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A CLOSED ORIENTABLE 3-MANIFOLD BOUNDS SO THAT ITS FUNDAMENTAL GROUP INJECTS

BY

TOM KNOBLAUCH

1. Introduction

If M^3 is a closed orientable 3-manifold, M^3 bounds a simply connected compact 4-manifold [1, p. 540]. We herein prove that M^3 bounds a compact (but not in general orientable) 4-manifold M^4 with the inclusion induced

$$i_{\#}: \Pi_1(M^3) \to \Pi_1(M^4)$$

injective. The theorem we will actually prove is slightly stronger than the statement above.

THEOREM 1.1. *If M^3 is a closed orientable 3-manifold, M^3 bounds M^4 such that if $f(M^2, \mathrm{Bd}M^2) \subseteq (M^4, \mathrm{Bd}M^4)$ is a singular disk with holds, then there is a map $g: M^2 \to M^3$ with $g|_{\mathrm{Bd}M^2} = f|_{\mathrm{Bd}M^2}$.*

2. Some Lens Space Analogs

Let D_r^2 be a disk with r handles. Let $A_r, B_r \subseteq (\mathrm{Bd}D_r^2) \times S^1$ be the oriented simple closed curves $(\mathrm{Bd}D_r^2) \times \{p\}$ and $\{q\} \times S^1$ respectively, where $p \varepsilon S^1$ and $q \varepsilon \mathrm{Bd}D_r^2$. If

$$h: (\mathrm{Bd}D_s^2) \times S^1 \to (\mathrm{Bd}D_r^2) \times S^1$$

is a homeomorphism with

$$h(A_s) = aA_r + bB_r \quad \text{and} \quad h(B_s) = cA_r + dB_r,$$

we let $M^3_{r,s,a,b,c,d,}$ denote $D_r^2 \times S^1 \cup_h D_s^2 \times S^1$. Notice that if $s = 0$, there is no need to specify c and d, and we can use the shorter notation $M^3_{r,0,a,b}$.

LEMMA 2.1. *The pair $(D_1^2, \mathrm{Bd}D_1^2)$ satisfies the conclusion of* (1.1).

Proof. Suppose $f(M^2, \mathrm{Bd}M^2) \subseteq (D_1^2, \mathrm{Bd}D_1^2)$ is a singular disk with holes. Let $J \subseteq \mathrm{Int}\ D_1^2$ be a nonseparating simple closed curve. General position

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$f(M^2)$ and J. If K is a component of $f^{-1}(J)$, $f(K)$ is trivial in J, since $f(K)$ bounds a singular disk with holes each of whose other boundary components lies in $\mathrm{Bd}D_1^2$ (use the closure of either half of $M^2 - K$). Then f can be redefined in a neighborhood of $f^{-1}(J)$ to make $f(M^2)$ miss J. Now $f(M^2) \subseteq D_1^2 - J$, which retracts onto $\mathrm{Bd}D_1^2$. ■

LEMMA 2.2. *For any topological space X, the pair $(X \times D_1^2, X \times \mathrm{Bd}D_1^2)$ satisfies the conclusion of* (1.1).

Proof. Let $f(M^2, \mathrm{Bd}M^2) \to (X \times D_1^2, X \times \mathrm{Bd}D_1^2)$ be a singular disk with holes. Suppose

$$p_1 : X \times D_1^2 \to X \quad \text{and} \quad p_2 : X \times D_1^2 \to D_1^2$$

are projection maps. Then by (2.1) there is a map $g \colon M^2 \to \mathrm{Bd}D_1^2$ with

$$g|_{\mathrm{Bd}M^2} = p_2 f|_{\mathrm{Bd}M^2}.$$

We define $h \colon M^2 \to X \times D_1^2$ by $h(m) = (p_1 f(m), g(m))$. If $m \in \mathrm{Bd}M^2$,

$$h(m) = (p_1 f(m), p_2 f(m)) = f(m).$$ ■

COROLLARY 2.3. *$M^3_{r,s,1,0,0,1}$ has property* 1.1 (*By this we mean $M^3_{r,s,1,0,0,1}$ bounds M^4 such that $(M^4, M^3_{r,s,1,0,0,1})$ satisfies the conclusion of* (1.1).)

LEMMA 2.4. *$M^3_{r,1,0,1,1,0}$ has property* 1.1.

Proof. Let $(\mathrm{Bd}D_r^2) \times [0, 1]$ be a collar for $\mathrm{Bd}D_r^2$ in D_r^2. To construct M^4, we attach $S_2^2 \times D_1^2$ (where S_2^2 is a 2-sphere with two handles) to

$$D_r^2 \times D_1^2 - ((\mathrm{Bd}D_r^2) \times (0, 1) \times \mathrm{Int}\, D_1^2)$$

by a homeomorphism

$$h \colon S_2^2 \times S^1 \to (\mathrm{Bd}D_r^2) \times [0, 1] \times S^1 \cup (\mathrm{Bd}D_r^2) \times \{0, 1\} \times D_1^2.$$

Suppose $f(M^2, \mathrm{Bd}M^2) \subseteq (M^4, \mathrm{Bd}M^4)$ is a singular disk with holes. Then we can use (2.3) to move $f(M^2)$ out of $\mathrm{Int}\,(S_2^2 \times D_1^2)$ so that

$$f(M^2) \subseteq D_r^2 \times D_1^2 - ((\mathrm{Bd}D_r^2) \times (0, 1) \times \mathrm{Int}\, D_1^2).$$

Then we can use (2.2) to move $f(M^2)$ out of

$$(D_r^2 - (\mathrm{Bd}D_r^2) \times [0, 1)) \times \mathrm{Int}\, D_1^2.$$ ■

We will need the following extension of (2.1). Let K be an oriented simple closed curve, $J \subseteq \mathrm{Int}\ D_1^2$ an oriented nonseparating simple closed curve, and $J \times [-\frac{1}{2}, \frac{1}{2}]$ a regular neighborhood of J in $\mathrm{Int}\ D_1^2$. Suppose $K \times [0, 1]$ is attached to D_1^2 by a homeomorphism

$$k\colon K \times \{0, 1\} \to J \times \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

with $k(K \times \{0\}) = J \times \{-\frac{1}{2}\}$ and $k(K \times \{1\}) = -J \times \{\frac{1}{2}\}$.

LEMMA 2.5. *The pair* $(D_1^2 \cup_k K \times [0, 1], \mathrm{Bd}D_1^2)$ *satisfies the conclusion of* (1.1).

Proof. Suppose $f(M^2, \mathrm{Bd}M^2) \subseteq (D_1^2 \cup_k K \times [0, 1], \mathrm{Bd}D_1^2)$ is a singular disk with holes. If we general position $f(M^2)$ and $K \times \{\frac{1}{2}\}$ and let $H \subseteq M^2$ be an oriented component of $f^{-1}(K \times \{\frac{1}{2}\})$, then $f(H)$ is trivial in $K \times \{\frac{1}{2}\}$. To see this, notice that $f(H)$ bounds a singular disk with holes in $D_1^2 \cup K \times [0, 1]$ each of whose other boundary components lies in $\mathrm{Bd}D_1^2$. If we attach a disk E^2 to $\mathrm{Bd}D_1^2$, $f(H)$ bounds a singular disk in $D_1^2 \cup K \times [0, 1] \cup E^2$ and in its retract

$$K \times [0, 1] \cup J \times \left[-\frac{1}{2}, \frac{1}{2}\right],$$

a Klein bottle; so $f(H)$ is trivial in $K \times \{\frac{1}{2}\}$. Next f can be redefined in a neighborhood of $f^{-1}(K \times \{\frac{1}{2}\})$ to miss $K \times \{\frac{1}{2}\}$ We can homotop $f(M^2)$ out of $K \times \{0, 1\}$ and apply (2.1) to move $f(M^2)$ out of Int D_1^2. ■

COROLLARY 2.6. *The pair* $((D_1^2 \cup K \times [0, 1]) \times S^1, (\mathrm{Bd}D_1^2) \times S^1)$ *satisfies the conclusion of* (1.1).

Proof. The proof is like that of (2.2). ■

LEMMA 2.7. $M_{1,0,1,1}^3$ *has property* 1.1.

Proof. We begin the construction of M^4 by forming a product $M_{1,0,1,1}^3 \times [0, 1]$. We start working in $M_{1,0,1,1}^3 \times \{1\}$.

Let $J \subseteq D_1^2$ be an oriented, nonseparating simple closed curve. Let $J \times [-1, 1]$ be a regular neighborhood of J in D_1^2 chosen so that the orientation of $J \times \{-1\} \times \{p\}$ agrees with that of A_1 in $D_1^2 - J \times (-1, 1) \times \{p\}$. Similarly, we choose the orientation of $\{j\} \times S^1$ to agree with that of B_1, where $j \in J$.

We attach $E_1^2 \times S^1$, $G_1^2 \times S^1$, and $K \times [0, 1] \times S^1$ (where E_1^2 and G_1^2 are disks with one handle, for which we have curves A_E, B_E and A_G, B_G, and K is an oriented simple closed curve) to $M_{1,0,1,1}^3$ by homeomorphisms

$$e\colon (\mathrm{Bd}E_1^2) \times S^1 \to J \times \{-1\} \times S^1,$$

$$g\colon (\mathrm{Bd}G_1^2) \times S^1 \to J \times \{1\} \times S^1,$$

$$k\colon K \times \{0, 1\} \times S^1 \to J \times \left\{-\frac{1}{2}, \frac{1}{2}\right\} \times S^1$$

satisfying

$$e(A_E) = J \times \{-1\} \times \{p\},\ e(B_E) = \{j\} \times \{-1\} \times S^1 + J \times \{-1\} \times \{p\},$$

$$g(A_G) = \{j\} \times \{1\} \times S^1 - J \times \{1\} \times \{p\},\ g(B_G) = J \times \{1\} \times \{p\},$$

$$k(K \times \{0\} \times \{p\}) = J \times \left\{-\frac{1}{2}\right\} \times \{p\},\ k(K \times \{1\} \times \{p\}) = -J \times \left\{\frac{1}{2}\right\} \times \{p\}.$$

By (2.3),

$$(M^3_{1,0,1,1} - J \times (-1, 1) \times S^1) \bigcup_e E^2_1 \times S^1 \cup_g G^2_1 \times S^1$$

which is homeomorphic to $M^3_{1,1,1,0,0,1}$, bounds P^4 satisfying the conclusion of (1.1).

Next, $J \times [-\frac{1}{2}, \frac{1}{2}] \times S^1 \cup_k K \times [0, 1] \times S^1$ is homeomorphic to $K^2 \times S^1$, where K^2 is a Klein bottle. By (2.2), $K^2 \times S^1$ bounds Q^4 satisfying the conclusion of (1.1).

Third,

$$J \times \left(\left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]\right) \times S^1 \bigcup_k (K \times [0, 1] \times S^1) \bigcup_e E^2_1 \times S^1 \bigcup_g G^2_1 \times S^1$$

is homeomorphic to $M^3_{1,1,0,1,1,0}$ and so bounds T^4 satisfying the conclusion of (1.1).

We let

$$M^4 = \left(M^3_{1,0,1,1} \times [0, 1] - D^2_1 \times S^1 \times \left[\frac{1}{3}, \frac{2}{3}\right]\right) \cup S^2_2 \times H^2_1 \cup P^4 \cup Q^4 \cup T^4$$

where H^2_1 is a disk with one handle.

Suppose $f(M^2, \mathrm{Bd}M^2) \subseteq (M^4, M^3_{1,0,1,1})$ is a singular disk with holes. We move $f(M^2)$ out of

$$S^2_2 \times (\mathrm{Int}\, H^2_1) \cup (\mathrm{Int}\, P^4) \cup (\mathrm{Int}\, Q^4) \cup (\mathrm{Int}\, T^4)$$

and then out of

$$(\mathrm{Int}\, E^2_1) \times S^1 \cup (\mathrm{Int}\, G^2_1) \times S^1.$$

We general position $f(M^2)$ and $(\mathrm{Bd}D^2_1) \times S^1 \times [\frac{2}{3}, 1]$ and let N^2 be a component of

$$f^{-1}\left(D^2_1 \times S^1 \times \left[\frac{2}{3}, 1\right] \cup_k K \times [0, 1] \times S^1\right).$$

We can homotop $f(M^2)$ so that

$$f(N^2) \subseteq D_1^2 \times S^1 \times \{1\} \bigcup K_k \times [0, 1] \times S^1.$$

By (2.6) we can move $f(N^2)$ out of

$$D_1^2 \times S^1 \times \{1\} \cup K \times [0, 1] \times S^1$$

so that $f(M^2) \subseteq M^3_{1,0,1,1} \times [0, 1]$, which retracts onto $M^3_{1,0,1,1} \times \{0\}$. ■

3. (1.1) for closed orientable 2-manifolds

LEMMA 3.1. *S_n^2 has property* 1.1.

Proof. We imbed $2n + 1$ simple closed curves in C_n^3 (a cube with n handles) as in Figure 1. We replace a regular neighborhood of each curve with a copy of $D_1^2 \times S^1$ to form K_n^3.

Let $f(M^2, \mathrm{Bd}M^2) \subseteq (K_n^3, S_n^2)$ be a singular disk with holes. By (2.2) we can make $f(M^2)$ miss each of the copies of $D_1^2 \times S^1$, and we may assume

$$f(M^2) \subseteq C_n^3 - \bigcup_{i=1}^{2n+1} C_i.$$

Letting

$$K^2 \subseteq (\mathrm{Int}\, C_n^3) - \bigcup_{i=n+2}^{2n+1} C_i$$

be a disk with n holes bounded by $\cup_{i=1}^{n+1} C_i$, we general position $f(M^2)$ and K^2. If J is a component of $f^{-1}(K^2)$, $f(J)$ bounds a singular disk in K^2, since $f(J)$ bounds a singular disk with holes (use the closure of either component of $M^2 - J$) in $C_n^3 - \cup_{i=1}^{2n+1} C_i$ each of whose other boundary components lies in $\mathrm{Bd}C_n^3$. Then $f(J)$ is homotopically trivial in $S^3 - \cup_{i=n+2}^{2n+1} C_i$ and hence in K^2.

We can change f on a neighborhood $N(J)$ of each such J to make $f(N(J))$ miss K^2 so that $f(M^2) \subseteq C_n^3 - K^2$, which retracts onto $\mathrm{Bd}C_n^3$. ■

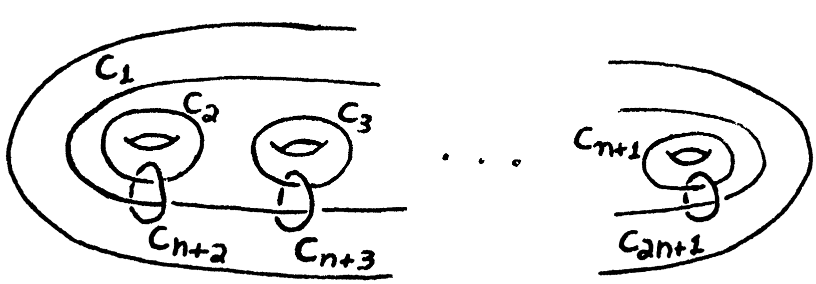


FIG. 1

4. Proof of the theorem

In the 4-tuple (M^3, J, R^3, k), M^3 is a closed orientable 3-manifold, $J \subseteq M^3$ is a simple closed curve, R^3 is a regular neighborhood of J in M^3, and k is an unknotted imbedding of R^3 in S^3. We perform m, an (M^3, J, R^3, k) modification, on M^3 as follows: Suppose $A, B \subseteq \mathrm{Bd}R^3$ are transverse simple closed curves intersecting in a single point, and suppose A and $k(B)$ bound disks in R^3 and $S^3 - k(\mathrm{Int}\, R^3)$ respectively. We attach a copy of $D_1^2 \times S^1$ to M^3 with a homeomorphism

$$h\colon (\mathrm{Bd}D_1^2) \times S^1 \to \mathrm{Bd}R^3$$

satisfying $h((\mathrm{Bd}D_1^2) \times \{p\}) = A$ and $h(\{q\} \times S^1) = B$ where $p \varepsilon S^1$ and $q \varepsilon \mathrm{Bd}D_1^2$. Let

$$m(M^3) = (M^3 - \mathrm{Int}\, R^3) \bigcup_h D_1^2 \times S^1.$$

Next suppose (M^3, J_1, R^3, k) is a 4-tuple as above, and $J_2 \subseteq \mathrm{Int}\, R^3$ bounds a disk in R^3 intersecting J_1 transversely in a single point. Let $R^3(1)$, $R^3(2) \subseteq \mathrm{Int}\, R^3$ be disjoint regular neighborhoods of J_1 and J_2 respectively, which inherit their imbeddings in S^3, k_1 and k_2, from k. Let m_i be the $(M^3, J_i, R^3(i), k_i)$ modification for $i = 1, 2$.

LEMMA 4.1. *$m_2(m_1(M^3))$ has property 1.1 if $m_1(M^3)$ does.*

Proof. Attach $E_1^2 \times S^1$ and $G_1^2 \times S^1$ to $m_2(m_1(M^3))$ by homeomorphisms

$$e\colon (\mathrm{Bd}E_1^2) \times S^1 \to \mathrm{Bd}R^3 \quad \text{and} \quad g\colon (\mathrm{Bd}G_1^2) \times S^1 \to \mathrm{Bd}R^3$$

so that

$$e((\mathrm{Bd}E_1^2) \times \{p\}) = g(\{q\} \times S^1) = A \quad \text{and} \quad e(\{t\} \times S^1) = g((\mathrm{Bd}G_1^2) \times \{p\}) = B,$$

where $q \varepsilon \mathrm{Bd}G_1^2$, $t \varepsilon \mathrm{Bd}E_1^2$, and $p \varepsilon S^1$.

Let N^3 be the closure of the component of $m_2(m_1(M^3)) - \mathrm{Bd}R^3$ that contains $D_1^2(1) \times S^1 \cup D_1^2(2) \times S^1$. Then

$$E_1^2 \times S^1 \cup (m_2(m_1(M^3)) - \mathrm{Int}\, N^3), \quad E_1^2 \times S^1 \cup G_1^2 \times S^1 \quad \text{and} \quad G_1^2 \times S^1 \cup N^3,$$

which are homeomorphic to $m_1(M^3)$, $M^3_{1,1,0,1,1,0}$, and $M^3_{2,1,0,1,1,0}$ respectively, bound P^4, Q^4, and R^4 satisfying the conclusion of Theorem 1.1 by hypothesis and (2.4).

Set $M^4 = P^4 \cup Q^4 \cup R^4$ and suppose $f(M^2, \mathrm{Bd}M^2) \subseteq (M^4, m_2(m_1(M^3)))$ is a singular disk with holes. Then $f(M^2)$ can be moved out of

$$(\mathrm{Int}\, P^4) \cup (\mathrm{Int}\, Q^4) \cup (\mathrm{Int}\, R^4)$$

into

$$m_2(m_1(M^3)) \bigcup_e E_1^2 \times S^1 \bigcup_g G_1^2 \times S^1$$

and then out of (Int E_1^2) $\times$ S^1 $\cup$ (Int G_1^2) $\times$ S^1 by (2.2). ■

If we call the preceding lemma an addition lemma, the following is a subtraction lemma. Suppose M^3 is a closed orientable 3-manifold and m is an (M^3, J, R^3, k) modification.

LEMMA 4.2. *M^3 has property* 1.1 *if $m(M^3)$ does.*

Proof. We omit the proof since it is similar to that of (4.1), but requires attaching only one copy of $D_1^2 \times S^1$ to M^3. ■

Suppose (M^3, J_3, R^3, k) is a 4-tuple as above; $J_4 \subseteq \text{Int } R^3$ is parallel to J_3; $R^3(3)$, $R^3(4) \subseteq \text{Int } R^3$ are disjoint regular neighborhoods of J_3 and J_4 which inherit their imbeddings k_3 and k_4 from k; and $k(J_3)$ and $k(J_4)$ are unlinked in S^3. We have another addition lemma.

LEMMA 4.3. *$m_4(m_3(M^3))$ has property* 1.1 *if $m_3(M^3)$ does.*

Proof. Again the proof is similar to that of (4.1) but involves attaching only one copy of $D_1^2 \times S^1$ to $m_4(m_3(M^3))$. ■

Suppose m_5 is an (M^3, J, R^3, k_5) modification, so that we may consider A and B as fixed. We attach a copy of $D_1^2 \times S^1$ to M^3 by a homeomorphism

$$g\colon (\text{Bd}D_1^2) \times S^1 \to \text{Bd}R^3$$

satisfying $g((\text{Bd}D_1^2) \times \{p\}) = B$ and $g((\{q\} \times S^1) = A$, where $p \varepsilon S^1$ and $q \varepsilon \text{Bd}D_1^2$. Let

$$N^3 = (M^3 - \text{Int } R^3) \bigcup_g (D_1^2 \times S^1).$$

LEMMA 4.4. *N^3 has property* 1.1 *if $m_5(M^3)$ does.*

Proof. Same comment as in the previous proof. ■

We are ready to use these tools to prove (1.1) for a key example.

Suppose X^3 is constructed using the diagram of Figure 2 as follows. We imbed two double solid tori $C_2^3(1)$ and $C_2^3(2)$ in S^3 using h_1 and h_2 as pictured in Figure 2. We than attach $C_2^3(2)$ to $C_2^3(1)$ by the homeomorphism

$$h = h_1^{-1}\, l h_2\colon \text{Bd}C_2^3(2) \to \text{Bd}C_2^3(1)$$

where l: $h_2(\text{Bd}C_2^3(2)) \to h_1(\text{Bd}C_2^3(1))$ is the homeomorphism obtained by isotoping $h_2(\text{Bd}C_2^3(2))$ in Figure 2 rigidly straight down into $h_1(\text{Bd}C_2^3(1))$. We

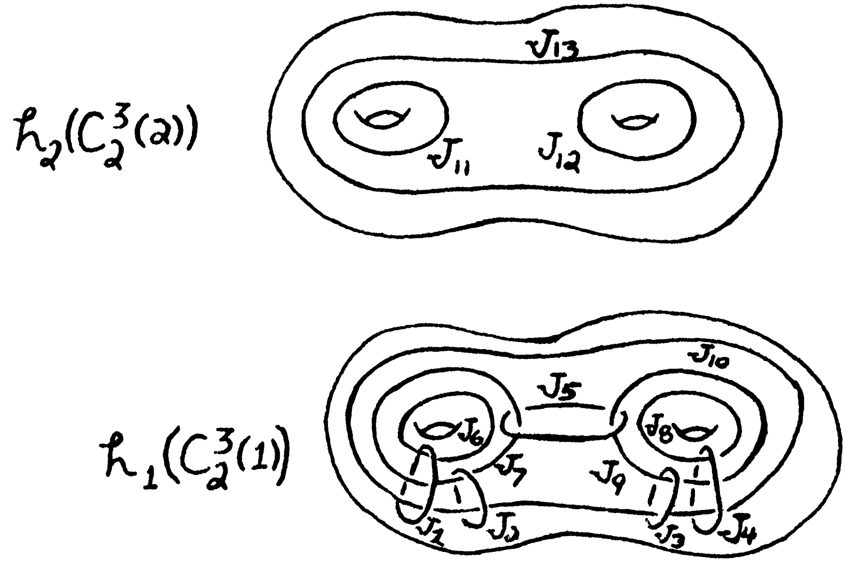


FIG. 2

use imbeddings

$$k_i = h_1|_{R^3(i)} \quad \text{for } 1 \leqslant i \leqslant 10,$$

$$k_i = h_2|_{R^3(i)} \quad \text{for } 11 \leqslant i \leqslant 13$$

to define m_1, a $(C^3_2(1) \cup_h C^3_2(2),\ J_i,\ R^3(i),\ k_i)$ modification. Let

$$X^3 = m_{13}\, m_{12} \ldots m_2\, m_1\left(C^3_2(1) \bigcup_h C^3_2(2)\right).$$

LEMMA 4.5. *X^3 has property* 1.1.

Proof. Construction of X^4. (1) To $X^3 \times [0, 1]$ we attach solid tori $Q^3(1)$, $Q^3(2)$, $Q^3(3)$, $Q^3(4)$, and $Q^3(5)$ by homeomorphisms

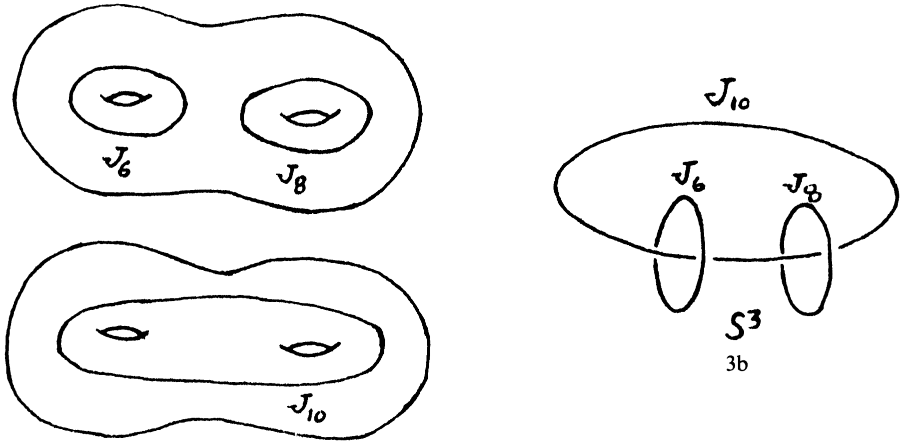
$$q_i\colon \mathrm{Bd}Q^3(i) \to (\mathrm{Bd}D^2_1(i)) \times S^1 \times \{1\}$$

where q_i satisfies

$$q_i(A(i)) = (\mathrm{Bd}D^2_1(i)) \times \{p\} \times \{1\} \quad \text{for } 1 \leqslant i \leqslant 5.$$

By (2.3), $D^2_1(i) \times S^1 \times \{1\} \cup_{q_i} Q^3(i)$ bounds $Q^4(i)$ satisfying the conclusion of (1.1).

(2) The 3-manifold $[X^3 \times \{1\} - \cup^5_{i=1} D^2_1(i) \times S^1 \times \{1\}] \cup (\cup^5_{i=1} Q^3(i))$ bounds a 4-manifold Z^4 satisfying the conclusion of (1.1), since its diagram is like that of X^3, but without J_1, J_2, J_3, J_4, and J_5. We can use (4.3) to amalgamate J_6, J_7, and J_{11}; J_8, J_9, and J_{12}; and J_{10} and J_{13}. We have the



3a 3b

FIG. 3

diagram of Figure 3a. The two sets $D_1^2(6) \times S^1$ and $D_1^2(8) \times S^1$ can be flipflopped using (4.4) to obtain the diagram of Figure 3b. We can remove J_6 and J_8 using (4.1), leaving $M^3_{0,1,0,1,1,0}$. We set

$$Y^4 = x^3 \times [0, 1] \cup \left(\bigcup_{i=1}^{5} Q^4(i) \right) \cup Z^4.$$

(3) Let K^3, $L^3 \subseteq X^3$ be the compact orientable 3-manifolds pictured in Figures 4a and 4b. In Figure 4b, L^3 is seen to be a subset of X^3 (see Figure 2) with five components. Each simple closed curve of Figure 2 represents a homeomorphic copy of $D_1^2 \times S^1$, as does each simple closed curve of Figure 4b. Each arc in Figure 4b represents a copy of $D_1^2 \times [0, 1]$. Similarly, in Figure 4a, K^3 is seen to be a subset of X^3 (see Figure 2). The top half of Figure 4a is identical to the top half of Figure 2, and the three simple closed curves represent copies of $D_1^2 \times S^1$. The bottom half of Figure 4a, a subset of the bottom half of figure 2, is made up of three disjoint solid tori. We describe the solid torus containing J_6 and J_7: This solid torus intersects $\mathrm{Bd}C_2^3(2)$ along the shaded annulus, and is therefore attached to the top half of figure 4a by attaching the shaded annulus to the corresponding shaded annulus in the top half of Figure 4a. The simple closed curves J_6 and J_7 represent copies of $D_1^2 \times S^1$. The three arcs are subarcs of J_1, J_2, and J_5. They represent copies of $D_1^2 \times [0, 1]$. We set

$$X^4 = \left[Y^4 - \left(N\left(K^3 \times \left\{\frac{1}{2}\right\}\right) \cup N\left(L^3 \times \left\{\frac{3}{4}\right\}\right) \right.\right.$$
$$\left.\left. \cup N\left(\bigcup_{i=11}^{13} D_1^2(i) \times S^1 \times \left\{\frac{1}{4}\right\}\right)\right)\right] \cup K^4 \cup L^4 \cup A^4$$

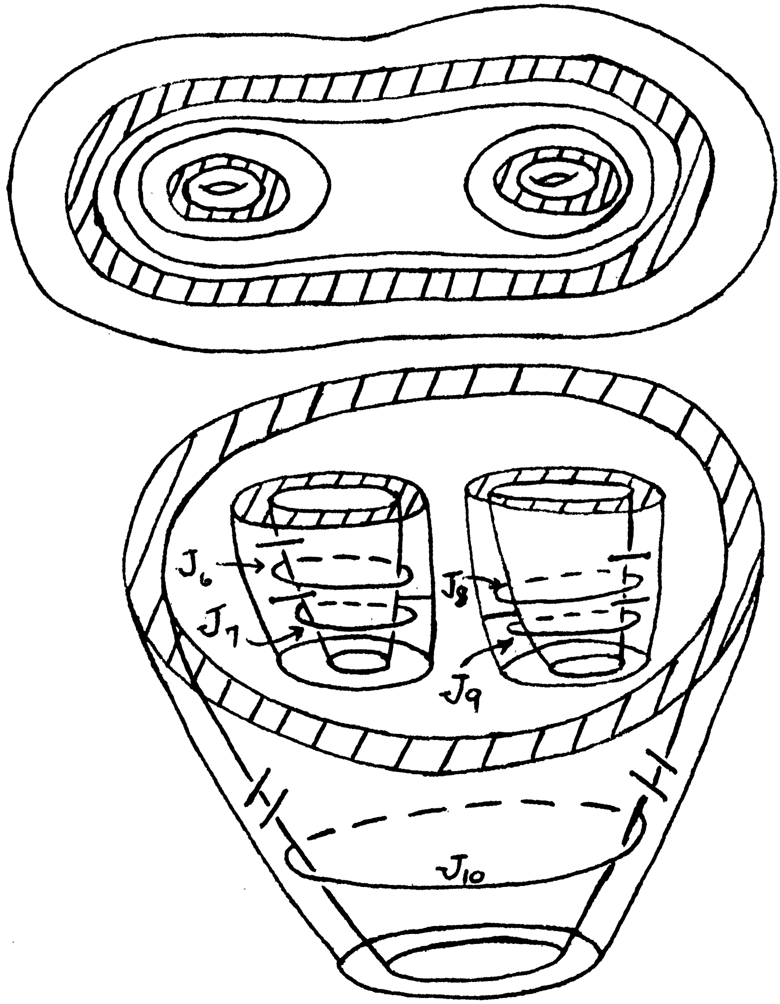


FIG. 4a $K^3 \subseteq X^3$

where each N is a regular neighborhood in Y^4; K^4, L^4, and A^4 have boundaries homeomorphic to $\mathrm{Bd}N(K \times \{\frac{1}{2}\})$, $\mathrm{Bd}N(L^3 \times \{\frac{3}{4}\})$, and $\mathrm{Bd}N(\cup_{i=11}^{13} D_1^2(i) \times S^1 \times \{\frac{1}{4}\})$ respectively; and each of the pairs $(\mathrm{Bd}K^4, K^4)$, $(\mathrm{Bd}L^4, L^4)$, and $(\mathrm{Bd}A^4, A^4)$ satisfies the conclusion of (1.1). Such an A^4 exists by (2.3) since

$$\mathrm{Bd}N\left(\bigcup_{i=11}^{13} D_1^2(i) \times S^1 \times \left\{\frac{1}{4}\right\}\right)$$

is homeomorphic to the disjoint union $\cup_{i=11}^{13}(S_2^2(i) \times S^1)$. To verify that such a manifold L^4 exists, notice that each component of $\mathrm{Bd}N(L^3 \times \{\frac{3}{4}\})$ can be constructed by doubling one of the five components of L^3 along its boundary.

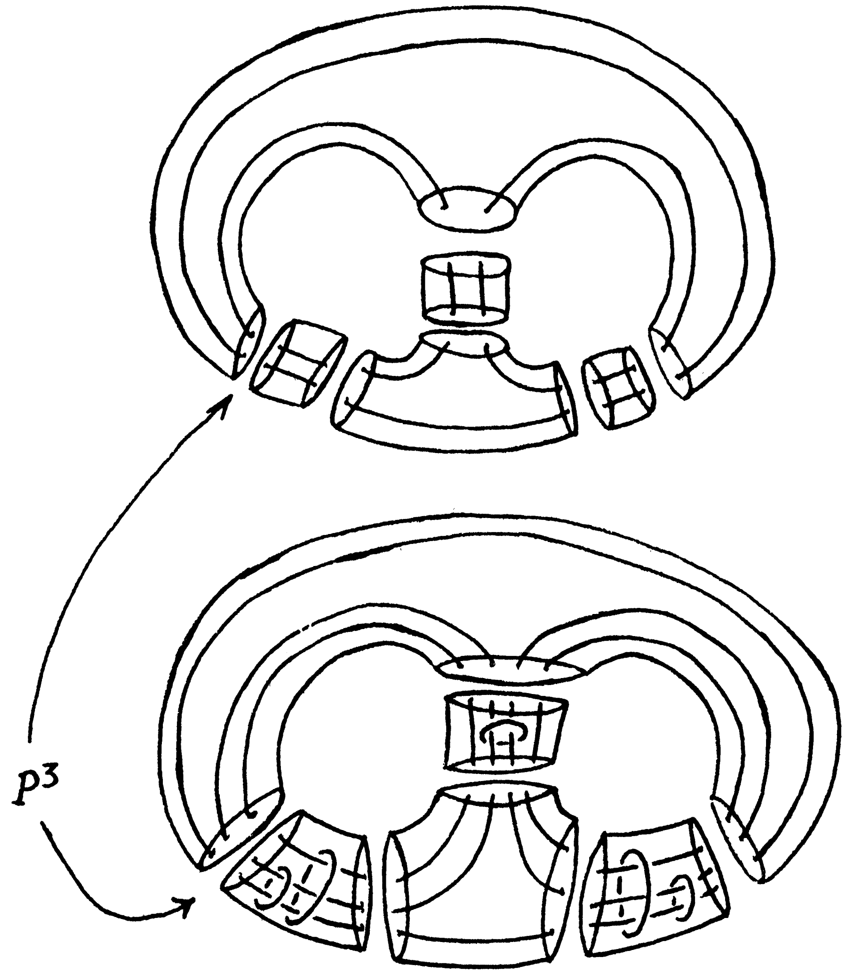


FIG. 4b $L^3 \subseteq X^3$

Then two of the components of $\mathrm{Bd}N(L^3 \times \{\frac{3}{4}\})$ can each be constructed from the diagram of Figure 2 using only J_6, J_7, J_8, J_9, J_{10}, J_{11}, J_{12}, and J_{13}, which has been dealt with previously. The other three components of $\mathrm{Bd}N(L^3 \times \{\frac{3}{4}\})$ can each be reduced, using (4.3) and (4.1), to $M^3_{1,0,1,0}$. Finally, we have $\mathrm{Bd}N(K^3 \times \{\frac{1}{4}\})$, which can be constructed by doubling K^3 along its boundary. The representation of K^3 in Figure 4b was useful for seeing K^3 as a subset of X^3. The representation of K^3 in Figure 5, which results from identifying the corresponding annuli of Figure 4a, helps us visualize half of $\mathrm{Bd}N(K^3 \times \{\frac{1}{4}\})$. The simple closed curves of Figure 5 represent copies of $D_1^2 \times S^1$. Each arc in Figure 4a appears in Figure 5 and again represents a copy of $D_1^2 \times [0, 1]$, which will be matched up, when K^3 is

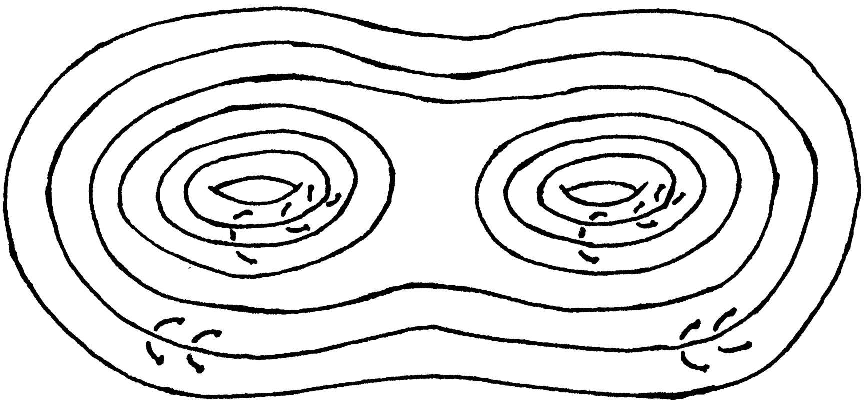


FIG. 5

doubled, with another copy of $D_1^2 \times [0, 1]$, to form a copy of $D_1^2 \times S^1$. The diagram for $\mathrm{Bd}N(K^3 \times \{\frac{1}{4}\})$ can again be reduced, using (4.3) and (4.1), to the diagram of Figure 3a.

Suppose $f(M^2, \mathrm{Bd}M^2) \subseteq (X^4, \mathrm{Bd}X^4)$ is a singular disk with holes. We can make $f(M^2)$ miss

$$K^4 \cup L^4 \cup A^4 \cup (\mathrm{Int}\, Z^4) \cup \left(\bigcup_{i=1}^{5} \mathrm{Int}\, Q^4(i)\right)$$

so that we may assume

$$f(M^2, \mathrm{Bd}M^2) \subseteq \left(X^3 \times [0, 1] \cup \left(\bigcup_{i=1}^{5} Q^3(i)\right)\right) - \left[\left(\bigcup_{i=11}^{13} D_1^2(i) \times S^1 \times \left\{\frac{1}{4}\right\}\right) \cup K^3 \times \left\{\frac{1}{2}\right\} \cup L^3 \times \left\{\frac{3}{4}\right\}\right], X^3 \times \{0\}\right) \subseteq (Y^4, \mathrm{Bd}Y^4).$$

We can make $f(M^2)$ miss $\bigcup_{i=11}^{13} D_1^2(i) \times S^1 \times [\frac{1}{4}, 1]$ by using (2.2) on the set

$$\left(\bigcup_{i=11}^{13} D_1^2(i) \times S^1 \times \left[\frac{1}{4}, 1\right]\right) - \left(K^3 \times \left\{\frac{1}{2}\right\} \cup L^3 \times \left\{\frac{3}{4}\right\}\right).$$

We general position $f(M^2)$ and $X^3 \times \{\frac{1}{2}, \frac{3}{4}\}$. Let N^2 be a component of

$$f^{-1}\left(X^3 \times \left[\frac{1}{2}, 1\right] \cup \left(\bigcup_{i=1}^{5} Q^3(i)\right)\right).$$

Notice that $f(\mathrm{Bd}N^2) \subseteq X^3 \times \{\frac{1}{2}\}$. Let J be a component of $f^{-1}(X^3 \times \{\frac{3}{4}\})$ and let K^2 be the closure of one of the two components of $N^2 - J$. Then $f(J) \subseteq P^3 \times \{\frac{3}{4}\}$ where P^3 is the closure of one of the six components of $X^3 - L^3$.

We now wish to construct a map

$$H\colon \left(X^3 - \bigcup_{i=11}^{13} D_1^2(i) \times S^1\right) \times \left[\frac{1}{2}, 1\right] \cup \left(\bigcup_{i=1}^{5} Q^3(i)\right) \to P^3$$

satisfying

(1) $H(\{p\} \times \{t\}) = p$ for $p \varepsilon P^3 - (\cup_{i=11}^{13} D_1^2(i) \times S^1)$, $t \varepsilon [\frac{1}{2}, 1]$
(2) $H((X^3 - K^3) \times [\frac{1}{2}, 1]) \subseteq P^3 - K^3$.

Construction of H. We assume, without loss of generality, that P^3 is the closure of the component of $X^3 - L^3$ indicated in Figure 4b. Consider the diagram of Figure 6 for $X^3 - D_1^2(12) \times S^1$. The tunnel formed by removing $D_1^2(12) \times S^1$ from X^3 has been enlarged until it runs over into the bottom half of Figure 6. Now we map $D_1^2(i) \times S^1$ onto $D^2(i) \times S^1$ for $i = 1, 2, 3, 4, 5, 8, 9$, and replace $D_1^2(12) \times S^1$ with $(\mathrm{Bd}D_1^2(12)) \times D^2$ so that we have a map

$$F\colon X^3 - D_1^2(12) \times S^1 \to Y^3 - (\mathrm{Bd}D_1^2(12)) \times D^2$$

where Y^3 is represented in Figure 7. Let

$$G\colon Y^3 \cong S_5^2 \times S^1 \to P^3 \cong S_5^2 \times I$$

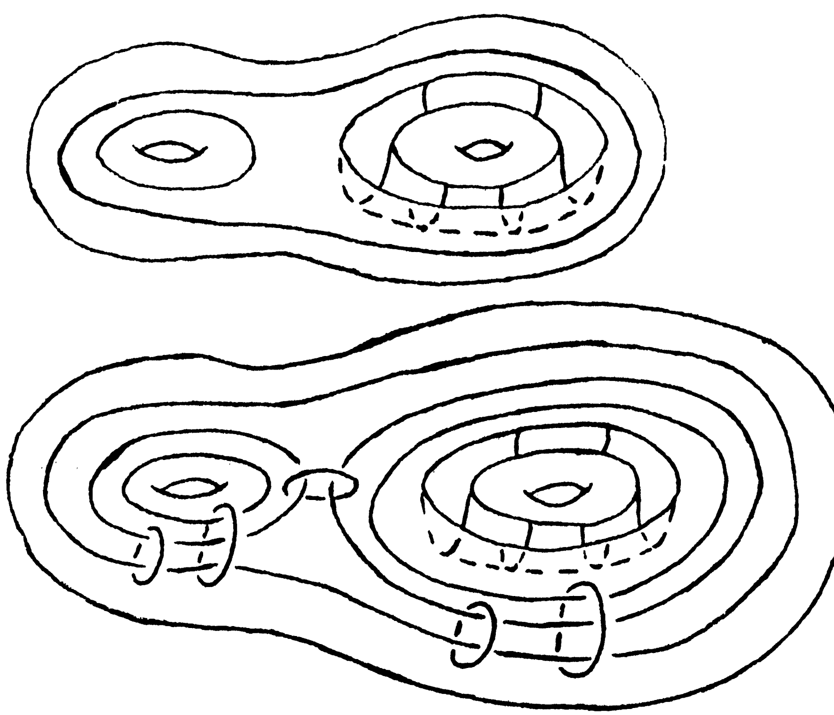


FIG. 6 $X^3 - D_1^2(12) \times S^1$

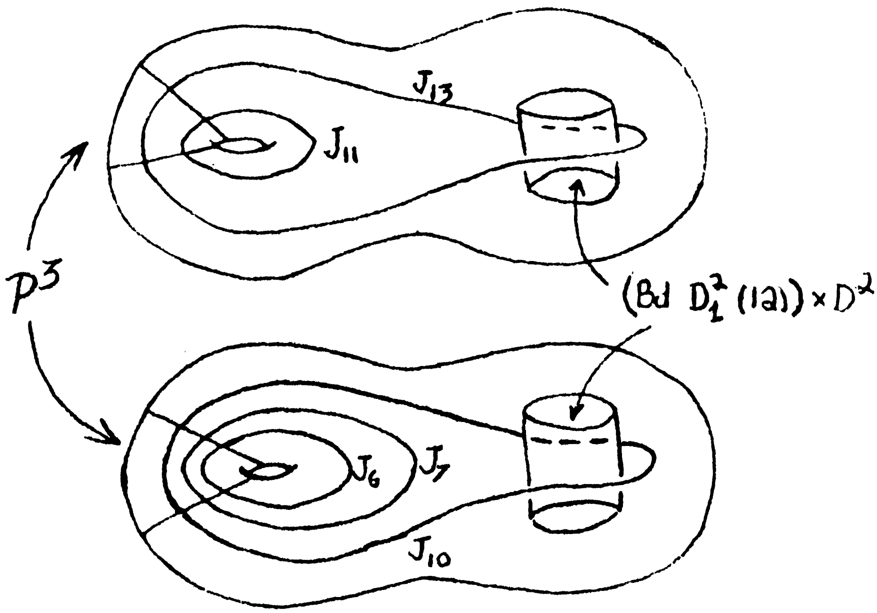


FIG. 7 Y^3

be a retraction such that $P_1G(s, j) = s$ where P_1 is projection onto the first factor, $s \varepsilon S_5^2$, and $j \varepsilon S^1$. Then

$$H = GF\colon X^3 - D_1^2(12) \times S^1 \to P^3$$

(the domain may be $X^3 - D_1^2(11) \times S^1$ or $X^3 - D_1^2(13) \times S^1$ if P^3 is one of the other components of $X^3 - L^3$) can be taken to be defined on

$$\left(X^3 - \bigcup_{i=11}^{13} D_1^2(i) \times S^1\right) \times \left[\frac{1}{2}, 1\right] \cup \left(\bigcup_{i=1}^{5} Q^3(i)\right).$$

The reader can check that H satisfies conditions (1) and (2) above. The reader may also wish to trace through the construction of H when P^3 is one of the two components of $X^3 - L^3$ pictured in the center of Figure 4b. This construction uses J_{13} in place of J_{12}.

The curve $H(f(J)) \subseteq P^3$ is a boundary component of the singular disk with holes $Hf(K^2)$, each of whose other boundary components is a subset of

$$H\left((X^3 - K^3) \times \left\{\frac{1}{2}\right\}\right) \subseteq P^3 - K^3$$

and therefore trivial in P^3. Thus $f(J)$ is trivial in $P^3 \times \{\frac{3}{4}\}$. Since $f(J)$ is trivial in $(X^3 - L^3) \times \{\frac{3}{4}\}$ for each component J of $f^{-1}(X^3 \times \{\frac{3}{4}\})$, we can remap $f^{-1}(X^3 \times [\frac{3}{4}, 1] \cup (\cup_{i=1}^5 Q^3(i))$ into $X^3 \times \{\frac{3}{4}\}$. Then $f(M^2) \subseteq X^3 \times [0, \frac{3}{4}]$, which retracts onto $X^3 \times \{0\}$.

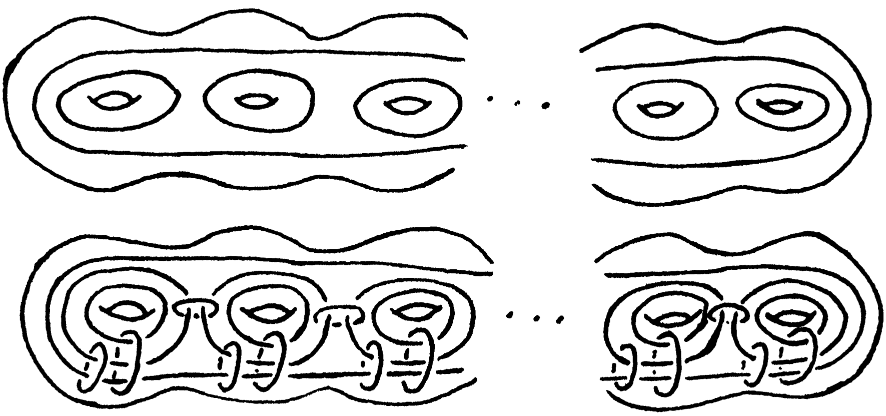


FIG. 8 $X^3(n)$

Suppose $X^3(n)$ is the 3-manifold constructed using the diagram of Figure 8.

LEMMA 4.6. *$X^3(n)$ has property* 1.1.

Proof. The proof is like that of (4.5). ■

Now suppose M^3 is any closed, orientable 3-manifold. M^3 can be constructed from S^3 as follows [2, p. 770]. Let $C_1, \ldots, C_n, D_1, \ldots, D_n, E_1, \ldots, E_{n-1} \subseteq \mathrm{Bd}C_n^3$ be the curves pictured in Figure 9, which shows an imbedding h of C_n^3 in S^3. Let $(\mathrm{Bd}C_n^3) \times [0, \infty)$ be a collar for $\mathrm{Bd}C_n^3$ in $S^3 - \mathrm{Int}\, h\,(C_n^3)$. There is a sequence $J_1, \ldots, J_m$ of simple closed curves in S^3 such that

$$J_i = C_j \times \{i\} \text{ or } D_j \times \{i\} \text{ or } E_j \times \{i\}$$

and

$$M^3 = \left[S^3 - \left(\bigcup_{i=1}^{m} R^3(i)\right)\right] \bigcup_{h'} \left(\bigcup_{i=1}^{m} R^{3\prime}(i)\right)$$

where $R^3(i)$ is a regular neighborhood of J_i in S^3 and $h'(\mathrm{Bd}R^{3\prime}(i)) = \mathrm{Bd}R^3(i)$ satisfies $h'(A_i') = A_i \pm B_i$.

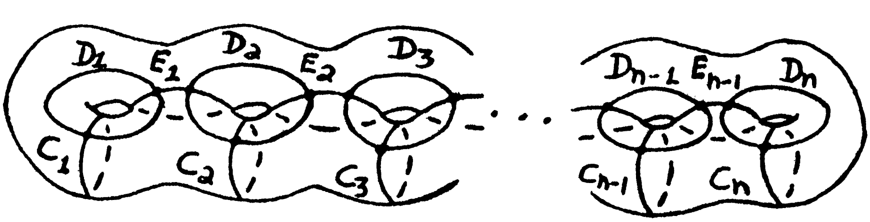


FIG. 9 $h(Bd\ C_n^3)$

We also need an auxiliary 3-manifold T^3. Let m_i be an $(S^3, J_i, R^3(i), k_i)$ modification, with $k_i = h|_{\mathrm{Bd}R^3(i)}$. We set

$$T^3 = m_m\, m_{m-1} \dots m_2\, m_1\, (S^3).$$

LEMMA 4.7. *M^3 has property 1.1 if T^3 does.*

Proof. We think of T^3 and M^3 as being attached along $S^3 - \bigcup_{i=1}^m \mathrm{Int}\, R^3(i)$. The 3-manifolds $R^{3\prime}(i) \cup D_1^2(i) \times S^1$ for $1 \leq i \leq m$, and T^3 bound 4-manifolds $M^4(i)$ and T^4 satisfying the conclusion of (1.1) by (2.7) and hypothesis, respectively. Let $M^4 = (\bigcup_{i=1}^m M^4(i)) \cup T^4$.

Suppose $f(M^2, \mathrm{Bd}M^2) \subseteq (M^4, M^3)$ is a singular disk with holes. We can move $f(M^2)$ out of $(\bigcup_{i=1}^m \mathrm{Int}\, M^4(i) \cup \mathrm{Int}\, T^4$ so that

$$f(M^2) \subseteq M^3 \cup \left(\bigcup_{i=1}^m D_1^2(i) \times S^1\right).$$

Then by (2.2) we can move $f(M^2)$ out of $\bigcup_{i=1}^m(\mathrm{Int}\, D_1^2(i)) \times S^1$.

LEMMA 4.8. *T^3 has property* 1.1.

Proof. We attach $K_n^3(1), K_n^3(2), \dots, K_n^3(m)$, where each $K_n^3(i)$ is homeomorphic to K_n^3 of Figure 1, to T^3 by homeomorphisms

$$l_i\colon \mathrm{Bd}K_n^3(i) \to (\mathrm{Bd}C_n^3) \times \left\{i - \frac{1}{2}\right\} \quad \text{for } 1 \leq i \leq m$$

where l_i is constructed using Figures 1 and 9. More precisely, one should imagine the collar $\mathrm{Bd}C_n^3 \times [0, \infty]$ included in Figure 9. Then Figure 1 should be superimposed on Figure 9 to see the map l_i. The 3-manifolds

$$C_n^3 \cup (\mathrm{Bd}C_n^3) \times \left[0, \frac{1}{2}\right] \cup K_n^3(1),$$

$$K_n^3(1) \cup \left[(BdC_n^3) \times \left[\frac{1}{2}, \frac{3}{2}\right] - R^3(i)\right] \cup D_1^2(1) \times S^1 \cup K_n^3(2),$$

$$\vdots$$

$$K_n^3(m-1) \cup \left[(\mathrm{Bd}C_n^3) \times \left[m - \frac{3}{2}, m - \frac{1}{2}\right] - R^3(m-1)\right]$$

$$\cup D_1^2(m-1) \times S^1 \cup K_n^3(m)$$

can each be constructed from a subdiagram of Figure 8 and so by (4.6) and (4.2) bound $T^4(1), \dots, T^4(m)$ respectively, satisfying the conclusion of (1.1). And

$$K_n^3(m) \cup \left[S^3 - \left(C_n^3 \cup (\mathrm{Bd}C_n^3) \times \left[0, m - \frac{1}{2}\right] \cup R^3(m)\right)\right] \cup D_1^2(m) \times S^1$$

can be altered by replacing $C_j \times \{m + 1\}$ by $D_1^2(m + j) \times S^1$ for $1 \leqslant j \leqslant n$, then flip-flopping $D_1^2(m + j) \times S^1$ to form a 3-manifold constructed from a subdiagram of Figure 8. So

$$K_n^3(m) \cup \left[S^3 - \left(C_n^3 \cup (\mathrm{Bd}C_n^3) \times \left[0, m - \frac{1}{2}\right] \cup R^3(m)\right)\right] \cup D_1^2(m) \times S^1$$

bounds $T^4(m + 1)$ satisfying the conclusion of (1.1).

Let $T^4 = \bigcup_{i=1}^{m+1} T^4(i)$. If $f(M^2, \mathrm{Bd}M^2) \subseteq (T^4, \mathrm{Bd}T^4)$ is a singular disk with holes, we can move $f(M^2)$ out of $\bigcup_{i=1}^{m+1}$ Int $T^4(i)$ into $T^3 \cup (\bigcup_{i=1}^{m} K_n^3(i))$, and by (3.1) out of $\bigcup_{i=1}^{m}$ Int $K_n^3(i)$ into T^3. ■

Theorem 1.1 follows from (4.7) and (4.8).

References

1. W. B. R. Lickorish, *A representation of orientable combinatorial three-manifolds*, Ann. of Math. (2), vol. 76 (1962), pp. 531–540.
2. ———, *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Cambridge. Philos. Soc., vol. 60 (1964), pp. 769–778.

University of Wisconsin at Marathon
Wausau, Wisconsin