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CRITERIA FOR ALGEBRAIC DEPENDENCE OF MEROMORPHIC MAPPINGS INTO ALGEBRAIC VARIETIES

BY

S. J. DROUILHET

In this paper we study the following problem: given nondegenerate meromorphic maps f and g from an affine algebraic variety M to projective algebraic varieties V', V'' of the same or lower dimension, if there are hypersurfaces A', A'' in V' and V'' such that $f^{-1}(A') = g^{-1}(A'')$, when are f and g algebraically related. Roughly our result says that if f and g satisfy the same algebraic relation at all points of $f^{-1}(A')$ and A' and A'' are sufficiently positive, then f and g must satisfy this relationship identically.

The main result of this paper is given below as Theorem 4. Before giving it, we will present some special cases, Propositions 1 and 3. They do not possess the same degree of sharpness and range of applicability as the general theorem, of which they are actually corollaries, but are conceptually somewhat clearer and easier to present at the outset. The principal applications of our main result are contained in the corollary to Proposition 1 and in Theorem 6.

PROPOSITION 1. *Let M be a smooth affine variety and $V_1, \ldots, V_k$ smooth projective algebraic varieties with* $\dim V_i \leq \dim M$ *for all i. Let S be a hypersurface in $V^k = \prod_{i=1}^k V_i$ such that the line bundle L on V^k defined by S has the form $L = \otimes_{i=1}^k \pi_i^* L_i$, where $\pi_i\colon V^k \to V_i$ is projection on ith factor and L_i is a holomorphic line bundle on V_i with $c_1(L_i) \geq 0$. For each i, let A_i be a hypersurface with normal crossings in V_i such that the line bundle L_{A_i} on V_i defined by A_i is positive. For each i, let $f_i\colon M \to V_i$ be a nondegenerate meromorphic map. If the following conditions are met, then*

$$(f_1 \times \cdots \times f_k)(M) \subset S:$$

(1) *either $M = \mathbf{C}^n$ or at least one f_i is transcendental;*
(2) *There is a set $E \subset M$ such that $f_i^{-1}(A_i) = E$ for all i;*
(3) $(f_1 \times \cdots \times f_k)(E) \subset S$;
(4) $d_i = 1 - [K_{V_i}^*/L_{A_i}] > 0$ *for all i;*

and either

$$(5) \qquad \left(d_j - k\left[\frac{L_j}{L_{A_j}}\right]\right) + \sum_{\substack{i=1\\ i\neq j}}^{k} \left(d_j d_i - \frac{k}{d_i}\left[\frac{L_i}{L_{A_i}}\right]\right) > 0$$

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for some j, or,

(5′) *for some j,* $d_i - k[L_i/L_{A_i}] > 0$ *whenever* $i \neq j$, *and*

$$d_j - k\left[\frac{L_j}{L_{A_j}}\right] + d_j \sum_{\substack{i=1 \\ i \neq j}}^{k} \left(d_i - k\left[\frac{L_i}{L_{A_i}}\right]\right) > 0.$$

Here, if L', L'' are holomorphic line bundles on a projective algebraic variety V, $[L'/L''] = \inf \{k \in \mathbf{R}: kc_1(L'') - c_1(L') > 0\}$. It should be mentioned that in cases where one is trying to use Theorem 1 to derive a specific numerical result, if both criteria (5) and (5′) are applicable, (5′) may give a sharper result, although it involves checking an extra condition. Neither of these criteria is conceptually simple. Later in the paper, a conceptually simpler version will be presented as Theorem 3, but it has the drawback of not leading to the sharpest possible results in specific cases.

The method of proof of Proposition 1 stems from an argument in [1], in which we gave one generalization of a one variable unicity theorem of R. Nevanlinna stated below as Theorem 2. The notation we use will be the same as that of [1]. With one exception it is the same as the several variables Nevanlinna theory notation employed by Shiffman in [6]; the exception is that our $\bar{N}_f(D, r)$ is defined to be $N(\text{supp } f^*D, r)$ in the notation of his paper. We will freely use results from [1] and [6].

Proof of Proposition 1. In this proof, as is usual in Nevanlinna theory, all inequalities involving the variable r which ultimately stem from the Second Main Theorem should be understood as holding for all positive r outside a set of finite measure.

In the case $M \neq \mathbf{C}^n$, by relabeling indices, we will assume on the basis of (1) that f_1 is transcendental. Shortly we will show that it follows that all the f_i are transcendental.

As in the reasoning leading to (3.3) in [1],

$$T_{f_i}(L_{A_i}, r) + T_{f_i}(K_{V_i}, r) \leq \bar{N}_{f_i}(A_i, r) + o(T_{f_i}(L_{A_i}, r)) \tag{6}$$

provided that $M = \mathbf{C}^n$ or f_i is transcendental. For all $b > [K_{V_i}^*/L_{A_i}]$,

$$bc_1(L_{A_i}) - c_1(K_{V_i}^*) > 0,$$

hence

$$bT_{f_i}(L_{A_i}, r) - T_{f_i}(K_{V_i}^*) \geq O(1),$$

so

$$T_{f_i}(K_{V_i}^*, r) \leq [K_{V_i}^*/L_{A_i}]T_{f_i}(L_{A_i}, r) + O(1).$$

This sort of argument will be used to justify analogous formulas in the sequel. On the basis of this calculation, (6) gives

$$(1 - [K_{V_i}^*/L_{A_i}])T_{f_i}(L_{A_i}, r) \leq \bar{N}_{f_i}(L_{A_i}, r) + o(T_{f_i}(L_{A_i}, r)). \tag{7}$$

LEMMA A. *For all i,*

(i) $O(T_{f_i}(L_{A_i}, r)) = O(T_{f_1}(L_{A_1}, r))$
(ii) $d_i T_{f_i}(L_{A_i}, r) \le T_{f_1}(L_{A_1}, r) + o(T_{f_1}(L_{A_1}, r))$
(iii) $d_1 T_{f_1}(L_{A_1}, r) \le T_{f_i}(L_{A_i}, r) + o(T_{f_1}(L_{A_1}, r))$.

Proof. By assumption (2) and the First Main Theorem from [6], for all i,

$$\bar{N}_{f_1}(A_1, r) = \bar{N}_{f_i}(A_i, r) \le T_{f_i}(L_{A_i}, r) + O(1). \tag{8}$$

By (7),

$$d_1 T_{f_1}(L_{A_1}, r) \le T_{f_i}(L_{A_i}, r) + o(T_{f_1}(L_{A_1}, r)). \tag{9}$$

If $M \ne \mathbf{C}^n$, then since f_1 is transcendental, $\log r = o(T_{f_1}(L_{A_1}, r))$. Since $d_1 > 0$, by (9), $\log r = o(T_{f_i}(L_{A_i}, r))$, implying f_i is transcendental. Thus (6) holds for all i. Suitable modification of the reasoning yielding (9) then gives

$$d_i T_{f_i}(L_{A_i}, r) \le T_{f_1}(L_{A_1}, r) + o(T_{f_i}(L_{A_i}, r)). \tag{10}$$

By (9) and (10),

$$0 < d_1 \le T_{f_i}(L_{A_i}, r)/T_{f_1}(L_{A_1}, r) + o(1), \tag{11}$$
$$0 < d_i \le T_{f_1}(L_{A_1}, r)/T_{f_i}(L_{A_1}, r) + o(1).$$

This pair of inequalities proves the lemma.

Note that by (7) and Lemma A,

$$\sum_{i=1}^{k} d_i T_{f_i}(L_{A_i}, r) \le \sum_{i=1}^{k} \bar{N}_{f_i}(A_i, r) + o(T_{f_1}(L_{A_1}, r)). \tag{12}$$

By assumption (2), for all i,

$$\bar{N}_{f_i}(A_i, r) = N(E, r). \tag{13}$$

Now assume that $(f_1 \times \cdots \times f_k)(M) \not\subset S$. We are going to obtain a contradiction.

LEMMA B. $N(E, r) \le \sum_{i=1}^{k} T_{f_i}(L_i, r) + O(1)$

Proof. V^k is a smooth projective algebraic variety. Define $h\colon M \to V^k$ by $h = f_1 \times \cdots \times f_k$. Since $f_1 \times \cdots \times f_k(E) \subset S$, $E \le \operatorname{supp}(h^*S)$, from which it follows that $N(E, r) \le N(h^*S, r)$. By the First Main Theorem in [6], since we are assuming $(f_1 \times \cdots \times f_k)(M) \not\subset S$,

$$N(h^*S, r) \le T_h(L, r) + O(1),$$

giving $N(E, r) \le T_h(L, r) + O(1)$. Since

$$T_h(L, r) = T_{f_1 \times \cdots \times f_k}(\pi_1^* L_1 \otimes \cdots \otimes \pi_k^* L_k, r) = \sum_{i=1}^{k} T_{f_i}(L_i, r),$$

the lemma is proved.

Combining (12) and Lemma B,

$$\sum_{i=1}^{k} d_i T_{f_i}(L_{A_i}, r) \le k \sum_{i=1}^{k} T_{f_i}(L_i, r) + o(T_{f_1}, L_{A_1}, r)). \tag{14}$$

By relabeling indices, we may assume that (5) holds for $j = 1$. For all i, $T_{f_i}(L_i, r) \le [L_i/L_{A_i}]T_{f_i}(L_{A_i}, r) + O(1)$. So

$$(d_1 - k[L_1/L_{A_1}])T_{f_1}(L_{A_1}, r) + \sum_{i=2}^{k} (d_i - k[L_i/L_{A_i}])T_{f_i}(L_{A_i}, r) \le o(T_{f_1}(L_{A_1}, r)). \tag{15}$$

Using Lemma A, (15) becomes

$$\left\{\left(d_1 - k\left[\frac{L_1}{L_{A_1}}\right]\right) + \sum_{i=2}^{k} \left(d_1 d_i - \left(\frac{k}{d_i}\right)\left[\frac{L_i}{L_{A_i}}\right]\right)\right\} T_{f_1}(L_{A_1}, r) \le o(T_{f_1}(L_{A_1}, r)). \tag{16}$$

Divide (16) by $T_{f_1}(L_{A_1}, r)$ and let r approach infinity. Then

$$\left(d_1 - k\left[\frac{L_1}{L_{A_1}}\right]\right) + \sum_{i=2}^{k} \left(d_1 d_i - \left(\frac{k}{d_i}\right)\left[\frac{L_i}{L_{A_i}}\right]\right) \le 0, \tag{17}$$

which contradicts (5), which, recall, we are assuming holds with $j = 1$. Therefore $(f_1 \times \cdots \times f_k)(M) \subset S$, and the theorem is proved in this case.

Now we will prove the theorem in the case that (5′) holds. By relabeling indices, we may assume it holds for $j = 1$. The inequality in (15) is still valid. However, because of the condition $d_i - k[L_i/L_{A_i}] \ge 0$ for all $i \ne 1$, Lemma A may be applied to (15) to conclude that

$$\left(d_1 - k\left[\frac{L_1}{L_{A_1}}\right]\right) T_{f_1}(L_{A_1}, r) + d_1 \sum_{i=2}^{k} \left(d_i - k\left[\frac{L_i}{L_{A_i}}\right]\right) T_{f_1}(L_{A_1}, r) \le o(T_{f_1}(L_{A_1}, r)). \tag{18}$$

The remainder of the proof of the theorem in this case is the same as in the first case.

COROLLARY. *Let f_1, f_2 be nonconstant meromorphic functions on $\mathbf{C}$. Let $P(x, y)$ be a polynomial of degree p_1 in x, p_2 in y. Let r be any integer greater than or equal to $p_1 + p_2 + 3$. Suppose there are two sets A_1, A_2 in $\mathbf{P}^1(\mathbf{C})$, each of which contains r points, such that $f_1^{-1}(A_1)$ and $f_2^{-1}(A_2)$ each equal to a common set E. Suppose for all z in E, $P(f_1(z), f_2(z)) = 0$. Then $P(f_1, f_2)$ is identically* 0.

Proof. Apply Proposition 1 using criterion (5′) with $M = \mathbf{C}$, $k = 2$, $V_1 = V_2 = \mathbf{P}^1(\mathbf{C})$, $S = \{(x, y) \in \mathbf{P}^1 \times \mathbf{P}^1 : P(x, y) = 0\}$, and A_1, A_2 are the two given sets. By relabeling indices, we may assume that $p_1 \ge p_2$. If H is the hyperplane bundle on $\mathbf{P}^1$, then $K^*_{V_1} = K^*_{V_2} = H^2$, $L_{A_i} = H^r$, and $L = \pi_1^* L_1 \otimes \pi_2^* L_2$ with

$L_i = H^{p_i}$. Conditions (1) through (4) are easily seen to be met. Condition (5′) remains to be checked. Note that here $d_1 = d_2 = 1 - (2/r) = (r-2)/r$. Thus

$$d_2 - k[L_2/L_{A_2}] = (r-2)/r - (2p_2)/r \geq (p_1 - p_2 + 1)/r > 0.$$

Now,

$$d_1 - k\left[\frac{L_1}{L_{A_1}}\right] + d_1\left(d_2 - k\left[\frac{L_2}{L_{A_2}}\right]\right) = \frac{r-2}{r} - \frac{2p_1}{r} + \frac{r-2}{r}\left\{\frac{r-2}{r} - \frac{2p_2}{r}\right\}.$$

This last term will be larger than 0 provided it is larger than 0 when multiplied by $r^2/2$, that is, provided that

$$r^2 - (p_1 + p_2 + 3)r + 2(p_2 + 1) > 0.$$

If $r \geq p_1 + p_2 + 3$, then this last inequality is valid, so (5′) holds. By Theorem 1, the corollary is proved.

To illustrate how the corollary is applied, we use it to derive a basic trigonometric identity. The motivation for this illustration is to provide an explanation for the phenomenon of Example 5.2 of our previous paper [1] in light of our present results.

Example. Let $f_1(z) = \sin(z), f_2(z) = \cos(z)$. In the corollary take

$$P(x, y) = x^2 + y^2 - 1 \quad \text{and} \quad A_1 = A_2 = \{0, \pm 1, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\sqrt{3}\}$$

All conditions of the corollary are met; in particular, A_1 and A_2 contain $7 \geq p_1 + p_2 + 3 = 7$ points. Thus $\sin^2 z + \cos^2 z - 1 = 0$ identically.

The next two examples deal with the sharpness of the corollary. The first shows it is sharp whenever $p_1 = p_2$, and the second illustrates its sharpness when $p_1 \neq p_2$.

Example. Let $f_1(z) = e^z, f_2(z) = e^{-z}$. In the corollary, take

$$P(x, y) = x^n - y^n \quad \text{and} \quad A_1 = A_2 = \{0, \infty\} \cup \{2n\text{th roots of unity}\}.$$

Here all conditions of the corollary are met except that A_1 and A_2 only contain $2n + 2$ points, whereas $p_1 + p_2 + 3 = 2n + 3$. Therefore, we cannot conclude that $e^{nz} - e^{-nz} = 0$ identically.

Example. Let $f_1(z) = e^z, f_2(z) = e^{-z}$. In the corollary, take $P(x, y) = x - 1$ and $A_1 = A_2 = \{0, 1, \infty\}$. All conditions are met except that A_1 and A_2 only contain 3 points and here $p_1 + p_2 + 3 = 4$. Thus we cannot conclude that $e^z - 1$ is identically 0.

Next we show that Nevanlinna's five point unicity theorem, originally proved in [4], can be recovered as a special case of the corollary.

THEOREM 2 (R. Nevanlinna). *Let f_1, f_2 be nonconstant meromorphic functions on* **C**. *Suppose for five points $a_1, \ldots, a_5 \in \mathbf{P}^1(\mathbf{C})$, $f_1^{-1}(a_i) = f_2^{-1}(a_i)$. Then f_1 and f_2 are identically equal.*

Proof. In the corollary, take

$$P(x, y) = x - y \quad \text{and} \quad A_1 = A_2 = \{a_1, \dots, a_5\}.$$

All conditions of the corollary are met since here $p_1 + p_2 + 3 = 5$.

The proof of Theorem 2 is essentially Nevanlinna's original proof. We use the central idea of his proof to obtain the results of this paper.

As was remarked in the discussion following the statement of Proposition 1, conditions (5) and (5′) can be replaced by a slightly clearer condition if one is willing to settle for results which might be less sharp and have narrower ranges of applicability. Here is one such result, which follows from Proposition 1.

PROPOSITION 3. *Suppose in the hypotheses of Proposition* 1, *all the* V_i *are the same and all the* A_i *are the same. Further suppose that* (5) *and* (5′) *are replaced by the following condition:*

(5″) *for all i*, $L_{A_i} \otimes K_{V_i} \otimes L_i^{-k} > 0$.

Then the conclusion is still valid.

This result is not as good as Proposition 1. If one tries to prove the corollary to Proposition 1 using Proposition 3, instead of the bound $r \geq p_1 + p_2 + 3$, one obtains the bound $r \geq 2 \max(p_1, p_2) + 3$, which is not as good.

In Proposition 1, S must be a hypersurface belonging to the complete linear system defined by an element of $\otimes_{i=1}^k \pi_i^* \operatorname{Pic}(V_i)$. In general,

$$\operatorname{Pic}(V^k) \neq \otimes_{i=1}^k \pi_i^* \operatorname{Pic}(V_i).$$

For example, if C is a smooth curve of genus greater than zero, then the diagonal in $C \times C$ defines a line bundle not belonging to

$$\pi_1^* \operatorname{Pic}(C) \otimes \pi_2^* \operatorname{Pic}(C).$$

(For a discussion of these facts, see the problem sets in [3].) It would be nice to find an analog of Proposition 1 in which S is allowed to be a more general hypersurface in V^k. Such an analog can be found, and is stated below as Theorem 4. Theorem 4 contains Propositions 1 and 3 as special cases.

THEOREM 4. *Let M be a smooth affine algebraic variety, and $V_1, \dots, V_k$ smooth projective algebraic varieties with* $\dim V_i \leq \dim M$ *for all i. For each i, let f_i: $M \to V_i$ be a nondegenerate meromorphic map. Let S be a hypersurface in $V^k = \prod_{i=1}^k V_i$ and L the line bundle on V^k it defines. For $i = 1$ to k, let L_i be any holomorphic line bundle on V_i with $c_1(L_i) \geq 0$. Letting π_i: $V^k \to V_i$ be projection on the ith factor, set*

$$\alpha = \left[L \Big/ \left(\bigotimes_{i=1}^{k} \pi_i^* L_i \right) \right].$$

For each i, let A_i be a hypersurface in V_i with normal crossings such that the line bundle on V_i is positive. If the following conditions are met, then $(f_1 \times \cdots \times f_k)(M) \subset S$:

(1) *either $M = \mathbf{C}^n$ or at least one f_i is transcendental;*
(2) *there is a set $E \subset M$ such that $f_i^{-1}(A_i) = E$ for all i;*
(3) $(f_1 \times \cdots \times f_k)(E) \subset S$;
(4) $d_i = 1 - [K_{V_i}^*/L_{A_i}] > 0$ *for all i;*

and either

(5) *for some j,*

$$\left(d_j - k\alpha\left[\frac{L_j}{L_{A_j}}\right]\right) + \sum_{\substack{i=1\\ i\neq j}}^{k}\left(d_i d_j - \frac{k\alpha}{d_i}\left[\frac{L_i}{L_{A_i}}\right]\right) > 0,$$

or

(5′) *for some j,*

$$d_i - k\alpha[L_i/L_{A_i}] > 0 \text{ for all } i \neq j$$

and

$$d_j - k\alpha\left[\frac{L_j}{L_{A_j}}\right] + d_j\sum_{\substack{i=1\\ i\neq j}}^{k}\left(d_i - k\alpha\left[\frac{L_i}{L_{A_i}}\right]\right) > 0.$$

Proof. Follow the proof of Proposition 1, replacing Lemma B with the following Lemma C where appropriate.

LEMMA C. $N(E, r) \leq \alpha \sum_{i=1}^{k} T_{f_i}(L_i, r) + O(1)$

Proof. Follow the proof of Lemma B, noting that

$$\text{(19)} \qquad T_h(L, r) \leq \left[L\Big/\left(\bigotimes_{i=1}^{k}\pi_i^* L_i\right)\right] T_{f_1\times\cdots\times f_k}(\pi_1^* L_1 \otimes \cdots \otimes \pi_k^* L_k)$$
$$= \alpha \sum_{i=1}^{k} T_{f_i}(L_i, r).$$

It should be noted that if Theorem 4 is applied repeatedly, it can be used to determine when $(f_1 \times \cdots \times f_k)(M)$ is contained in an intersection of hypersurfaces in V^k. In particular, this method can be used to prove the principal unicity theorem in our previous paper [1], which gave a criterion for two nondegenerate meromorphic maps from an affine variety to a projective variety to be identically equal.

As an application of Theorem 4, we will give a generalization of the following theorem, originally due to E. M. Schmid [5]:

THEOREM 5 (E. M. Schmid). *Let V be a smooth elliptic curve, and $f, g\colon \mathbf{C} \to V$ nonconstant holomorphic maps. Suppose there exist r points $a_1, \ldots, a_r \in V$ such that $f^{-1}(a_i) = g^{-1}(a_i)$. If $r \geq 5$, then $f \equiv g$.*

Our result is:

THEOREM 6. *Let V be a smooth elliptic curve, and $f, g\colon \mathbf{C} \to V$ nonconstant holomorphic maps. Let $A = \{a_1, \ldots, a_r\}$ be a set of r points in V. Let n be a nonzero integer, $nA = \{na_1, \ldots, na_r\}$, and m the number of distinct points in nA. Suppose $f^{-1}(A) = g^{-1}(nA)$ and for all i, for all $z \in f^{-1}(a_i)$, $g(z) = na_i$. If*

$$rm - (r + m)(n^2 + 1) > 0, \tag{20}$$

then $g \equiv nf$.

Proof. It suffices to prove the theorem for $n > 0$. Let

$$S = \{(x, y) \in V \times V\colon y = nx\}.$$

We want to show that $f \times g(\mathbf{C}) \subset S$. Let p be any point on V, and L_p the line bundle it defines on V. As usual, let $\pi_i\colon V \times V \to V$ be projection on the ith factor, and L the line bundle on $V \times V$ defined by S.

LEMMA. $\alpha = [L/(\pi_1^* L_p \otimes \pi_2^* L_p)] = n^2 + 1$.

Proof. (For background information on the part of the proof involving correspondences on curves, see [2], pages 282–290.) Write $V = \mathbf{C}/\Lambda$, where Λ is the lattice generated by 1 and τ, where $\operatorname{Im} \tau > 0$. Let z be a coordinate on V arising from the projection $\pi\colon \mathbf{C} \to V$. Let (z_1, z_2) be the corresponding coordinates on $V \times V$. Since

$$c_1(L_p) \in H^{1,1}(V, \mathbf{Z}) \quad \text{and} \quad \int_V c_1(L_p) = \deg (L_p) = 1,$$

we have

$$c_1(L_p) = \frac{i}{2 \operatorname{Im} \tau}\, dz \wedge d\bar{z}.$$

This gives

$$c_1(\pi_1^* L_p \otimes \pi_2^* L_p) = \frac{i}{2 \operatorname{Im} \tau} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$$

Now we calculate $c_1(L)$. As a first step we find $c_1(L_\Delta)$, where L_Δ is the line bundle defined by Δ, the diagonal in $V \times V$. Let $E = \pi_1^{-1}(p)$, $F = \pi_2^{-1}(p)$. Since $c_1(L_\Delta) \in H^{1,1}(V \times V, \mathbf{Z})$, by a symmetry argument we may write

$$c_1(L_\Delta) = \frac{i}{2 \operatorname{Im} \tau} [\delta dz_1 \wedge d\bar{z}_1 + \beta dz_2 \wedge d\bar{z}_2 + \gamma(dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_1)].$$

Now,

$$1 = E \cdot \Delta = \int_{V \times V} c_1(\pi_1^* L_p) \wedge c_1(L_\Delta) = \beta,$$

so $\beta = 1$. Similarly, $\delta = 1$. In general, if C is a curve of genus g, then the self-intersection number of the diagonal on $C \times C$ is $2 - 2g$. Here then,

$$0 = \Delta \cdot \Delta = \int_{V \times V} [c_1(L_\Delta)]^2 = 2(\delta\beta - \gamma)^2,$$

so $\gamma^2 = \delta\beta = 1$. In short,

$$c_1(L_\Delta) = \frac{i}{2 \operatorname{Im} \tau} [dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + \gamma(dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_1)],$$

where $\gamma^2 = 1$ (its exact value is of no concern to us). We next use this result to find $c_1(L)$. L is the line bundle on $V \times V$ defined by the curve of the correspondence $S = \{(x, y) \in V \times V: y = nx\}$. The valence of this correspondence is $-n$. Hence S is linearly equivalent to a divisor of the form $aE + bF + n\Delta$, where $a, b \in Z$. We compute a and b using the facts that $S \cdot E = 1$, $S \cdot F = n^2$, and $S \cdot \Delta = (n - 1)^2$, obtaining

$$S \sim (n^2 - n)E + (1 - n)F + n\Delta.$$

As a result,

$$\begin{aligned} (21) \quad c_1(L) &= (n^2 - n)c_1(\pi_1^* L_p) + (1 - n)c_1(\pi_2^* L_p) + nc_1(L_\Delta) \\ &= \frac{i}{2 \operatorname{Im} \tau} [n^2 dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + n\gamma(dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_1)] \end{aligned}$$

where $\gamma^2 = 1$. Now

$$\begin{aligned} \alpha &= \inf \{k \in \mathbf{R}: kc_1(\pi_1^* L_p \otimes \pi_2^* L_p) - c_1(L) > 0\} \\ &= \inf \Big\{k \in \mathbf{R}: \frac{i}{2 \operatorname{Im} \tau} [(k - n^2)dz_1 \wedge d\bar{z}_1 \\ &\qquad + (k - 1)dz_2 \wedge d\bar{z}_2 - n\gamma(dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_1)] > 0\Big\} \\ &= \inf \left\{k \in \mathbf{R}: \begin{pmatrix} k - n^2 & -n \\ -n & k - 1 \end{pmatrix} \text{ is positive definite}\right\}. \end{aligned}$$

For a fixed k, the eigenvalues of the matrix in question are the roots λ of the equation $\lambda^2 - (2k - n^2 - 1)\lambda + [k^2 - (n^2 + 1)k] = 0$, and will be positive if $k > n^2 + 1$. Thus $\alpha = n^2 + 1$, and the lemma is proved.

Now we return to the proof of Theorem 6. Conditions (1) through (4) of Theorem 4 are met. Here K_V is trivial, so $d_1 = d_2 = 1$. Let $k = 2$,

$$L_1 = L_2 = L_p, \quad A_1 = \{a_1, \ldots, a_r\}, \quad A_2 = \{b_1, \ldots, b_m\} = \{na_1, \ldots, na_r\}.$$

Apply condition (5) with $j = 1$. Then the left hand side of (20) becomes

$$2 - \frac{2(n^2+1)}{r} - \frac{2(n^2+1)}{m}$$

which is greater than 0 since $rm - (r+m)(n^2+1) > 0$.

Note that if $n = 1$, then Theorem 6 reduces to Theorem 5. Schmid showed that Theorem 5 is sharp. We will comment on her proof later. The following example does not quite demonstrate sharpness of Theorem 6 in a case where $n > 1$, but does show that if condition (20) is not satisfied, then it is not necessarily true that $g = nf$.

Example. Let $V = \mathbf{C}/\Lambda$ where Λ is the lattice generated by 1 and τ with $\operatorname{Im} \tau > 0$. Let $\pi: \mathbf{C} \to V$ be projection. Define $f, g: \mathbf{C} \to V$ by

$$f(z) = \pi(z), \quad g(z) = -2\pi(z).$$

Let $A = \{x \in V: 4x = 0\}$. For $n = 2$, $nA = 2A = \{x \in V: 2x = 0\}$. Here $r = 16$, $m = 4$. All conditions of Theorem 6 are met except (20) since here

$$rm - (r+m)(n^2+1) = -36 < 0.$$

So we cannot use Theorem 6 to conclude that $g(z) = 2f(z)$.

To show that Theorem 5 is sharp, Schmid gave the following example. Let V be as in the preceding example. Let $f, g: \mathbf{C} \to V$ be $f(z) = \pi(z)$, $g(z) = -\pi(z)$. Let $r = 4$ and $\{a_1, \ldots, a_r\} = \{0, 1/2, \tau/2, (1+\tau)/2\}$. All conditions of Theorem 5 are met except that $r = 4 < 5$. So we cannot conclude that $g \equiv f$.

A full investigation of what happens in Theorem 5 when $r < 5$ has yet to be undertaken. However, as a matter of interest, we record the fact that Schmid's example is basically the only one of its type for that particular choice of $a_1, \ldots, a_4$. To prove this assertion, stated precisely below as Proposition 8, we will use the following theorem of Nevanlinna from [4]:

THEOREM 7 (R. Nevanlinna). *Suppose $f, g: \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ are non-constant holomorphic maps. Suppose there are four points $a_1, \ldots, a_4 \in \mathbf{P}^1(\mathbf{C})$ such that $f^{-1}(a_i) = g^{-1}(a_i)$ with multiplicities. Then either $f \equiv g$, or by relabeling points, $f^{-1}(a_1) = f^{-1}(a_2) = \phi$, and $g = L \circ f$, where L is a fractional linear transformation of $\mathbf{P}^1$.*

PROPOSITION 8. *Let $V = \mathbf{C}/\Lambda$ where Λ is the lattice generated by 1 and τ with* $\operatorname{Im} \tau > 0$. *Let $f, g: \mathbf{C} \to V$ be nonconstant holomorphic maps. Let*

$$\{a_1, \ldots, a_4\} = \{0, 1/2, \tau/2, (1+\tau)/2\}.$$

Suppose $f^{-1}(a_i) = g^{-1}(a_i)$ with multiplicities for all i. Then either $g = f$ or $g = -f$.

Proof. Let $\mathscr{P}: V \to \mathbf{P}^1$ be the Weierstrass $\mathscr{P}$-function. Let $\tilde{f}, \tilde{g}: \mathbf{C} \to P^1$ be $\tilde{f} = \mathscr{P} \circ f$, $\tilde{g} = \mathscr{P} \circ g$. Let $e_i = \mathscr{P}(a_i)$. Then for $i = 1$ to 4, $\tilde{f}^{-1}(e_i) = \tilde{g}^{-1}(e_i)$ with multiplicities. By the defect relations, f, hence $\tilde{f}$, is surjective. So no $\tilde{f}^{-1}(e_i) = \phi$. Thus by Theorem 7, $\mathscr{P} \circ f \equiv \mathscr{P} \circ g$. Thus $g = f$ or $g = -f$.

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Yankton College
Yankton, South Dakota
Moorhead State University
Moorhead, Minnesota