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FORMAL FUNCTIONS OVER GRASSMANNIANS

BY

ARI BABAKHANIAN AND HEISUKE HIRONAKA

Introduction

This work contains the theorem: *The field of formal-rational functions* $\hat{K}$ *along a connected closed subscheme* X *of positive dimension in a Grassmannian* Grass (n, r) *is exactly the field of rational functions on* Grass (n, r).

The study of formal-rational functions (or holomorphic functions) was first begun by Zariski [5], and was later extended to the theory of formal functions and formal schemes by Grothendieck [1]. Our study here is based on the aforementioned works and is a continuation of investigations made by Hironaka [3], Hironaka–Matsumura [4], and Hartshorne [2]. Our results are dependent on the determination of the fields of formal-rational functions in the special cases of formal functions along $\mathbf{P}^1$ in $\mathbf{P}^n$, and along $\mathbf{P}^n$ in Grass (n, r). In [4], among other results, the field of formal-rational functions $\hat{K}$ along a closed algebraic variety of positive dimension in a projective space $\mathbf{P}^n$ was determined to be exactly the field of rational functions on $\mathbf{P}^n$. The proof of this theorem was based upon a crucial lemma (Hironaka–Matsumura, Lemma (3.1), [4]) in which the same conclusion was reached in the case of formal functions along $\mathbf{P}^1$ in $\mathbf{P}^n$. Our Lemma (3.2) shows that this result holds in the case of formal functions along $\mathbf{P}^n$ in Grass (n, r). Summarizing, we have shown that the field of formal-rational functions along the subvariety of the ambient space in each of the cases considered is equal to the field of rational functions over the ambient space.

Notations. The rings involved here are polynomial rings over a field k. When R is a ring, we shall denote the total ring of fractions by $[R]_0$. A point of Grass (n, r), $n < r$ is represented by an $n \times r$ matrix (κ_{ij}) of rank n, and two such matrices (κ_{ij}), (η_{ij}) represent the same point if there is a nonsingular $n \times n$ matrix σ such that $\sigma(\kappa_{ij}) = (\eta_{ij})$; i.e., Grass (n, r) is the quotient modulo the action of $GL(n, k)$ on the Stiefel manifold St (n, r) of n frames in $\mathbf{A}^r$. We define the structure sheaf of rings $\mathcal{O}_{\mathrm{Grass}(n,r)}$ via the Plücker imbedding

$$\pi\colon \mathrm{Grass}\ (n, r) \to \mathbf{P}^{\binom{r}{n}-1}.$$

Let (κ_{ij}) be the $n \times r$ matrix representing a point x of Grass (n, r), then $\kappa_{i_1 \cdots i_n}$ is the $(i_1, \ldots, i_n)$-th Plücker coördinate of x. $\mathcal{U}_{i_1 \cdots i_n}$ is the open affine in Grass (n, r) such that any point in it can be represented by a matrix (t_{ij}) where the matrix of the columns $i_1, \ldots, i_n$ is the identity matrix.

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1. The blow-up $B_P(\text{Grass }(n, r))$

Consider the set P of points

$$x = \begin{pmatrix} \kappa_{11} & \cdots & \kappa_{1r} \\ \vdots & & \vdots \\ \kappa_{n1} & \cdots & \kappa_{nr} \end{pmatrix}$$

in Grass (n, r) described as follows: $x \in P$ if whenever the rank of the columns $i_1, \ldots, i_n$ in x is n, then $n + 1$ is one of $i_1, \ldots, i_n$. We define the projection p_{n+1} with center at P,

$$p_{n+1}: \text{Grass }(n, r) - P \to \text{Grass }(n, r-1),$$

by

$$p_{n+1} \begin{pmatrix} \kappa_{1,1} & \cdots & \kappa_{1,r} \\ \vdots & & \vdots \\ \kappa_{n,1} & & \kappa_{n,r} \end{pmatrix} = \begin{pmatrix} \kappa_{1,1} & \cdots & \kappa_{1,n} & \kappa_{1,n+2} & \cdots & \kappa_{1r} \\ \vdots & & \vdots & \vdots & & \vdots \\ \kappa_{n1} & \cdots & \kappa_{nn} & \kappa_{n,n+2} & \cdots & \kappa_{nr} \end{pmatrix};$$

and extend p_{n+1} to a correspondence

$$Z \subset \text{Grass }(n, r) \times \text{Grass }(n, r-1),$$

where

$$Z = V(\ldots, \kappa_{i_1 \cdots i_n} \eta_{j_1 \cdots j_n} - \kappa_{j_1 \cdots j_n} \eta_{i_1 \cdots i_n}, \ldots)$$

with $\kappa_{i_1 \cdots i_n}$ the $(i_1, \ldots, i_n)$-th Plücker coördinate of x in $\mathbf{P}^{\binom{r}{n}-1}$, and $\eta_{j_1 \cdots j_n}$ the $(j_1, \ldots, j_n)$-th Plücker coördinate of

$$\begin{pmatrix} \eta_{1,1} & \cdots & \eta_{1,n} & \eta_{1,n+2} & \cdots & \eta_{1,r} \\ \vdots & & \vdots & \vdots & & \vdots \\ \eta_{n,1} & \cdots & \eta_{n,n} & \eta_{n,n+2} & \cdots & \eta_{nr} \end{pmatrix}$$

in the imbedding

$$\text{Grass }(n, r-1) \to \mathbf{P}^{\binom{r-1}{n}-1},$$

and where $n + 1$ does not occur among $i_1, \ldots, i_n, j_1, \ldots, j_n$.

Thus the center P is blown up by Z to the whole of $P \times \text{Grass }(n, r-1)$. Moreover Z is irreducible, hence a variety itself. We denote Z by $B_P(\text{Grass }(n, r))$ and call it the blow-up of Grass (n, r) at P. (We remark that in the case $n = 1$, $B_P(\text{Grass }(1, r))$ is the usual blow-up of $\mathbf{P}^{r-1}$ at one point.) We can cover $B_P(\text{Grass }(n, r))$ by $\binom{r-1}{n}[\binom{r-1}{n-1} + 1]$ open affines as follows:

(i) $\binom{r-1}{n}$ open affines

$$U_{i_1 \cdots i_n} = B_P(\text{Grass }(n, r)) \cap \{\text{Grass }(n, r) - V(\kappa_{i_1 \cdots i_n})\} \times \{\text{Grass }(n, r-1) - V(\eta_{i_1 \cdots i_n})\}$$

where $n + 1$ does not occur among $i_1, \ldots, i_n$.

(ii) $\binom{r-1}{n-1}\binom{r-1}{n}$ open affines

$$V_{i_1\cdots i_n;\, j_1\cdots j_n} = B_P(\text{Grass }(n, r)) \cap \{\text{Grass }(n, r) - V(\kappa_{i_1\cdots i_n})\} \times \{\text{Grass }(n, r-1) - V(\eta_{j_1\cdots j_n})\}$$

where $n+1$ occurs among $i_1, \ldots, i_n$ and does not occur among $j_1, \ldots, j_n$.

(I) Under the first projection $\pi_1 : B_P(\text{Grass }(n, r)) \to \text{Grass }(n, r)$, $U_{i_1\ldots i_n}$ goes isomorphically to the affine Grass $(n, r) - V(\kappa_{i_1\cdots i_n})$. Moreover

$$\bigcup_{\substack{i_1\cdots i_n,\\ i_\alpha \neq n+1}} U_{i_1\cdots i_n}$$

covers that part of $B_P(\text{Grass }(n, r))$ which is isomorphic to Grass $(n, r) - P$.

(II) Affine coordinates in the ambient space containing $V_{i_1\ldots i_n;\, j_1\ldots j_n}$ are

$$z_{k_1\cdots k_n} = \kappa_{i_1\cdots i_n}^{-1}\kappa_{k_1\cdots k_n}$$

where $(k_1, \ldots, k_n) \neq (i_1, \ldots, i_n)$, and $n+1$ occurs among $i_1, \ldots, i_n$; and

$$W_{k_1\cdots k_n} = \eta_{j_1\cdots j_n}^{-1}\eta_{k_1\cdots k_n}$$

where $n+1$ does not occur among $j_1, \ldots, j_n$.

2. Covering of $\mathbf{P}^n \subset$ Grass (n, r) in the blow-up

The projective space $\mathbf{P}^n \subset$ Grass (n, r) consisting of all the points in Grass (n, r) represented by the $n \times r$ matrices (κ_{ij}) of rank n with $\kappa_{ij} = 0$ for $j \geq n+2$ can be covered by the $n+1$ affines $A_1, \ldots, A_{n+1}$ where A_i consists of the points which can be represented by matrices (κ_{ij}) with $\kappa_{ij} = 0$ for $j \geq n+2$, and the columns $1, 2, \ldots, \hat{i}, \ldots, n+1$ form the identity matrix. A point $x \in A_i \cap A_{n+1}$ is represented by

$$\begin{pmatrix} 1 & \cdots & 0 & v_{1,i}^{(i)} & 0 & \cdots & 0 & \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \\ \vdots & & 1 & \vdots & 0 & & \vdots & \boxed{0} \\ \vdots & & 0 & \vdots & 1 & & \vdots & \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & \cdots & 0 & v_{n,i}^{(i)} & 0 & \cdots & 1 & \end{pmatrix}$$

in A_i, and by

$$\begin{pmatrix} 1 & \cdots & 0 & t_{1,n+1} & \\ \vdots & \ddots & \vdots & \vdots & \boxed{0} \\ 0 & \cdots & 1 & t_{n,n+1} & \end{pmatrix}$$

in A_{n+1}, and the v's and the t's are related by

$$\text{(I)}\qquad \begin{aligned} t_{\alpha,n+1} &= -\frac{v^{(i)}_{\alpha,i}}{v^{(i)}_{n,i}} \quad \text{for } \alpha = 1, \ldots, i-1, \\ t_{i,n+1} &= \frac{1}{v^{(i)}_{n,i}}, \\ t_{\beta,n+1} &= -\frac{v^{(i)}_{\beta-1,i}}{v^{(i)}_{n,i}} \quad \text{for } \beta = i+1, \ldots, n. \end{aligned}$$

LEMMA (2.1). *For $r > n + 1$, the subset*

$$U_{1,\ldots,n} \cup \bigcup_{i=1}^{n} V_{1,\ldots,\hat{i},\ldots,n,n+1;1,\ldots,n}$$

of $B_P(\text{Grass}\ (n, r))$ is isomorphic to $\mathbf{P}^n \times S^{n(r-n-1)}$, and is a covering of $\mathbf{P}^n$ under the projection map

$$\pi_1 \colon B_P(\text{Grass}\ (n, r)) \to \text{Grass}\ (n, r),$$

where $S^{n(r-n-1)}$ is the affine space

$$\text{Spec}\ (k[\{t_{i,n+j}\}_{i=1,\ldots,n, j=2,\ldots,r-n}]).$$

Proof. The affine coordinates $z_{k_1 \cdots k_n}$ and $W_{k_1 \cdots k_n}$ in the ambient space containing $V_{1,\ldots,\hat{i},\ldots,n+1;1,\ldots,n}$ are:

(i) $v^{(i)}_{1,i}, \ldots, v^{(i)}_{n,i}, v^{(i)}_{1,n+2}, \ldots, v^{(i)}_{n,r}$, and certain homogeneous polynomials in these where

$$z_{1,\ldots,\hat{\alpha},\ldots,n,n+j} = \kappa^{-1}_{1,\ldots,\hat{i},\ldots,n+1} \kappa_{1,\ldots,\hat{\alpha},\ldots,n,n+j} = v^{(i)}_{\alpha,n+j}$$

and the $v^{(i)}_{\alpha,i}$ are the affine coordinates of the cover A_i of $\mathbf{P}^n \subset \text{Grass}\ (n, r)$.

$$\text{(ii)}\qquad \left\{ v^{(i)}_{\alpha,n+j} - \frac{v^{(i)}_{\alpha,i} v^{(i)}_{n,n+j}}{v^{(i)}_{n,i}} \right\}_{\substack{\alpha=1,\ldots,n-1;\\ j=2,\ldots,r-n}}, \quad \left\{ \frac{v^{(i)}_{n,n+j}}{v^{(i)}_{n,i}} \right\}_{j=2,\ldots,r-n},$$

and certain homogeneous polynomials in these, where

$$W_{1,\ldots,\hat{\alpha},\ldots,i,\ldots,n,n+j} = \eta^{-1}_{1,\ldots,n} \eta_{1,\ldots,\hat{\alpha},\ldots,i,\ldots,n,n+j} = v^{(i)}_{\alpha,n+j} - \frac{v^{(i)}_{\alpha,i} v^{(i)}_{n,n+j}}{v^{(i)}_{n,i}} \quad \text{for } \alpha < i,$$

$$W_{1,\ldots,\hat{i},\ldots,n,n+j} = \eta^{-1}_{1,\ldots,n} \eta_{1,\ldots,\hat{i},\ldots,n,n+j} = \frac{v^{(i)}_{n,n+j}}{v^{(i)}_{n,i}},$$

$$W_{1,\ldots,i,\ldots,\hat{\alpha},\ldots,n,n+j} = v^{(i)}_{\alpha-1,n+j} - \frac{v^{(i)}_{\alpha-1,i} v^{(i)}_{n,n+j}}{v^{(i)}_{n,i}} \quad \text{for } i < \alpha.$$

Thus $V_{1,\ldots,\hat{i},\ldots,n+1;1,\ldots,n}$ is isomorphic to the affine space $S^{n(r-n)}$ with coordinates

$$\left(v_{1,i}^{(i)}, \ldots, v_{n,i}^{(i)}, \left\{v_{\alpha,n+j}^{(i)} - \frac{v_{\alpha,i}^{(i)} v_{n,n+j}^{(i)}}{v_{n,i}^{(i)}}\right\}_{\alpha=1,\ldots,n-1;j-2,\ldots,r-n}, \left\{\frac{v_{n,n+j}^{(i)}}{v_{n,i}^{(i)}}\right\}_{j=2,\ldots,r-n}\right)$$

where the first n entries are the affine coordinates of the open cover A_i of $\mathbf{P}^n \subset \text{Grass}(n, r)$. Finally $U_{1,\ldots,n}$ is isomorphic to the affine space with affine coordinates $(t_{1,n+1}, \ldots, t_{n,n+1}, t_{1,n+2}, \ldots, t_{n,r})$ where the first n entries are the affine coordinates of the open cover A_{n+1} of $\mathbf{P}^n \subset \text{Grass}(n, r)$. The relationships between the t's and the v's on the intersection

$$U_{1,\ldots,n} \cap V_{1,\ldots,\hat{i},\ldots,n+1;1,\ldots,n}$$

are those given in (I) together with

$$\text{(II)} \qquad \begin{aligned} t_{\alpha,n+j} &= v_{\alpha,n+j}^{(i)} - \frac{v_{\alpha,i}^{(i)} v_{n,n+j}^{(i)}}{v_{n,i}^{(i)}}, \quad \alpha = 1, \ldots, i-1, \\ t_{i,n+j} &= \frac{v_{n,n+j}^{(i)}}{v_{n,i}^{(i)}} \\ t_{\beta,n+j} &= v_{\beta-1,n+j}^{(i)} - \frac{v_{\beta-1,i}^{(i)} v_{n,n+j}^{(i)}}{v_{n,i}^{(i)}}, \quad \beta = i+1, \ldots, n. \end{aligned}$$

Thus $U_{1,\ldots,n} \cup \bigcup_{i=1}^{n} V_{1,\ldots,\hat{i},\ldots,n+1;1,\ldots,n} \cong \mathbf{P}^n \times S^{n(r-n-1)}$.

3. Formal functions over Grass (n, r)

In this section we will need the following algebraic lemma.

LEMMA (3.1). *Let Φ and L be fields, $\Phi \subset L$, and let $v_1, \ldots, v_n$ be indeterminates over L. Then*

$$\Phi((v_1, \ldots, v_n)) \cap L(v_1, \ldots, v_n) = \Phi(v_1, \ldots, v_n).$$

Proof. Let $\nu = (\nu_1, \ldots, \nu_n)$, ν_i non-negative integers and let $\deg \nu = \sum_{i=1}^{n} \nu_i$. Let

$$\xi \in \Phi((v_1, \ldots, v_n)) \cap L(v_1, \ldots, v_n), \quad \xi \neq 0.$$

As an element of $\Phi((v_1, \ldots, v_n))$, ξ can be written as

$$\xi = \frac{\sum_{\deg \nu = 0}^{\infty} c_\nu v^\nu}{\sum_{\deg \nu = 0}^{\infty} d_\nu v^\nu} \quad \text{where } v^\nu = v_1^{\nu_1} \cdot \ldots \cdot v_n^{\nu_n}, \ c_\nu, d_\nu \in \Phi.$$

On the other hand, as an element of $L(\upsilon_1, \ldots, \upsilon_n)$,

$$\xi = \frac{\sum_{\deg i=0}^{m} a_i \upsilon^i}{\sum_{\deg i=0}^{l} b_i \upsilon^i} \quad \text{with } a_i, b_i \in L.$$

Thus

$$\sum_{\deg i+\deg \nu=0}^{\infty} b_i c_\nu \upsilon^{i+\nu} = \sum_{\deg i+\deg \nu=0}^{\infty} a_i d_\nu \upsilon^{i+\nu},$$

where $i + \nu = (i_1 + \nu_1, \ldots, i_n + \nu_n)$. Therefore for each fixed $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_\alpha \geq 0$, we have

$$\sum_{i+\nu=\mu} b_i c_\nu = \sum_{i+\nu=\mu} a_i d_\nu.$$

Let (w_λ) be a linear basis of L over Φ and write

$$a_i = \sum f_{i\lambda} w_\lambda, \; b_i = \sum g_{i\lambda} w_\lambda \quad \text{with } f_{i\lambda}, g_{i\lambda} \in \Phi.$$

So for a fixed w_λ which is involved in $b_i = \Sigma\, g_{i\lambda} w_\lambda$ for some i,

$$\sum_{i+\nu=\mu} g_{i\lambda} c_\nu = \sum_{i+\nu=\mu} f_{i\lambda} d_\nu,$$

and so

$$\sum_{\deg \mu=0}^{\infty} \sum_{i+\nu=\mu} g_{i\lambda} c_\nu \upsilon_1^{\mu_1} \cdot \ldots \cdot \upsilon_n^{\mu_n} = \sum_{\deg \mu=0}^{\infty} \sum_{i+\nu=\mu} f_{i\lambda} d_\nu \upsilon_1^{\mu_1} \cdot \ldots \cdot \upsilon_n^{\mu_n}.$$

This identity can be written as

$$\left(\sum_{\deg \nu=0}^{\infty} c_\nu \upsilon^\nu\right)\left(\sum_{\deg i}^{l} g_{i\lambda} \upsilon^i\right) = \left(\sum_{\deg \nu=0}^{\infty} d_\nu \upsilon^\nu\right)\left(\sum_{\deg i=0}^{m} f_{i\lambda} \upsilon^i\right).$$

Thus,

$$\xi = \frac{\sum_{\deg \nu=0}^{\infty} c_\nu \upsilon^\nu}{\sum_{\deg \nu=0}^{\infty} d_\nu \upsilon^\nu} = \frac{\sum_{\deg i=0}^{m} f_{i\lambda} \upsilon^i}{\sum_{\deg i=0}^{l} g_{i\lambda} \upsilon^i} \in \Phi(\upsilon_1, \ldots, \upsilon_n).$$

We will need the following two results from Hironaka–Matsumura [4] in the proof of the next lemma. The first is the theorem on birational invariance of the field of formal rational functions (Theorem 2.6 in [4]) which states that if $f\colon Z' \to Z$ is a proper birational morphism of schemes and X is a closed subset of Z, $X' = f^{-1}(X)$ then $\hat{f}\colon \hat{Z}' \to \hat{Z}$ induces an isomorphism $K(\hat{Z}) \xrightarrow{\sim} K(\hat{Z}')$, where $\hat{Z}$ (resp. $\hat{Z}'$) is the completion of Z (resp. Z') along X (resp. X'). The next result we need (Theorem (2.7) in [4]) is that, with the notations above, and under the assumptions of the next lemma, there is a canonical isomorphism (2.7.3 in [4])

$$[K(Z') \otimes_{K(Z)} K(\hat{Z})]_0 \simeq K(\hat{Z}').$$

LEMMA (3.2). *The field of formal-rational functions* $K(\mathrm{Grass}^\wedge(n, r))$ *of* Grass (n, r) *along the projective subspace* $\mathbf{P}^n$ *is exactly the field of rational functions on* Grass (n, r).

Proof. Let Q be the set of points $Q_1, \ldots, Q_n$ in Grass (n, r) such that Q_i is represented by the $n \times r$ matrix (κ_{ij}) where the columns $1, \ldots, \hat{i}, \ldots, n+1$ form the identity matrix and all the remaining columns are zero. Let

$$\pi_1 \colon B_P(\mathrm{Grass}\ (n, r)) \to \mathrm{Grass}\ (n, r)$$

be the first projection as in (I) of Section 1. Let $E = \pi_1^{-1}(Q)$ and let G be the strict transform of $\mathbf{P}^n$ in $B_P(\mathrm{Grass}\ (n, r))$. Thus

$$\pi_1^{-1}(\mathbf{P}^n) = E \cup G.$$

Let $B_P(\mathrm{Grass}^\wedge(n, r))$ (resp. $\hat{B}_1, \ldots, \hat{B}_n, \hat{G}_G$) be the completion of $B_P(\mathrm{Grass}\ (n, r))$ along $E \cup G$ (resp. along $\pi_1^{-1}(Q_1), \ldots, \pi_1^{-1}(Q_n), G$). By the birational invariance theorem of Hironaka–Matsumura (2.6 in [4]), quoted above, it is enough to show

$$K(B_P(\mathrm{Grass}^\wedge(n, r)) \simeq K(B_P(\mathrm{Grass}\ (n, r)).$$

Since G is covered by the $n + 1$ affines

$$U_{1,\ldots,n},\ \{V_{1,\ldots,\hat{i},\ldots,n+1;1,\ldots,n}\}_{i=1,\ldots,n},$$

by Lemma (2.1), $G \approx \mathbf{P}^n \times S^{n(r-n-1)}$. Therefore by the Hironaka–Matsumura Theorem 2.7 [4] quoted above.

$$K(\hat{B}_G) = [k[[\{t_{i,n+j}\}_{i=1,\ldots,n;j=2,\ldots,r-n}]]]_0(t_{1n}, \ldots, \hat{t}_{in}, \ldots, t_{nn}, v_{ni}^{(i)}).$$

(Note. From here on we will abbreviate the indices. Unless otherwise specified, in the sequel, α runs through the set $\{1, \ldots, n\}$ and j runs through the set $\{1, \ldots, r-n\}$. For example, $\{t_{\alpha,n+j}\}_{\alpha\neq i}$ will stand for $\{t_{\alpha,n+j}\}_{\alpha=1,\ldots,n;\alpha\neq i;j=1,\ldots,r-n}$.) On the other hand, by the identies (I) and (II) in Section 2,

$$\begin{aligned} K(\hat{B}_i) &= [k[[v_{1i}^{(i)}, \ldots, v_{ni}^{(i)}, \{v_{i,n+j}^{(i)}\}_{i=1,\ldots,n}]]]_0 \\ &= [k[[t_{1,n+1}v_{ni}^{(i)}, \ldots, t_{i,n+1}v_{ni}^{(i)}, \ldots, t_{n,n+1}v_{ni}^{(i)}, v_{n,i}^{(i)}, \\ &\qquad \{t_{\alpha,n+j} - t_{\alpha,n+1}t_{i,n+j}v_{ni}^{(i)}\}, \{t_{i,n+j}v_{ni}^{(i)}\}]]]_0 \\ &\subset [k[[\{t_{\alpha,n+j}\}_{\alpha\neq i}]][\{t_{\alpha,n}\}_{\alpha\neq i}, \{t_{i,n+j}\}]]_0((v_{ni}^{(i)})). \end{aligned}$$

So by the Hironaka–Matsumura Lemma 3.2 in [4], or Lemma (3.1) in the case $n = 1$, and the identity $t_{i,n+1} = 1/v_{ni}^{(i)}$,

$$K(\hat{B}_i) \cap K(\hat{B}_G) \subset [k[[\{t_{\alpha,n+j}\}_{\alpha \neq i}]][\{t_{\alpha,n}\}, \{t_{i,n+j}\}]]_0.$$

Similarly, for $l \neq i$,

$$\begin{aligned} K(\hat{B}_l) &= [k[[\{t_{\alpha,n+j}\}_{\alpha \neq l}]][\{t_{\alpha,n}\}_{\alpha \neq l}, \{t_{l,n+j}\}]]_0((v_{nl}^{(l)})) \\ &\subset \Phi((\{t_{i,n+j}\}, v_{n,l}^{(l)})) \end{aligned}$$

where

$$\Phi = [k[[\{t_{\alpha,n+j}\}_{\alpha \neq l,i}]][\{t_{\alpha,n}\}_{\alpha \neq l}, \{t_{l,n+j}\}]]_0.$$

Let $L = [k[[\{t_{\alpha,n+j}\}]][\{t_{\alpha,n}\}_{\alpha \neq l}]]_0$. Then $\Phi \subset L$, and, by Lemma (3.1) and the identity $t_{nl} = 1/v_{nl}^{(l)}$,

$$\begin{aligned} K(\hat{B}_l) \cap K(\hat{B}_i) \cap K(\hat{B}_G) &\subset \Phi((\{t_{i,n+j}\}, v_{nl}^{(l)})) \cap L(\{t_{i,n+j}\}, v_{nl}^{(l)}) \\ &= [k[[\{t_{\alpha,n+j}\}_{\alpha \neq l,i}]][\{t_{\alpha,n}\}, \{t_{i,n+j}\}, \{t_{l,n+j}\}]]_0. \end{aligned}$$

It is now clear that by induction we can show

$$K(B_P(\mathrm{Grass}\,(n,r)) = k(t_{\alpha\beta}) \subset K(B_P(\mathrm{Grass}^\wedge(n,r)) \subset \bigcap_{i=1}^{n} K(\hat{B}_i) \cap K(\hat{B}_G) \subset k(t_{\alpha\beta}).$$

In the next lemma let $\mathbf{P}^1 \subset$ Grass (n, r) be given by the two affines l_0, l_1 where l_0 is the subset of Grass (n, r) consisting of all the points which can be represented by the $n \times r$ matrices (v_{ij}) where the columns $1, \ldots, n$ form the identity matrix and the entries $v_{1,n+1}, \ldots, v_{n-1,n+1}, v_{i,j}$, $i = 1, \ldots, n$, $j = n + 2$ $\ldots, r$ are zeros, and l_1 is the set of all points (t_{ij}) where the columns $1, \ldots, n-1$, $n+1$ form the identity matrix and the entries $t_{1,n}, \ldots, t_{n-1,n}, t_{ij}$, $i = 1, \ldots, n$, $j = n + 2, \ldots, r$ are zeros, and where the relation between $t_{n,n}$ and $v_{n,n+1}$ over $l_0 \cap l_1$ is $t_{n,n} = 1/v_{n,n+1}$.

LEMMA (3.3). *The field of formal-rational functions $\hat{K}$ of* Grass (n, r) *along the projective subspace $\mathbf{P}^1$ is exactly the field of rational functions on* Grass (n, r).

Proof. The field of formal-rational functions along the subspace $\mathbf{P}^1$ is

$$\hat{K} = k(v_{n,n+1})((\{v_{i,n+1}\}_{i=1,\ldots,n-1}, \{v_{ij}\}_{i=1,\ldots,n;j=n+2,\ldots,r})).$$

Then $\xi \in \hat{K}$ can be written $\xi = f/g$ where f and g are elements of

$$k(v_{n,n+1})[[\{v_{i,n+1}\}_{i=1,\ldots,n-1}, \{v_{i,j}\}_{i=1,\ldots,n;j=n+2,\ldots,r}]].$$

We can rewrite f/g in the form

$$\xi = \sum c \prod_{\substack{i=1,\ldots,n;\\ j=2,\ldots,r-n}} v_{i,n+j}^{\nu(i,j)} \Big/ \sum d \prod_{\substack{i=1,\ldots,n;\\ j=2,\ldots,r-n}} v_{i,n+j}^{\nu(i,j)}$$

where $\prod v_{i,n+j}^{v(i,j)}$ is a product in powers $v(i,j) \geq 0$ of the variables

$$\{v_{i,n+j}\}_{i=1,\ldots,n;j=2,\ldots,r-n},$$

and

$$c, d \in k(v_{n,n+1})[[v_{1,n+1}, \ldots, v_{n-1,n+1}]] \subset k(v_{n,n+1})((v_{1,n+1}, \ldots, v_{n-1,n+1})).$$

By the Hironaka–Matsumura Lemma (3.1) in [4], $\mathbf{P}^1$ is universally G_3 in $\mathbf{P}^n$, that is,

$$k(v_{n,n+1})((v_{1,n+1}, \ldots, v_{n-1,n+1})) = K(v_{1,n+1}, \ldots, v_{n,n+1}).$$

Thus

$$\xi \in k(v_{1,n+1}, \ldots, v_{n,n+1})((v_{i,n+j,i=1,\ldots,n;j=2,\ldots,r-n})).$$

It now follows by our Lemma (3.2) that ξ is a rational function over Grass (n, r).

THEOREM (3.4). *Let X be a connected closed subscheme of dimension greater than or equal to* 1 *in* Grass (n, r). *Then the field of formal-rational functions of* Grass (n, r) *along X is exactly the field of rational functions on* Grass (n, r).

Proof. We proceed as in the proof of Theorem (3.3) in Hironaka–Matsumura [4] using Lemma (3.3) above. Let Grass$^\wedge$ (n, r) be the completion of Grass (n, r) along X. Let C be an irreducible reduced curve contained in X. Since X is connected, $K(\text{Grass}^\wedge(n, r))$ is contained in $K(\text{Grass}\,(n, r)_{/C})$, where Grass $(n, r)_{/C}$ is the completion of Grass (n, r) along C. Therefore it is enough to assume $X = C$. Recalling that Grass (n, r) is the Grassmannian of n-planes E in k^r, then $C = \{E_t\}_{t \in C} \subset \text{Grass}\,(n, r)$. Given an $r - n - 1$-plane $S \in \text{Grass}\,(r - n - 1, r)$, consider the Schubert cycle

$$\Sigma_S = \{E \in \text{Grass}\,(n, r), \dim\,(E \cap S) \geq 1\}.$$

By choosing S generically, we may assume that $C \cap \Sigma_S = \phi$. For this S, set $Q = k^r/S$, and let $\bar{E}$ be projection to Q of $E \in \text{Grass}\,(n, r) - \Sigma_S$. Then $\dim Q = n + 1$, $\dim \bar{E} = n$, and $\bar{C} = \{\bar{E}_t\}_{t \in C}$ is the image of C in Grass $(n, n + 1)$. Consider the projection

$$\text{Grass}\,(n, r) - \Sigma_S \xrightarrow{\pi_1} \text{Grass}\,(n, n + 1).$$

Let $\mathbf{P}^1 \subset \text{Grass}\,(n, n + 1)$ and $L^{n-2} \subset \text{Grass}\,(n, n + 1)$ be such that $L \cap \mathbf{P}^1 = \phi$ and $L \cap \bar{C} = \phi$, and let

$$\text{Grass}\,(n, n + 1) - L \xrightarrow{\bar{\pi}_1} \mathbf{P}^1$$

be the projection with center L. By Hironaka's Lemma (2.2) in [3], Grass $(n, n + 1) - L$ can be given a unique structure of vector bundle such that the inclusion s: $\mathbf{P}^1 \subset \text{Grass}\,(n, n + 1)$ is the zero section. Let

$$B_L(\text{Grass}\,(n, n + 1)) \xrightarrow{\beta} \text{Grass}\,(n, n + 1)$$

be the blowing up of Grass $(n, n+1)$ with center L. Then, by Hironaka, Section 2 in [3], there is a morphism

$$B_L(\text{Grass }(n, n+1)) \xrightarrow{p} \mathbf{P}^1.$$

Consider the diagram

$$\begin{CD} V @>{\pi}>> B_L(\text{Grass }(n, n+1)) \\ @VVV @VV{\beta}V \\ \text{Grass }(n, r) - \Sigma_S @>>> \text{Grass }(n, n+1) \end{CD}$$

where V is the fibred product of Grass $(n, r) - \Sigma_S$ with $B_L(\text{Grass }(n, n+1))$ over Grass $(n, n+1)$. We know Grass $(n, r) - \Sigma_S$ has a structure of vector bundle over Grass $(n, n+1)$ so π inherits a structure of vector bundle which induces the structure of a vector bundle on $p \circ \pi$ whose zero section is the inclusion $\mathbf{P}^1 \subset V$. Next consider the fibred product

$$\begin{CD} W @>{\gamma}>> V \\ @V{\pi'}VV @VV{p \circ \pi}V \\ C @>{\lambda}>> \mathbf{P}^1 \end{CD}$$

where λ is the restriction of $p \circ \pi$ to C. As π' inherits a vector bundle structure, let C_1 be the zero section of π' which is equal to $\gamma^{-1}(\mathbf{P}^1)$. We have another section C_2 of π' which induces the inclusion $C \subset V$. Then there is an automorphism σ of W such that $\sigma(C_1) = C_2$. Let $\hat{W}_i (i = 1, 2)$ be the completion of W along C_i. Then σ extends to an isomorphism

$$\hat{W}_1 \xrightarrow{\sim} \hat{W}_2$$

which induces an isomorphism

$$K(\hat{W}_2) \xrightarrow{\sim} K(\hat{W}_1)$$

and $K(W)$ is mapped onto itself under this isomorphism. By our Lemma (3.3) and Theorem (2.7) in [4], we have $K(\hat{W}_1) = K(W)$. So $K(\hat{W}_2) = K(W)$. Since $\gamma(C_2) = C$ we have a map φ: $W_2 \to \text{Grass}^{\wedge}(n, r)$ which induces a monomorphism

$$K(\text{Grass}^{\wedge}\ (n, r)) \longrightarrow K(\hat{W}_2) = K(W).$$

Since $K(W)$ is a finite algebraic extension of $K(\text{Grass }(n, r))$, we have $K(\text{Grass}^{\wedge}(n, r))$ is finite algebraic over $K(\text{Grass } n, r))$, and its branch locus in Grass (n, r) is contained in that of $K(W)$ over $K(\text{Grass }(n, r))$. By the purity of branch locus $K(\text{Grass}^{\wedge}(n, r))$ is unramified over $K(\text{Grass }(n, r))$. Since Grass (n, r) is simply connected, we conclude that $K(\text{Grass}^{\wedge}(n, r)) = K(\text{Grass }(n, r))$.

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University of Illinois
Urbana, Illinois
Harvard University
Cambridge, Massachusetts