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WEIGHTED KERNEL FUNCTIONS AND CONFORMAL MAPPINGS

BY

WILLIAM S. COHN

Introduction

Let D be a domain in the plane bounded by $n + 1$ analytic Jordan curves. Garabedian [5] and Nehari [6] consider the following extremal problem. Suppose h is positive and continuous on ∂D. For $\zeta \in D$ let $S = \{f, f$ holomorphic and bounded on $D, f(\zeta) = 0$, and $|f| < h$ on $\partial D\}$. What is $\sup_{f \in S}|f'(\zeta)|$?

Within the framework of this problem certain functions arise naturally. These are the "reproducing kernels" $B(z, \zeta, h^2)$, holomorphic in $z \in D$ which satisfy

$$f(\zeta) = \int_{\partial D} f(\eta)\overline{B(\eta, \zeta, h^2)}h^2|d\eta|$$

for f holomorphic on $\bar{D}$, the closure of D.

It is the purpose of this paper to study these kernels from the point of view of the Hardy class, $H^2(D)$. The basic technique is to make simple changes in h^2 and calculate the resulting change in $B(z, \zeta, h^2)$. This amounts to varying the inner product on $H^2(D)$.

Our main results are Theorem 5.2 and 5.4. Theorem 5.4 may be regarded as a generalization of the identity

$$\frac{2(1 - \bar{\zeta}z)}{(1 - \bar{\zeta}e^{i\theta})(1 - ze^{-i\theta})} = \frac{e^{i\theta} + z}{e^{i\theta} - z} + \frac{e^{-i\theta} + \bar{\zeta}}{e^{-i\theta} - \bar{\zeta}} \tag{1}$$

which holds for $|\zeta| < 1$, $|z| < 1$.

This identity expresses a relationship between the H^2 reproducing kernel and the kernel

$$\frac{e^{i\theta} + z}{e^{i\theta} - z}$$

used in the integral representation of a singular inner function defined on the unit disk. We recall that

$$s(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\sigma(\theta)\right)$$

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is a singular inner function when σ is a positive measure on $[0, 2\pi)$ which is singular with respect to $d\theta$.

The identity (1) proves to be very useful in the work of Ahern and Clark [1], in which an isometry of $H^2 \ominus sH^2$ and $L^2(d\sigma)$ is constructed, which is natural with respect to the restricted shift operator on $H^2 \ominus sH^2$. For $f \in H^2 \ominus sH^2$, $Tf = Pzf$ is the restricted shift. Here, P denotes orthogonal projection onto $H^2 \ominus sH^2$.

In particular, Ahern and Clark show that T is unitarily equivalent to multiplication by z plus a Volterra operator, on $L^2(d\sigma)$. Thus, Ahern and Clark give a "concrete" example of the Nagz-Foias model theory.

Theorem 5.4, which generalizes (1), relates $H^2(D)$ reproducing kernels to a kernel $P(z, \eta)$ used in representing singular inner functions $s(z)$ defined on a multiply connected domain D. See [4]. Again,

$$s(z) = \exp\left(-\int_{\partial D} P(z, \eta)\, d\sigma(\eta)\right)$$

where σ is positive and singular with respect to arclength on the boundary of D.

Theorem 5.4 can then be used to construct an isometry of $H^2(D) \ominus sH^2(D)$ and $L^2(d\sigma)$. This isometry gives a concrete example of the Abrahamse-Douglas model theory. Once again, the restricted shift on $H^2(D) \ominus sH^2(D)$ is unitarily equivalent to multiplication by z plus a compact integral operator, on $L^2(d\sigma)$. See [3].

The construction of the isometry and the study of the restricted shift will appear in the Indiana Journal of Mathematics in a separate paper.

1. We begin by recalling some basic facts about $H^2(D)$. For details see Rudin [8].

A holomorphic function f on D belongs to $H^2(D)$ if $|f|^2$ has a harmonic majorant on D. Let $L^2(\partial D)$ be the L^2 space of functions on the boundary of D with respect to arclength measure, ds. In the usual way, $H^2(D)$ may be identified with a closed subspace of $L^2(\partial D)$ and is therefore a Hilbert space.

We define equivalent inner products on $H^2(D)$: let $h > 0$ be a continuous function on ∂D and let $dm = h^2\, ds$. By $H^2(D, dm)$ we mean the space $H^2(D)$ with inner product

$$\langle f, g\rangle_{dm} = \langle f, g\rangle_{h^2} = \int_{\partial D} f\bar{g}\, dm.$$

We also write

$$\|f\|_{dm}^2 = \|f\|_{h^2}^2 = \int_{\partial D} |f|^2 h^2\, ds.$$

The following special case will be important. Let $G(z, p)$ be Green's function for D with pole at p. Define harmonic measure for p:

$$dm_p = \frac{-\partial G}{\partial n}(z, p)\frac{ds}{2\pi}$$

(As always, $\partial/\partial n$ denotes differentiation along the outward normal.)

Observe that

(1.1) $$f(p) = \langle f, 1\rangle_{dm_p}, \quad f \in H^2(D).$$

Finally, let $h_1^2\, ds$ and $h_2^2\, ds$ define two inner products. The following proposition is easily checked.

PROPOSITION 1.1. *Let $f \in H^2(D)$. Then* $\|f\|_{h_1^2} \le \max\,(h_1 h_2^{-1})\|f\|_{h_2^2}$.

2. In this section we define the kernels $B(\cdot, \zeta, h^2)$ and prove they are "continuous as a function of h^2".

Let $\zeta \in D$. Then it is well known that $\Lambda f = f(\zeta)$ defines a bounded linear form on any $H^2(D, dm)$. See [8]. This yields:

PROPOSITION 2.1. *For $\zeta \in D$ there is a unique function $B(\cdot, \zeta, dm) \in H^2$ such that $f(\zeta) = \langle f, B(\cdot, \zeta, dm)\rangle_{dm}$, for all $f \in H^2$.*

We often write $B(z, \zeta, dm) = B(z, \zeta, h^2)$ for $h^2\, ds = dm$.

We have the usual properties of reproducing kernels:

(a) $\|B(\cdot, \zeta, dm)\|_{dm}^2 = B(\zeta, \zeta, dm)$
(b) $B(z, \zeta, dm) = \overline{B(\zeta, z, dm)}$ for $z, \zeta \in D$
(c) For $f \in H^2$, $|f(\zeta)| \le \|f\|_{dm}\|B(\cdot, \zeta, dm)\|_{dm}$.

We need the following lemma relating the kernel functions for ζ and the different measures $h^2\, ds$.

LEMMA 2.1. *Let $\{h_n\}$ be a sequence of continuous positive functions on ∂D converging uniformly to a positive h. Then $B(\cdot, \zeta, h_n^2)$ converges in H^2 to $B(\cdot, \zeta, h^2)$.*

Proof. We show convergence in $H^2(D, h^2)$ by proving that

$$\sup_{\|f\|_{h^2}^2 \le 1} |\langle f, B(\cdot, \zeta, h_n^2) - B(\cdot, \zeta, h^2)\rangle_{h^2}|$$

tends to zero as n tends to ∞. Now,

$$\begin{aligned}
\langle f, B(\cdot, \zeta, h_n^2) &- B(\cdot, \zeta, h^2)\rangle_{h^2} \\
&= \langle f, B(\cdot, \zeta, h_n^2)\rangle_{h_n^2} - \langle f, B(\cdot, \zeta, h^2)\rangle_{h^2} \\
&\quad + \langle f, B(\cdot, \zeta, h_n^2)\rangle_{h^2} - \langle f, B(\cdot, \zeta, h_n^2)\rangle_{h_n^2} \\
&= f(\zeta) - f(\zeta) + \langle f, B(\cdot, \zeta, h_n^2)\rangle_{h^2} - \langle f, B(\cdot, \zeta, h_n^2)\rangle_{h_n^2} \\
&= \int f\overline{B(\cdot, \zeta, h_n^2)}(h^2 - h_n^2)\, ds.
\end{aligned}$$

Thus the modulus of this last expression is less than or equal to

$$\max |h - h_n^2| \, \|f\|_{ds} \|B(\cdot, \zeta, h_n^2)\|_{ds}$$

which by Prop. 1.1 is less than or equal to

$$\max |h^2 - h_n^2| \max h^{-1} \max h_n^{-1} \|B(\cdot, \zeta, h_n^2)\|_{h_n 2}$$

if $\|f\|_{h^2} \leq 1$. Clearly, we need only show that $\|B(\cdot, \zeta, h_n^2)\|_{h_n 2}$ remains bounded as $n \to \infty$.

For this, define

$$\phi_k(\eta) = B(\eta, \zeta, h_k^2)/\|B(\cdot, \zeta, h_k^2)\|_{h^2}.$$

Obviously $\|\phi_k\|_{h^2} = 1$. Now

$$\begin{aligned} |\phi_k(\zeta)| &= |\langle \phi_k, B(\cdot, \zeta, h^2)\rangle_{h^2}| \\ &\leq \|\phi_k\|_{h^2} \|B(\cdot, \zeta, h^2)\|_{h^2} \\ &= \|B(\cdot, \zeta, h^2)\|_{h^2}. \end{aligned}$$

So $\{|\phi_k(\zeta)|\}$ is a bounded sequence. On the other hand

$$|\phi_k(\zeta)| = B(\zeta, \zeta, h_k^2)/\|B(\cdot, \zeta, h_k^2)\|_{h^2} = \|B(\cdot, \zeta, h_k^2)\|_{h_k 2}^2/\|B(\cdot, \zeta, h_k^2)\|_{h^2}.$$

By Prop. 1.1,

$$\|B(\cdot, \zeta, h_k^2)\|_{h^2} \leq \max (hh_k^{-1})\|B(\cdot, \zeta, h_k^2)\|_{h_k 2}$$

Thus

$$|\phi_k(\zeta)| \geq \|B(\cdot, \zeta, h_k^2)\|_{h_k 2}/\max (hh_k^{-1})$$

or

$$\|B(\cdot, \zeta, h_k^2)\|_{h_k 2} \leq \max (hh_k^{-1})|\phi_k(\zeta)|.$$

The right hand stays bounded as $k \to \infty$, completing the proof.

3. Lemma 2.1 showed that $B(z, \zeta, h^2)$ was "continuous as a function of h^2". This section will show that $B(z, \zeta, h^2)$ is "differentiable in h^2" in an appropriate sense.

Let Γ denote ∂D and let $\Gamma = \gamma_1 \cup \cdots \cup \gamma_{n+1}$ where γ_i is a component of Γ. We suppose γ_{n+1} is the outer boundary. Let $dm = h^2\, ds$ be a measure on Γ as in the previous section.

If $\Lambda = (\lambda_1, \ldots, \lambda_{n+1})$ is an $(n + 1)$-tuple with $\lambda_i > 0$, $i = 1, \ldots, n + 1$, then the function $h_\Lambda(z) = \lambda_i^{1/2}h(z)$, $z \in \gamma_i$, is positive and continuous on Γ.

DEFINITION. With $dm = h^2\, ds$, and Λ as above, $\Lambda\, dm$ is defined to be the measure $h_\Lambda^2\, ds$. That is, $\Lambda\, dm$ is a perturbation of dm by the weight factor λ_i on γ_i.

Suppose z and $\zeta \in D$. Define $G(\Lambda) = G(\lambda_1, \ldots, \lambda_{n+1}) = B(z, \zeta, \Lambda\, dm)$.

LEMMA 3.1. *G is differentiable. Precisely,*

$$\frac{\partial G}{\partial \lambda_i}(\Lambda) = -\int_{\gamma_i} B(\cdot, \zeta, \Lambda\, dm)\overline{B(\cdot, z, \Lambda\, dm)}\, dm.$$

Proof. Let $\Lambda' = (\lambda_1, \ldots, \lambda_i + \Delta\lambda, \ldots, \lambda_{n+1})$. Then

$$\begin{aligned}(\Delta\lambda)^{-1}[G(\Lambda') - G(\Lambda)] &= (\Delta\lambda)^{-1}[B(z, \zeta, \Lambda'\, dm) - B(z, \zeta, \Lambda\, dm)] \\ &= (\Delta\lambda)^{-1}[\langle B(\cdot, \zeta, \Lambda'\, dm), B(\cdot, z, \Lambda\, dm)\rangle_{\Lambda\, dm} \\ &\quad - \langle B(\cdot, \zeta, \Lambda'\, dm), B(\cdot, z, \Lambda\, dm)\rangle_{\Lambda'\, dm}] \\ &= \int_{\gamma_i} B(\cdot, \zeta, \Lambda'\, dm)\overline{B(\cdot, z, \Lambda\, dm)}\left[\frac{\lambda_i - (\lambda_i + \Delta\lambda)}{\Delta\lambda}\right] dm \\ &= -\int_{\gamma_i} B(\cdot, \zeta, \Lambda'\, dm)\overline{B(\cdot, z, \Lambda\, dm)}\, dm.\end{aligned}$$

As $\Delta\lambda \to 0$, $h^2_{\Lambda'} \to h^2$ uniformly on Γ, and Lemma 2.1 gives the result. Observe that the partial derivatives are continuous in Λ, again a consequence of Lemma 2.1.

Lemma 3.1 prompts the following definition.

DEFINITION. $K_j(z, \zeta, dm) \equiv \int_{\gamma_j} B(\cdot, \zeta, dm)\overline{B(\cdot, z, dm)}\, dm.$

LEMMA 3.2. *$K_i(z, \zeta, dm)$ is holomorphic in z and belongs to $H^2(D)$.*

Proof. Let T be the linear form $Tf = \int_{\gamma_i} f\overline{B(\cdot, \zeta, dm)}\, dm$, $f \in H^2$. T is bounded. So there is a unique $g \in H^2$ such that $Tf = \langle f, g\rangle_{dm}$, for all $f \in H^2$. In particular,

$$TB(\cdot, z, dm) = \langle B(\cdot, z, dm), g\rangle_{dm},$$

or

$$\overline{g(z)} = \int_{\gamma_i} B(\cdot, z, dm)\overline{B(\cdot, \zeta, dm)}\, dm,$$

which proves the lemma.

This characterization of $K_i(\cdot, \zeta, dm)$ leads to the next result.

LEMMA 3.3. *Fix $\zeta \in D$. Let $\Lambda' = (1, \ldots, 1 + \Delta\lambda, \ldots, 1)$, where $1 + \Delta\lambda$ occurs in the ith place. Then the functions*

$$F(\Delta\lambda) = (\Delta\lambda)^{-1}[B(\cdot, \zeta, \Lambda'\, dm) - B(\cdot, \zeta, dm)]$$

converge in H^2 to $-K_i(\cdot, \zeta, dm)$ as $\Delta\lambda \to 0$.

Proof. We show that

$$\sup_{\|f\|_{dm}\le 1} |\langle f, F(\Delta\lambda) + K_i(\cdot, \zeta, dm)\rangle_{dm}|$$

tends to zero as $\Delta\lambda$ goes to zero.

As in the proof of Lemma 2.1,

$$\langle f, F(\Delta\lambda)\rangle_{dm} = \int_\Gamma f\overline{B(\cdot, \zeta, \Lambda' dm)} \left[\frac{h^2 - h^2_{\Lambda'}}{\Delta\lambda}\right] ds = -\int_{\gamma_i} f\overline{B(\cdot, \zeta, \Lambda' dm)}\, dm.$$

Furthermore,

$$\langle f, K_i(\cdot, \zeta, dm)\rangle_{dm} = \int_{\gamma_i} f\overline{B(\cdot, \zeta, dm)}\, dm.$$

Thus

$$|\langle f, F(\Delta\lambda) + K_i(\cdot, \zeta, dm)\rangle_{dm}| = \left|\int_{\gamma_i} f(\overline{B(\cdot, \zeta, \Lambda' dm)} - \overline{B(\cdot, \zeta, dm)})\ dm\right|$$
$$\le \|f\|_{dm}\|B(\cdot, \zeta, \Lambda' dm) - B(\cdot, \zeta, dm)\|_{dm}.$$

Since $\Delta\lambda \to 0$ implies $\Lambda' \to (1, 1, \ldots, 1)$, Lemma 2.1 gives the result.

4. Conformal mappings of D onto the unit disk with circular slits. Most of the material in this section can be found in the books by Bergman [2] and Nehari [6].

Recall that $G(z, \zeta)$ is the Green's function for D with pole at ζ. Precisely, $G(z, \zeta) = h(z, \zeta) - \log |z - \zeta|$ where $h(z, \zeta)$ is the harmonic function on D whose boundary values equal $\log |z - \zeta|$, $z \in \partial D$. Set

$$H(z, \zeta) = \int_{[z_0, z]} \frac{\partial G}{\partial n_\eta}(\eta, \zeta)\, ds(\eta),$$

where $[z_0, z]$ denotes a path in D from a fixed point z_0 to z.

$G(z, \zeta) + iH(z, \zeta)$ is holomorphic in z, but in general is not single valued.

Let $w_i(z)$ be the harmonic measure for γ_i, that is, the harmonic function on D which vanishes on γ_j, $j \ne i$, and is identically 1 on γ_i. Denote by W_i a (multiple valued) holomorphic function whose real part is w_i.

For $i, j = 1, \ldots, n + 1$ let

$$p_{ij} = \int_{\gamma_j} \frac{\partial w_i}{\partial n} \frac{ds}{2\pi}. \tag{4.1}$$

That is, p_{ij} is the period of w_i around γ_j. The following properties of the p_{ij} are well known:

(a) $p_{ij} = p_{ji}$.
(b) The $n \times n$ matrix $[p_{ij}]_{i,j=1,\ldots,n}$ has non-vanishing determinant.

If u is harmonic on D, then u will not necessarily have a single valued harmonic conjugate. However, as a consequence of (b), for some choice of α_i, $i = 1, \ldots, n$, $u - \sum_{i=1}^{n} \alpha_i w_i$ will have a single valued conjugate. This is the idea behind the next definition.

DEFINITION. For $a \in D$ and $\zeta \in D$,

$$L(\zeta, a) \equiv \exp\left(-G(\zeta, a) - iH(\zeta, a) - \sum_{i=1}^{n} \alpha_i(a) W_i(\zeta)\right)$$

where $\alpha_i(a)$ are chosen so

$$\int_{\gamma_j} -\frac{\partial G}{\partial n}(\eta, a)\frac{ds}{2\pi}(\eta) = \sum_{i=1}^{n} \int_{\gamma_j} \alpha_i(a)\frac{\partial w_i}{\partial n}(\eta)\frac{ds}{2\pi}(\eta). \tag{4.2}$$

(This says that $L(\zeta, a)$ is a single valued function of ζ; its periods around the γ_j vanish.) Formula (4.2) says

$$w_j(a) = \sum_{i=1}^{n} \alpha_j(a) p_{ij} \tag{4.3}$$

where we have used Green's formula. Thus

$$\sum_{i=1}^{n} w_i(a)\pi_{ij} = \alpha_j(a) \quad \text{where} \quad [\pi_{ij}] = [p_{ij}]^{-1}. \tag{4.4}$$

We state the following theorem which identifies the $L(\cdot, a)$s as the "Blaschke factors" for D.

THEOREM 4.1. *$L(\cdot, a)$ is a conformal map of D onto the unit disk with circular slits which sends a to the origin, and maps γ_{n+1} onto the unit circle.*

Some further properties of the $L(\cdot, a)$s will be needed. It is known that as $a \to \gamma_{n+1}$, $L(\zeta, a) \to 1$ for fixed ζ, and as $a \to \gamma_k$, $k \neq n+1$, $L(\cdot, a)$ converges uniformly on compact subsets to a conformal map of D onto an annulus centered at the origin with circular slits. (We denote this map by $L(\cdot, a^*)$, where $a^* \in \gamma_k$.) We also have the fact that $|L(z_i, a)|$ remains constant as z_i ranges over γ_i. Precisely,

$$|L(z_i, a)|^2 = \begin{cases} 1 & \text{if } i = n+1, \\ \exp\left(-2\sum_{j=1}^{n} w_j(a)\pi_{ij}\right) & \text{if } i \neq n+1, \end{cases} \tag{4.5}$$

and these formulas are valid for $a \in \partial D$.

Finally, we remark that the choice of the outer boundary as γ_{n+1} is irrelevant. Any boundary component may be taken as γ_{n+1} and a conformal map constructed as above will take γ_{n+1} onto the unit circle.

5. In this section we derive the fundamental identity that relates reproducing kernels for different measures to the maps $L(\cdot, a)$. We use this to prove that

$$\lim_{a \to a^*} (\|B(\cdot, a, dm)\|_{dm}^2 |a - a^*|)^{-1} = -2 \frac{\partial G}{\partial n_a}(a^*, t),$$

where $dm = dm_t$ and a tends to $a^* \in \partial D$ along a normal line to ∂D at a^*. (We say "$a \to a^* \in \partial D$, normally".) We then construct $P(z, a)$, the kernel used by Coiffman and Weiss [4] and prove

$$\overline{B(\zeta, a^*, dm)} B(z, a^*, dm)\left(-2 \frac{\partial G}{\partial n}(a^*, t)\right)$$

$$= B(z, \zeta, dm)\{P(z, a^*) + \overline{P(\zeta, a^*)}\} + \sum 2\pi_{ij} K_j(z, \zeta, dm) \frac{\partial w_i}{\partial n}(a^*)$$

where $dm = dm_t$ and $a^* \in \partial D$. Most of the rest of the section is devoted to removing the restriction $dm = dm_t$ and proving the correct results.

Let $dm = h^2\, ds$. For $a \in \bar{D}$ we consider the following "special" perturbation of dm.

DEFINITION. By $\Lambda(a)\, dm$, we mean the measure

$$\Lambda(a)\, dm(z) = |L(z, a)|^2\, dm(z), \quad \text{for } z \in \partial D.$$

That is, $\Lambda(a) = (\lambda_1(a), \ldots, \lambda_{n+1}(a))$ where

$$\lambda_i(a) = \begin{cases} 1 & \text{if } i = n + 1; \\ \exp\left(-2 \sum_{j=1}^{n} w_j(a)\pi_{ij}\right) & \text{if } i \neq n + 1. \end{cases}$$

Suppose $f \in H^\infty(D)$. By fH^2 we mean $\{fg : g \in H^2\}$. Obviously $fH^2 \subseteq H^2$. We have the following easy results.

PROPOSITION 5.1. *Let $a \in D$. Then $L(\cdot, a)H^2 = \{f : f \in H^2$ and $f(a) = 0\}$.*

PROPOSITION 5.2. *Let $a \in \partial D$. Then $L(\cdot, a)H^2 = H^2$.*

Whether $a \in D$ or ∂D we see that $L(\cdot, a)H^2$ is a closed subspace of H^2. The following observation is important.

PROPOSITION 5.3. *Let $a \in \bar{D}$ and let $M = L(\cdot, a)H^2$. Let P denote orthogonal projection onto M in $H^2(D, dm)$. Then*

$$PB(z, \zeta, dm) = \overline{L(\zeta, a)} L(z, a) B(z, \zeta, \Lambda(a)\, dm).$$

Proof. First, let $a \in D$. Since the right hand side belongs to M we need only show it is the reproducing kernel for ζ in M. If $f \in M$ then $f(z) = L(z, a)g(z)$, where $g \in H^2$. Thus

$$\begin{aligned}
&\langle f, \overline{L(\zeta, a)}L(\cdot, a)B(\cdot, \zeta, \Lambda(a)\, dm)\rangle_{dm} \\
&\quad = L(\zeta, a)\langle L(\cdot, a)g, L(\cdot, a)B(\cdot, \zeta, \Lambda(a)\, dm)\rangle_{dm} \\
&\quad = L(\zeta, a)\langle g, B(\cdot, \zeta, \Lambda(a)\, dm)\rangle_{\Lambda(a)\, dm} \\
&\quad = L(\zeta, a)g(a) = f(a)
\end{aligned}$$

as desired. If $a \in \partial D$, the same proof works, since any $f \in H^2$ may be written as $f = L(\cdot, a)g$, where $g \in H^2$.

This leads to:

LEMMA 5.1. *Let $a \in \bar{D}$ and $z, \zeta \in D$. If $a \in D$ then*

$$B(z, \zeta, dm) - \overline{L(\zeta, a)}L(z, a)B(z, \zeta, \Lambda(a)\, dm) = \frac{\overline{B(\zeta, a, dm)}B(z, a, dm)}{\|B(\cdot, a, dm)\|_{dm}^2}. \tag{5.1.1}$$

If $a \in \partial D$ then

$$B(z, \zeta, dm) = \overline{L(\zeta, a)}L(z, a)B(z, \zeta, \Lambda(a)\, dm). \tag{5.1.2}$$

Proof. For the first part, observe that the left hand side is $P_{M^\perp} B(\cdot, \zeta, dm)$ evaluated at z, where $P_{M^\perp}$ denotes orthogonal projection in $H^2(D, dm)$ onto $H^2 \ominus M$ where $M = L(\cdot, a)H^2$. This is a consequence of Proposition 5.3. On the other hand $H^2 \ominus M$ is a one dimensional subspace spanned by $B(\cdot, a, dm)$. Thus

$$\begin{aligned}
P_{M^\perp} B(z, \zeta, dm) &= \left\langle B(\cdot, \zeta, dm), \frac{B(\cdot, a, dm)}{\|B(\cdot, a, dm)\|_{dm}} \right\rangle_{dm} \frac{B(z, a, dm)}{\|B(\cdot, a, dm)\|_{dm}} \\
&= \frac{\overline{B(\zeta, a, dm)}B(z, a, dm)}{\|B(\cdot, a, dm)\|_{dm}^2}.
\end{aligned}$$

This proves (5.1.1). (5.1.2) follows from Proposition 5.3, since for $a \in \partial D$, $M = L(\cdot, a)H^2 = H^2$, and P_M is the identity.

THEOREM 5.1. *Let $t \in D$. Set $dm = dm_t$ and let $a \to a^* \in \partial D$ normally. Then*

$$\lim_{a \to a^*} \frac{1}{\|B(\cdot, a, dm)\|^2 |a - a^*|} = -2 \frac{\partial G}{\partial n_a}(a^*, t).$$

Proof. By (5.1.1) with $z = \zeta = t$ we have

$$B(t, t, dm) - |L(t, a)|^2 B(t, t, \Lambda(a)\, dm) = |B(t, a, dm)|^2 / \|B(\cdot, a, dm)\|_{dm}^2.$$

Since $B(\cdot, t, dm) = B(\cdot, t, dm_t) = 1$, we get

$$1 - |L(t, a)|^2 B(t, t, \Lambda(a)\, dm) = \{\|B(\cdot, a, dm)\|_{dm}^2\}^{-1}$$

Thus

$$\frac{1}{\|B(\cdot, a, dm)\|_{dm}^2 \, |a - a^*|} = \frac{1 - L(t, a)^2 B(t, t, \Lambda(a)\, dm)}{|a - a^*|}.$$

By (5.1.2), $1 = L(t, a^*)^2 B(t, t, \Lambda(a^*)\, dm)$, so

$$\lim_{a \to a^*} \frac{1}{\|B(\cdot, a, dm)\|_{dm}^2 \, |a - a^*|}$$
$$= \lim_{a \to a^*} \frac{|L(t, a^*)|^2 B(t, t, \Lambda(a^*)\, dm) - |L(t, a)|^2 B(t, t, \Lambda(a)\, dm)}{|a - a^*|}$$
$$= \frac{\partial}{\partial n_a} \{ |L(t, a^*)|^2 B(t, t, \Lambda(a^*)\, dm) \}$$

where we know the limits exist by the differentiability of $B(t, t, \Lambda\, dm)$ in Λ.

By the product rule, the last expression equals

$$B(t, t, \Delta(a^*)\, dm) \frac{\partial}{\partial n_a} |L(t, a^*)|^2 + |L(t, a^*)|^2 \frac{\partial}{\partial n_a} B(t, t, \Lambda(a^*)\, dm).$$

From equation (4.4) we see

$$|L(t, a)|^2 = \exp\left(-2G(t, a) - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} w_i(t) w_j(a) \right)$$

yielding

$$\frac{\partial}{\partial n_a} |L(t, a^*)|^2 = |L(t, a^*)|^2 \left(-2 \frac{\partial G}{\partial n_a}(a^*, t) - 2 \sum_{i, j} \pi_{ij} w_i(t) \frac{\partial w_j}{\partial n}(a^*) \right).$$

Since $B(t, t, \Lambda(a^*)) = |L(t, a^*)|^{-2}$, we have shown that

$$B(t, t, \Lambda(a^*)\, dm) \frac{\partial}{\partial n_a} |L(t, a^*)|^2 = -2 \frac{\partial G}{\partial n_a}(a^*, t) - 2 \sum_{i, j}^{n} \pi_{ij} w_i(t) \frac{\partial w_j}{\partial n}(a^*).$$

Now observe that

$$\frac{\partial}{\partial n_a} B(t, t, \Lambda(a^*)\, dm) = \sum_{i=1}^{n+1} \frac{\partial}{\partial \lambda_i} B(t, t, \Lambda(a^*)\, dm) \cdot \frac{\partial \lambda_i}{\partial n}(a^*)$$
$$= \sum_{i=1}^{n} \left(-\int_{\gamma_i} |B(\cdot, t, \Lambda(a^*))|^2 \, dm \right) \frac{\partial \lambda_i}{\partial n}(a^*)$$

by the chain rule, Lemma 3.1, and the fact that $\lambda_{n+1}(a) \equiv 1$. Since, for $i \neq n + 1$,

$$\lambda_i(a) = \exp\left(-2 \sum_{j=1}^{n} w_j(a) \pi_{ij} \right),$$

we have

$$\frac{\partial \lambda_i}{\partial n}(a^*) = \exp\left(-2\sum_{j=1}^{n} w_j(a^*)\pi_{ij}\right)\left(-2\sum_{j=1}^{n}\pi_{ij}\frac{\partial w_j}{\partial n}(a^*)\right)$$
$$= |L(z_i, a^*)|^2\left(-2\sum_{j=1}^{n}\frac{\partial w_j}{\partial n}(a^*)\pi_{ij}\right), \quad z_i \in \gamma_i.$$

Using $|B(\eta, t, \Lambda(a^*)\, dm)|^2 = |L(t, a^*)|^{-2}|L(\eta, a^*)|^{-2}$, we see that

$$|L(t, a^*)|^2 \frac{\partial}{\partial n_a}(B(t, t, \Lambda(a^*)\, dm))$$
$$= |L(t, a^*)|^2 \sum_{i=1}^{n}\left(-\int_{\gamma_i}|B(\cdot, t, \Lambda(a^*)\, dm)|^2\, dm\right)\cdot\frac{\partial \lambda_i}{\partial n}(a^*)$$
$$= |L(t, a^*)|^2 \sum_{i=1}^{n}\left(-\int_{\gamma_i}|L(t, a^*)|^{-2}|L(\cdot, a^*)|^{-2}\, dm\right)|L(z_i, a^*)|^2$$
$$\cdot\left\langle -2\sum_{j=1}^{n}\pi_{ij}\frac{\partial w_j}{\partial n}(a^*)\right\rangle$$
$$= 2\sum_{i=1}^{n}\int_{\gamma_i} dm \cdot \sum_{j=1}^{n}\pi_{ij}\frac{\partial w_j}{\partial n}(a^*)$$
$$= 2\sum_{i=1}^{n} w_i(t)\sum_{j=1}^{n}\pi_{ij}\frac{\partial w_j}{\partial n}(a^*)$$

where we have used the fact that dm is harmonic measure for t. Adding this to the result of the first calculation proves the theorem.

DEFINITION. Let $a \in \partial D$ and $z \in D$. Using the notation of Section 4 we define the function

$$P(z, a) = -\frac{\partial G}{\partial n_a}(z, a) - i\frac{\partial H}{\partial n_a}(z, a) - \sum_{i,j}^{n}\pi_{ij}W_j(z)\frac{\partial w_i}{\partial n}(a).$$

For each $a \in \partial D$, $P(z, a)$ is holomorphic in z. $P(z, a)$ is the kernel used by Coiffman and Weiss in [4]. In case D is the unit disk it is $(e^{i\theta} + z)/(e^{i\theta} - z)$.

The formula

$$2(1 - \bar{\zeta}e^{i\theta})^{-1}(1 - ze^{-i\theta})^{-1} = (1 - \bar{\zeta}z)^{-1}\left\{\frac{e^{i\theta} + z}{e^{i\theta} - z} + \frac{e^{-i\theta} + \bar{\zeta}}{e^{-i\theta} - \bar{\zeta}}\right\}$$

is easily checked and may be rewritten as

$$2\overline{B(\zeta, e^{i\theta}, dm)}B(z, e^{i\theta}, dm) = B(z, \zeta, dm)\{P(z, e^{i\theta}) + \overline{P(\zeta, e^{i\theta})}\}$$

for $dm = d\theta/2\pi$ and $z, \zeta \in U$. A similar formula holds in general.

THEOREM 5.2. *Let $z, \zeta \in D$ and let $a \to a^* \in \partial D$ normally. Then if $dm = dm_t$,*
$$\lim_{a\to a^*} \overline{B(\zeta, a, dm)}B(z, a, dm)$$
exists and is continuous as a function of a^*. Precisely,
$$\overline{B(\zeta, a^*, dm)}B(z, a^*, dm)\left(-2\frac{\partial G}{\partial n}(a^*, t)\right)$$
$$= B(z, \zeta, dm)\{P(z, a^*) + \overline{P(\zeta, a^*)}\} + \sum_{i,j}^{n} 2\pi_{ij}k_j(z, \zeta, dm)\frac{\partial w_i}{\partial n}(a^*).$$

Proof. Proceed as in Theorem 5.1.

(5.2.1)
$$\frac{B(z, \zeta, dm) - \overline{L(\zeta, a)}L(z, a)B(z, \zeta, \Lambda(a)\, dm)}{|a - a^*|} = \frac{\overline{B(\zeta, a, dm)}B(z, a, dm)}{\|B(\cdot, a, dm)\|_{dm}^2\,|a - a^*|}.$$

Rewrite the left hand side as
$$\frac{\overline{L(\zeta, a^*)}L(z, a)B(z, \zeta, \Lambda(a^*)\, dm) - \overline{L(\zeta, a)}L(z, a)B(z, \zeta, \Lambda(a)\, dm)}{|a - a^*|}$$
$$= B(z, \zeta, \Lambda(a^*)\, dm)\left[\frac{\overline{L(\zeta, a^*)}L(z, a^*) - \overline{L(\zeta, a)}L(z, a)}{|a - a^*|}\right]$$
$$+ \overline{L(\zeta, a)}L(z, a)\left[\frac{B(z, \zeta, \Lambda(a^*)\, dm) - B(z, \zeta, \Lambda(a)\, dm)}{|a - a^*|}\right]$$

Claim: the first term converges to $B(z, \zeta, dm)\{P(z, a^*) + \overline{P(\zeta, a^*)}\}$. For this, we note $B(z, \zeta, \Lambda(a^*)\, dm) = B(z, \zeta, dm)/\overline{L(\zeta, a^*)}L(z, a)$. Next,
$$\lim_{a\to a^*}\frac{\overline{L(\zeta, a^*)}L(z, a^*) - \overline{L(\zeta, a)}L(z, a)}{|a - a^*|}$$
$$= \frac{\partial}{\partial n_a}\overline{L(\zeta, a^*)}L(z, a)$$
$$= \overline{L(\zeta, a^*)}L(z, a^*)$$
$$\times\left\{-\frac{\partial G}{\partial n}(\zeta, a^*) + i\frac{\partial H}{\partial n_a}(\zeta, a^*) - \sum\frac{\partial \alpha_i}{\partial n_a}(a^*)\overline{W_i}(\zeta) - \frac{\partial G}{\partial n_a}(z, a^*)\right.$$
$$\left. - i\frac{\partial H}{\partial n_a}(z, a^*) - \sum\frac{\partial \alpha_i}{\partial n_a}(a^*)W_i(z)\right\}$$
$$= \overline{L(\zeta, a^*)}L(z, a)$$
$$\times\left\{-\frac{\partial G}{\partial n}(\zeta, a^*) + i\frac{\partial H}{\partial n_a}(\zeta, a^*) - \sum_{i,j}\frac{\partial w_i}{\partial n}(a^*)\pi_{ij}W_i(\zeta) - \frac{\partial G}{\partial n_a}(z, a^*)\right.$$
$$\left. - i\frac{\partial H}{\partial n_a}(z, a^*) - \sum_{i,j}\frac{\partial w_i}{\partial n}(a^*)\pi_{ij}W_j(z)\right\}$$
$$= \overline{L(\zeta, a^*)}L(z, a)\{\overline{P(\zeta, a^*)} + P(z, a^*)\}.$$

This proves the claim.

For the second term we must calculate

$$\frac{\partial}{\partial n_a} B(z, \zeta, \Lambda(a)\, dm)$$

and evaluate at $a = a^*$. By the chain rule this equals

$$\sum_{i=1}^{n+1} \frac{\partial}{\partial \lambda_i} B(z, \zeta, \Lambda(a^*)\, dm) \cdot \frac{\partial \lambda_i}{\partial n}(a^*)$$
$$= \sum_{i=1}^{n} - \int_{\gamma_i} B(\cdot, \zeta, \Lambda(a^*)\, dm)\overline{B(\cdot, z, \Lambda(a^*)\, dm)}\, dm \cdot \frac{\partial \lambda_i}{\partial n}(a^*).$$

As in Theorem 5.1,

$$\frac{\partial \lambda_i}{\partial n}(a^*) = |L(z_i, a^*)|^2\left(-2\sum_{j=1}^{n} \frac{\partial w_j}{\partial n}(a^*)\pi_{ij}\right) \quad \text{where} \quad z_i \in \gamma_i.$$

Using this, and again the relation

$$B(z, \zeta, \Lambda(a^*)\, dm) = B(z, \zeta, dm)/\overline{L(\zeta, a^*)}L(z, a)$$

we get

$$\frac{\partial}{\partial n_a} B(z, \zeta, \Lambda(a^*)\, dm)$$
$$= -2\sum_{i,j=1}^{n} \int_{\gamma_i} \frac{B(\cdot, \zeta, dm)\overline{B(\cdot, z, dm)}}{\overline{L(\zeta, a^*)}L(\cdot, a^*) \cdot L(z, a^*)\overline{L(\cdot, a^*)}}$$
$$\times\ |L(\cdot, a^*)|^2\, dm \cdot \frac{\partial w_j}{\partial n}(a^*)\pi_{ij}$$
$$= \frac{-2}{\overline{L(\zeta, a^*)}L(z, a^*)} \sum_{i,j=1}^{n} \int_{\gamma_i} B(\cdot, \zeta, dm)\overline{B(\cdot, z, dm)}\, dm \frac{\partial w_j}{\partial n}(a^*)\pi_{ij}$$
$$= \frac{2}{\overline{L(\zeta, a^*)}L(z, a^*)} \sum_{i,j} K_i(z, \zeta, dm)\pi_{ij} \frac{\partial w_j}{\partial n}(a^*).$$

Multiplying by $\overline{L(\zeta, a^*)}L(z, a^*)$ shows that the left hand side of (5.2.1) converges to

$$B(z, \zeta, dm)\{P(z, a^*) + \overline{P(\zeta, a^*)}\} + \sum_{i,j}^{n} 2\pi_{ij} K_i(z, \zeta, dm) \frac{\partial w_j}{\partial n}(a^*)$$

and the theorem follows by applying Theorem 5.1 to the right hand side of (5.2.1).

This theorem has several implications. If $a^* \in \gamma_k$, then $P(z, a^*)$ is continuous in z for $z \in \bar{D} \backslash \gamma_k$. For $dm = dm_t$ we define

$$B(z, a^*, dm) = \lim_{a \to a^*, \text{normally}} B(z, a, dm).$$

Then:

COROLLARY 5.1. *If $a^* \in \gamma_k$, then $B(\cdot, a^*, dm_t) \in L^2(\Gamma\backslash\gamma_k)$. By $L^2(\Gamma\backslash\gamma_k)$ we mean the L^2 space with respect to ds on $\Gamma\backslash\gamma_k$. Furthermore,*

$$B(\cdot, a, dm) \to B(\cdot, a^*, dm) \text{ in } L^2(\Gamma\backslash\gamma_k).$$

Proof. For the first assertion we use Theorem 5.2 with $\zeta = t$:

$$2B(z, a^*, dm)\left(-\frac{\partial G}{\partial n}(a^*, t)\right)$$

$$= \{P(z, a^*) + \overline{P(t, a^*)}\} + \sum 2\pi_{ij} K_j(z, t, dm)\frac{\partial w_i}{\partial n}(a^*).$$

$\partial G(a^*, t)/\partial n$ never vanishes on Γ, and the right side is in $L^2(\Gamma\backslash\gamma_k)$.

For the second assertion we use (5.2.1) with $\zeta = t$:

$$\frac{B(z, a, dm)}{\|B(\cdot, a, dm)\|^2 |a - a^*|}$$

$$= B(z, t, \Lambda(a^*)\ dm)\left\{\frac{\overline{L(t, a^*)}L(z, a^*) - \overline{L(t, a)}L(z, a)}{|a - a^*|}\right\}$$

$$+ \overline{L(t, a)}L(z, a)\left\{\frac{B(z, t, \Lambda(a^*)\ dm) - B(z, t, \Lambda(a)\ dm)}{|a - a^*|}\right\}$$

$$= \overline{L(t, a^*)}^{-1} L(z, a^*)^{-1}\left\{\frac{\overline{L(t, a^*)}L(z, a^*) - \overline{L(t, a)}L(z, a)}{|a - a^*|}\right.$$

$$\left.+ \overline{L(t, a)}L(z, a)\left\{\frac{B(z, t, \Lambda(a^*)\ dm) - B(z, t, \Lambda(a)\ dm)}{|a - a^*|}\right\}\right.$$

Now, the first expression converges uniformly on $\Gamma\backslash\gamma_k$ to $\{P(z, a^*) + \overline{P(t, a^*)}\}$ as $a \to a^*$ normally. We deal with the second term:

$$L(z, a) \to L(z, a^*) \text{ uniformly for } z \in \gamma_i,\ i \neq k.$$

Thus we need only show that the expression in brackets converges in H^2, as $a \to a^*$ normally. In fact,

$$\frac{B(z, t, \Lambda(a^*)\ dm) - B(z, t, \Lambda(a)\ dm)}{|a - a^*|} \xrightarrow{H^2} \sum_{i,j}^{n} 2\pi_{ij} K_i(z, t, \Lambda(a^*)\ dm)\frac{\partial w_j}{\partial n}(a^*).$$

The proof of this is a straightforward adaption of the proof of Lemma 3.3, and will be omitted.

Briefly then, for $dm = dm_t$, Theorem 5.1 implies Theorem 5.2. We want to eliminate the restriction that $dm = dm_t$.

Suppose $dm = h^2\ ds$. The correct result is:

THEOREM 5.3. *Let $a \to a^* \in \partial D$ normally. Then*

$$\lim_{a \to a^*} (\|B(\cdot, a, h^2)\|_{h^2}^2 |a - a^*|)^{-1} = 2h(a^*).$$

Once we have Theorem 5.3 for a measure $h^2\, ds$, we can derive:

THEOREM 5.4. *Let $z, \zeta \in D$ and $a \to a^* \in \partial D$ normally. Then $\overline{B(\zeta, a, h^2)}B(z, a, h^2)$ converges to a continuous limit on Γ. Precisely,*

$$2\overline{B(\zeta, a^*, h^2)}B(z, a^*, h^2)h^2(a^*)$$
$$= B(z, \zeta, h^2)\{P(z, a^*) + \overline{P(\zeta, a^*)}\} + \sum_{i,j}^{n} 2\pi_{ij} K_j(z, \zeta, h^2) \frac{\partial w_i}{\partial n}(a^*).$$

Proof of Theorem 5.3. Let $a \in D$. We prove the theorem for $dm = \Lambda(a)\, dm_t$. By Lemma 5.1,

$$\|B(\cdot, \zeta, dm_t)\|_{dm_t}^2 - |L(\zeta, a)|^2 \|B(\cdot, \zeta, \Lambda(a)\, dm_t)\|_{\Lambda(a)dm_t}^2 = \frac{|B(\zeta, a, dm_t)|^2}{\|B(\cdot, a, dm_t)\|_{dm_t}^2}.$$

Thus

$$1 - |L(\zeta, a)|^2 \|B(\cdot, \zeta, \Lambda(a)\, dm_t)\|_{\Lambda(a)dm_t}^2 = \frac{|B(\zeta, a, dm_t)|}{\|B(\cdot, a, dm_t)\|_{dm_t}^2} \cdot \|B(\cdot, \zeta, dm_t)\|_{dm_t}^{-2}.$$

Let $\zeta \to \zeta^* \in \Gamma$ normally. By Theorems 5.1 and 5.2 the right side goes to zero. Applying Theorem 5.1 to the left hand side gives the theorem for the measure $\Lambda(a)\, dm_t$.

Thus Theorem 5.4 is also proved for $dm = \Lambda(a)\, dm_t$.

Now induction establishes Theorem 5.2 and Theorem 5.4 for any measure in the form $dm = \Lambda(a_1)\Lambda(a_2) \cdots \Lambda(a_m)\, dm_t$, for $a_i \in D$.

To prove the result for the general $h^2\, ds$ we need the following theorem.

THEOREM 5.5. *Let $0 < h$ be continuous on Γ. Then there is a function $H \in H^\infty(D)$ such that $|H|^2 = h^2$ on Γ, $H(\zeta) = 0$ for a preassigned ζ, and H has at most n zeros on D. Further,*

$$|H(z)| = \exp\left(-\int_\Gamma \frac{\partial G}{\partial n_\eta}(\eta, z) \log h(\eta)\, ds(\eta) - \sum_{i=1}^{n} G(z, a_i)\right)$$

where the a_i are the zeros of H.

Indication of proof. H arises as the solution to the following extremal problem. Let $f \in H^\infty(D)$, $|f| \le h$ on Γ, and $f(\zeta) = 0$. Find f so that $|f'(\zeta)|$ is a maximum. This matter is also dealt with in [8].

Observe that H is kind of a finite Blaschke product.

We finish the proof of Theorem 5.3 and 5.4:

Let $M = H(z)H^2 = \{f: f \in H^2, f(a_i) = 0$, where the a_i are the zeros of $H\}$. The subspace $H^2(D, dm) \ominus M$ is spanned by $\{B(\cdot, a_i\ dm)\}_{i=1}^n$. It is easy to check that if

$$\phi_k(z, dm) = \frac{B(z, a_k, \Lambda(a_1) \cdots \Lambda(a_{k-1})\ dm)}{\|B(\cdot, a_k, \Lambda(a_1) \cdots \Lambda(a_{k-1})\ dm)\|} \prod_{i=1}^{k-1} L(z, a_i),$$

then $\{\phi_k\}_{k=1}^n$ is an orthonormal basis for $H^2(D, dm) \ominus M = M^\perp$. Let $P_{M^\perp}$ denote orthogonal projection onto $M^\perp$. Then

$$P_{M^\perp} B(z, \zeta, dm) = \sum_{k=1}^n \overline{\phi_k(\zeta, dm)} \phi_k(z, dm).$$

On the other hand,

$$P_{M^\perp} B(z, \zeta, dm) = B(z, \zeta, dm) - \overline{H(\zeta)} H(z) B(z, \zeta, h^2\ dm),$$

which may be verified along the lines of Proposition 5.3. Thus, letting $z = \zeta$ and $dm = dm_t$ we have

$$\|B(\cdot, \zeta, dm_t)\|_{dm_t}^2 - |H(\zeta)|^2 \|B(\cdot, \zeta, h^2\ dm_t)\|_{h^2 dm_t}^2 = \sum_{k=1}^n |\phi_k(\zeta, dm)|^2.$$

Divide both sides of this equation by $\|B(\cdot, \zeta, dm_t)\|_{dm_t}^2$. Apply Theorems 5.3 and 5.4 to the right side and deduce that it tends to zero as $\zeta \to \zeta^* \in \partial D$ normally. This gives the desired result for a measure $h^2\ dm_t$ and hence for any measure $h^2\ ds$.

REFERENCES

1. P. R. AHERN and D. N. CLARK, *On functions orthogonal to invariant subspaces.* Acta Math., vol. 124 (1970), pp. 191–204.
2. S. BERGMAN, *The kernel function and conformal mapping*, Math Surveys, Number V, American Math. Soc., 1950.
3. W. COHN, *Orthogonal complements to invariant subspaces*, thesis, University of Wisconsin, Madison, 1978.
4. R. COIFFMAN and G. WEISS, *A kernel associated with certain multiply connected domains and its applications to factorization theorems*, Studia Math., vol. 28 (1966), pp. 31–68.
5. P. R. GARABEDIAN, *Schwarz's lemma and the Szegö kernel function*, Trans. Amer. Math. Soc., vol. 67 (1949), pp. 1–35.
6. Z. NEHARI, *Conformal mapping*, McGraw-Hill, N.Y., 1952.
7. ———, *A class of domain functions and some allied extremal problems*, Trans. Amer. Math. Soc., vol. 69 (1950), pp. 161–178.
8. W. RUDIN, *Analytic functions of class H_p*, Trans. Amer. Math. Soc., vol. 78 (1955), pp. 46–66.
9. G. C. TUMARKIN and S. YA HAVINSON, *On the existence in multiply connected regions of single valued analytic functions with boundary values of given modulus*, Izv. Akad. Nauk SSR Ser Mat., 22 (1958) (in Russian).

THE UNIVERSITY OF OKLAHOMA
NORMAN, OKLAHOMA