attaching a compact codimension-one submanifold on each topological end of U. The topological ends of U arise from cutting M along certain components of N. If an end of U arises from cutting along a component N' of N with trivial normal bundle, then the attached boundary submanifold on T is diffeomorphic to N'. However, if the normal bundle of N' is nontrivial, then the boundary component of T attached at this end of U will correspond to some two sheeted cover of N'. The boundary components of T have fixed orientations induced from the orientation on the associated components of N. Therefore T is a compact P.L. manifold with fixed orientations on each boundary component.

Case 4. The normal bundle to N is trivial, and $M-N$ is connected and nonorientable. Let T be the end closure of $M-N$ and suppose that P_1 and

P_2 are two distinct points on the boundary of T which correspond to the same point on N. By Lemma 1, there is a non-special connecting path joining P_1 to P_2. Let α be the associated loop in M which has a trivial normal bundle.

By choice of α, a tubular neighborhood of α is homeomorphic to $D^{n-1}\times S^1$. Now apply the Thom construction in [7] (especially pages 47 and 48) to get a mapping $T_\alpha: M^n \to S^{n-1}$. Clearly, $(T_\alpha)_*([N])$ generates $H_{n-1}(S^{n-1}, Z) = Z$. Hence, $[N]$ is a primitive non-torsion class in M.

Since one of the above four cases must occur, the lemma is proved.

Proof of Theorem 1. (1) Let T be the end closure of a nonorientable path component of $M-N$ with more than one end. Since $M-N$ is not connected by assumption, two boundary components of T, say E_1 and E_2, come from cuts along distinct path components N_i and N_j of N. By Lemma 1, there is a special connecting path joining E_1 to E_2. Hence, we can take the oriented connected sum of N_i and N_j to reduce the number of path components of N. However, this construction contradicts the minimality of N, proving (1).

(2) Suppose T is the end closure of a path component of $M-N$ with three ends. Recall that the boundary of T is given the orientation induced by N, not by an orientation of T. By Lemma 2, there is a special connecting path joining two boundary components of T. If these boundary components come from cuts along distinct path components N_1 and N_2 of N, then we can take the oriented connected sum of N_1 and N_2 to reduce the number of path components of N. Since this construction contradicts the minimality of N, two boundary components of T must come from a cut along the same path component of N, proving (2).

(3) Suppose T is the end closure of an orientable path component of $M-N$ with more than four ends. In this case, at least three of the boundary components of T, say E_1, E_2, E_3, arise from cuts along distinct path components of N. By Lemma 2, there is a special connecting path joining two of these ends. As above, the existence of such a special connecting path contradicts the minimality of N, proving the first part of (3). If T has exactly four boundary components, the above argument shows the ends come from cuts along two distinct path components of N. The second part of (3) follows immediately from this observation.

Suppose that T is the end closure of an orientable path component of $M-N$. Before finishing the proof of the theorem, we remark on the relationship between the number of boundary components of T and the homology classes associated to these boundary components.

Case 1. T has four boundary components. By (3) we know that the four boundary components E_1, E_2, E_3, E_4 of T arise from cuts along path components N_1 and N_2 of N. Suppose that the ends E_1, E_2 of T come from a cut along N_1 and the ends E_3, E_4 come from a cut along N_2. Since N is minimal, there are no special connecting paths joining either E_1, E_2 to either of E_3, E_4. This implies T induces a homology between $2N_1$ and $2N_2$ in M. Therefore $2[N_1]=2[N_2]$.

Let $[N_1]_2$ and $[N_2]_2$ denote the associated classes in $H_{n-1}(M, Z_2)$. It is easy to construct an embedded circle $\alpha: S^1 \to M$ which intersects $N_1 \cup N_2$ transversely in one point of N_1. If $\cap: H_1(M, Z_2) \times H_{n-1}(M, Z_2) \to Z_2$ denotes the intersection pairing on homology (see [8]), then

$$[\alpha]_2 \cap ([N_1]_2 + [N_2]_2) = 1 \in Z_2$$

where $[\alpha]_2$ is the class in $H_1(M, Z_2)$ associated to the loop α. This implies $[N_1]=[N_2]+\sigma$. Since N is minimal there is no special connecting curve joining E_1 to E_3 or joining E_3 to E_2. Therefore, Lemma 2 implies that there is a special connecting path β joining E_1 and E_2 with end points corresponding to the same point on N_1. The path β induces a loop $\alpha: S^1 \to M$ which intersects N_1 in one point. By the choice of α, the tubular neighborhood of α is homeomorphic to $D^{n-1} \times S^1$. As in the proof of Lemma 3, the Thom construction applied to α shows that $[N_1]$ is a primitive class of infinite order. Hence, $[N]=2\gamma+\sigma$.

Case 2. T has one boundary component. Since T is orientable, we may assume that the single boundary component of T arises from a cut along a component of N with nontrivial normal bundle. This shows $|N|=1$ and $M-N$ is connected. Since the cycle $2N$ bounds the cycle T, $[N]$ has order two in $H_{n-1}(M, Z)$ and hence $[N]=\sigma$.

Case 3. T has two boundary components. If these boundary components arise from a single cut, then $|N|=1$ and $[N]=\sigma$. If the boundary components of T come from cuts along path components N_1 and N_2 with trivial normal bundle, then clearly $[N_1]=\pm[N_2]$. But by minimality of N, $[N_1]\neq -[N_2]$ and so $[N_1]=[N_2]$. If N_1 has trivial normal bundle and N_2 has nontrivial normal bundle, then the minimality of N similarly implies $2[N_2]=[N_1]$. If N_1 and N_2 both have nontrivial normal bundle, then $M-N$ is path connected and $2[N_1]=2[N_2]$. In this last case, there is a loop α in a neighborhood of N_2 and α intersects $N_1 \cup N_2$ transversally in a single point on N_2. Therefore,

$$[\alpha]_2 \cap ([N_1]_2 + [N_2]_2) = 1 \in Z_2.$$

This implies $[N_1]=[N_2]+\sigma$ in the case where both normal bundles are nontrivial.

Case 4 T has three boundary components. Suppose two of the boundary components, say E_1 and E_2, of T come from cuts along N_1 and N_2 respectively. The argument given in the proof of (3) shows that the third boundary component of T arises from a cut along either N_1 or N_2, say N_1. If $M-N$ is connected, then the normal bundle to N_2 is nontrivial. In this case,

$2[N_1]=2[N_2]$. Since there is a loop α with

$$[\alpha]_2 \cap ([N_1]_2+[N_2]_2)=1\in Z_2,$$

we must have $[N_1]=[N_2]+\sigma$. By minimality of N, $[N_2]\neq\sigma$ and so N_1 and N_2 must have infinite order. By Case 2 of Lemma 3, $[N_2]$ is primitive, which implies $[N]=2[N_2]+\sigma=2\gamma+\sigma$ for the primitive class $\gamma=[N_2]$ of infinite order. If $M-N$ is not path connected then T gives a homology between $2N_1$ and N_2. Hence, $2[N_1]=[N_2]$ when $M-N$ is not connected.

We will now prove the forward implications of (4), (5) and (6). It is a direct consequence of the ordering process described below that there are never more than two nonorientable path components in $M-N$. Since the forward implications of (4), (5) and (6) are mutually exclusive and are inclusive, the converse implications are also true.

(4) Suppose $M-N$ is connected with one nonorientable component. By the proof of part (1) of the Theorem and Case 2 and Case 4 of Lemma 3, N is connected and represents a primitive class. Since the mod two reduction of σ is the Poincare dual to the first Stiefel-Whitney class of M, the duality theorem implies that $[N]_2\neq\sigma_2\in H_1(M, Z_2)$ and hence $[N]\neq\sigma$. This shows $N=\gamma$ where γ is a primitive class of infinite order.

If $M-N$ is not connected, then order the path components U_1, $U_2, \ldots, U_{k+1}$ of $M-N$ and the components $N_1, N_2, \ldots, N_l$ of N as follows: Let U_1 be the one nonorientable path component of $M-N$. By part (1), U_1 has one end arising from a cut along a component N_1 of N with trivial normal bundle. Since $M-N$ is not path connected, N_1 arises as the end to some other component U_2 of $M-N$. We have one of the following cases.

(i) If U_2 has three ends or if U_2 has two ends and the second end of U_2 arises from a cut along a component N_2 of N with nontrivial normal bundle, then U_2 is the last component of $M-N$ and $|N|=2$.

(ii) If U_2 has two ends and the second end arises from a cut along a component N_2 of N with trivial normal bundle, then N_2 arises as the end of another component U_3 of $M-N$.

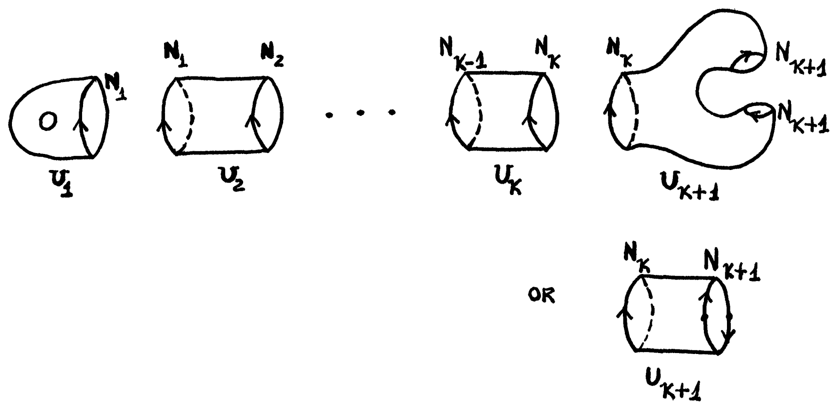
Either situation (i) or (ii) as above holds for U_3. If (i) holds then U_3 is the last component of $M-N$ and label the remaining component of N by N_3. If (ii) holds then the other end of U_3 arises from a cut along a component N_3 of N with trivial normal bundle. Now N_3 arises as the end of another component U_4 of $M-N$.

Continue labeling the components of $M-N$ and N sequentially, as in the last paragraph, until all of the components of $M-N$ and of N are numbered. See the figure on p. 206.

It follows from the above labeling process that $|N|=k+1$ and that $[N_1]=[N_2]=\cdots=[N_k]$. By the earlier Cases 2 and 4, we have $[N_k]=2[N_{k+1}]$. Hence

$$[N]=(2k+1)[N_{k+1}]=(2k+1)\gamma$$

for the primitive class $\gamma=[N_{k+1}]$ of infinite order.



(5) Suppose $M-N$ has two non-orientable path components. A similar ordering argument as in the proof of (4) shows that all the components of N are homologous. By Case 1 of Lemma 3, each component of N represents twice a primitive class. Hence, $[N]=2k\gamma$ with $|N|=k$.

(6) Suppose $M-N$ is orientable. If there is a component of $M-N$ with three ends or with an end arising from a cut along a component of N with nontrivial normal bundle, then call this component U_1. Let N_1 be the associated component of N with nontrivial normal bundle or the component of N_1 which gives rise to two ends of U_1. If U_1 has one end, then $[N]=\sigma$ and $|N|=1$. Suppose U_1 has another end N_2. Continue the ordering process of components of $M-N$ and of N as in the proof of (4). In this case, we have $[N_2]=[N_3]=\cdots=[N_{k+1}]$ and $2[N_1]=[N_2]=[N_{k+1}]=2[N_{k+2}]$ where N_{k+2} is the last component of N. Hence

$$[N]=2(k+1)[N_1] \quad \text{or} \quad [N]=2(k+1)[N_1]+\sigma,$$

where $k\geq 0$. Since there is a loop α intersecting N transversally at one point on N_1, we have $[N]=2(k+1)\gamma+\sigma$ where $\gamma=[N_1]$. Since $2[N_1]=2[N_{k+2}]$, γ is a primitive class of infinite order.

If there is a component of $M-N$ with four ends, then by earlier remarks $M-N$ is path connected, $|N|=2$ and $[N]=2\gamma+\sigma$.

If every component of $M-N$ has two ends and if the normal bundle to N is trivial and N_i is a component of N, then $2[N_i]=0$. Hence, $[N]=\sigma$ and $|N|=1$.

The above three cases are inclusive which completes the proof of (6) and by the remark before the proof of (4) completes the proof of the Theorem.

We now prove that every element of $H_{n-1}(M, Z)$ can be represented by a closed embedded orientable submanifold.

Theorem 2. *Suppose M is a closed nonorientable P.L. n-dimensional manifold. Then every element of $H_{n-1}(M, Z)$ can be represented by an orientable P.L. submanifold.*

Proof. Let $x \in H_{n-1}(M, Z)$ and let $x^* \in H^1(M, Z[w_1])$ be the image under the Poincare duality isomorphism. Here $Z[w_1]$ denotes the integer sheaf twisted by the first Stiefel Whitney class $w_1 \in H^1(M, Z_2)$. Since $H^1(M, Z_2)$ has as a classifying space the $(n+1)$-dimensional projective space $\mathbf{P}^{n+1}$, we can consider w_1 to be represented by a P.L. map $\sigma: M \to \mathbf{P}^{n+1}$.

In [3] (see pages 171–176), it is shown that $H^1(M, Z[w_1])$ has a "classifying space". In fact, the elements of $H^1(M, Z[w_1])$ are in natural one to one correspondence with the fiber homotopy classes of liftings of $\sigma: M \to \mathbf{P}^{n+1}$ to the twisted circle bundle $\rho: K \to \mathbf{P}^{n+1}$. Here K is the generalized Klein bottle formed by $S^{n+1} \times S^1/\tau$ where $\tau(x, y) = (-x, \tilde{y})$ and $\sim$ denotes complex conjugation on S^1. Thus an element $\gamma \in H^1(M, Z[w_1])$ can be "represented" by a lifting σ_γ of σ:

$$\begin{tikzcd} & K \arrow[d, "\rho"] \\ M \arrow[ur, "\sigma_\gamma"] \arrow[r, "\sigma"'] & \mathbf{P}^{n+1}. \end{tikzcd}$$

Since complex conjugation on S^1 has two fixed points ± 1, there are two cross sections $c_{-1}, c_1: \mathbf{P}^{n+1} \to K$ of the bundle $\rho: K \to \mathbf{P}^{n+1}$.

Claim. If σ_γ is transverse regular to $c_1(\mathbf{P}^{n+1})$, then $\sigma_\gamma^{-1}((c_1(\mathbf{P}^{n+1})))$ is an orientable P.L. submanifold which represents the Poincare dual of γ.

Recall that $H^1(M, Z[w_1])$ can be computed from a certain cochain complex as follows. After picking a P.L. triangulation of M, we can lift this triangulation to a triangulation of the oriented two sheeted covering space $\alpha: \tilde{M} \to M$ of M with covering transformation $T: \tilde{M} \to \tilde{M}$. Given an oriented k-simplex β in M, it will lift to two oriented k-simplices β_1 and β_2 of $\tilde{M}$ with $T(\beta_1) = \beta_2$. This implies for the dual cochains β_1^* and β_2^*, we have $T^*(B_1^*) = -\beta_2^*$. This is because T is orientation reversing as a map on $\tilde{M}$.

The map T has a $+1$ and a -1 eigen space on both the k-chain and a k-co chain complexes of $\tilde{M}$. Now $H^k(M, Z[w_1])$ is defined to be the k-*th* cohomology group of skew cochain complex associated to the -1 eigen spaces of T in the cochain complex of $\tilde{M}$.

We now consider the following commutative diagram

$$\begin{tikzcd} \tilde{M} \arrow[r, "\tilde{\sigma}_\gamma"] \arrow[dd, "\alpha"'] & \tilde{K} = S^{n+1} \times S^1 \arrow[r, "\pi_2"] \arrow[d] & S^1 \\ & K \arrow[d, "\rho"'] & \\ M \arrow[ur, "\sigma_\gamma"] \arrow[r, "\sigma"'] & \mathbf{P}^{n+1} \arrow[u, bend right, "c_1"'] & \end{tikzcd}$$

where $\tilde{\sigma}_\gamma$ is the lift of σ_γ. It is straightforward to check that

$$\pi_2 \circ \tilde{\sigma}_\gamma: \tilde{M} \to S^1$$

is a Z_2 equivariant map where Z_2 acts by T on $\tilde{M}$, and by complex conjugation on S^1.

Let $I_\theta^+ = \{e^{i\gamma} \in S^1 \mid -\theta \le \gamma \le \theta\}$ and let

$$I_\theta^- = \overline{S^1 - \mathring{I}_\theta^+}$$

where $\circ$ denotes interior and an overbar denotes closure in S^1. Define the subsets K_0, K_+, K_- of the Klein bottle K, by

$$K_0 = (S^{n+1} \times \{e^{i\theta}, e^{-i\theta}\})/\tau, \quad K_+ = (S^{n+1} \times I_\theta^+)/\tau \quad \text{and} \quad K_- = (S^{n+1} \times I_\theta^-)/\tau.$$

Then it is straightforward to check that

$$H^1(K, K_-, Z[u]) \simeq H^1(K_+, K_0, Z[u]) = Z$$

where u is the pull back of the generator of $H^1(\mathbf{P}^{n+1}, Z_2)$ to the appropriate subsets of the Klein bottle K. Hence $H^1(K, K_-, Z[u])$ is free and generated by a class we will denote by ι. We will denote the pullback of ι to the Klein bottle by the same letter $\iota \in H^1(K, Z[u])$. Now given any lift h of σ

$$\begin{tikzcd} & K \subset (K, K_+) \arrow[d, "\rho"] \\ M \arrow[ur, "h"] \arrow[r, "\sigma"'] & \mathbf{P}^{n+1} \end{tikzcd}$$

there is an element $h^*(\iota) \in H^1(M, Z[w_1])$ which only depends on the fiber homotopy equivalence class of h with respect to σ. This is the correspondence between $H^1(M, Z[w_1])$ and fiber homotopy classes of liftings of σ.

Since we may assume that $\tilde{\sigma}_\gamma$ is transverse regular to $c_1(\mathbf{P}^{n+1})$, we may pick θ small enough so that $\sigma^{-1}(K_+) = V$ is a regular neighborhood of N in M. From the diagram

$$\begin{tikzcd} (M, M - \mathring{V}) \arrow[r, "g"] & (K, K_+) \\ & K \arrow[u, hook] \arrow[d] \\ M \arrow[uu, phantom, "\cup"] \arrow[ur, "\sigma_\gamma"] \arrow[r, "\sigma"'] & \mathbf{P}^{n+1} \end{tikzcd}$$

it is clear that the class $\gamma = \sigma_\gamma^*(\iota)$ comes from a class γ^* in

$$H^1(M, M - \mathring{V}, Z[g^*u]).$$

By excision this gives a class γ^* in $H^1(V, \partial V, Z[g^*u])$.

Now let $\tilde{V} = \alpha^{-1}(V)$ and $\tilde{N} = \alpha^{-1}(N)$. Since $\pi_2 \circ \tilde{\sigma}_\gamma \mid \tilde{V}: \tilde{V} \to S^1$ is a Z_2 equivariant map and S^1 is a $K(Z, 1)$, we may consider $\pi_2 \circ \tilde{\sigma}_\gamma$ to represent an element in $H^1(\tilde{V}, Z)$ or the first cohomology of $\tilde{V}$ arising from the skew complex (the -1 eigen space for T).

We also have the following commutative diagram

$$\begin{array}{ccc} H_{n-1}(V, Z) & \xrightarrow{i^*} & H_{n-1}(M, Z) \\ \wr\| & & \wr\| \\ H^1(V, \partial V, Z[g^*u]) & \xrightarrow{J^*} & H^1(M, Z[w_1]) \end{array}$$

where J^* is induced by excision to $H^1(M, M-\mathring{V}, Z[g^*u])$ and inclusion to $H^1(M, Z[w_1])$. Of course i^* is just the map induced by inclusion. The vertical isomorphisms are Poincare duality and the diagram commutes by naturality.

There is another commutative diagram where the rows arise from part of the Gysin sequence for the Z_2 bundle $\alpha: \tilde{N} \to N$. Here the vertical isomorphisms are given by Poincare duality.

$$\begin{array}{ccccccc} H_n(V, Z) & \longrightarrow & H_{n-1}(V, Z) & \xrightarrow{\theta} & H_{n-1}(\tilde{V}, Z) & \longrightarrow & H_{n-1}(V, Z) \\ \wr\| & & \wr\| & & \wr\| & & \wr\| \\ H^0(V, \partial V, Z[g^*u]) & \xrightarrow[\cup g^*u]{} & H^1(V, \partial V, Z[g^*u]) & \xrightarrow{\alpha^*} & H^1(\tilde{V}, \partial\tilde{V}, Z) & \longrightarrow & H^1(V, \partial V, Z[g^*u]). \end{array}$$

One can compute that on the chain, cochain level the above diagram is commutative. Since $H^0(V, \partial V, Z[g^*u]) = H_n(V, Z) = 0$, the maps θ and α^* are injective.

Now recall that originally we picked an element $\gamma \in H^1(M, Z[w_1])$ and have shown that γ was in the image of an element $\gamma^* \in H^1(V, \partial V, Z[g^*u])$. By usual Poincare duality, we know that the Poincare dual of the class $\alpha^*(\gamma^*)$ can be represented by the submanifold $\tilde{N} \subset \tilde{V}$. The manifold N is oriented and by definition of θ, $\theta[N] = [\tilde{N}]$. Since θ and α are injective and the diagram commutes, $[N]$ must be the Poincare dual to γ^*. From the previous commutative diagram, it now follows that the Poincare dual of γ is the class $[N] \in H_{n-1}(M, Z)$ which proves the claim.

Given a class $\delta \in H_{n-1}(M, Z)$ we can clearly represent δ by a submanifold by considering the Poincare dual of δ as a lift $P(\delta): M \to K$ of σ so that $P(\delta)$ is transverse to $c_1(P^{n+1})$ and then take the submanifold representative $(P(\delta))^{-1}(c_1(P^{n+1}))$ for δ. Note that N has a well defined orientation. To see this first note that $\tilde{N}$ has a well defined orientation and $T \mid \tilde{N}: \tilde{N} \to \tilde{N}$ is orientation preserving. Hence the orientation induced on N is the orientation induced as the quotient space $\tilde{N}/(T \mid \tilde{N})$. This completes the proof of Theorem 2.

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