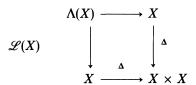
## THE EILENBERG-MOORE SPECTRAL SEQUENCE AND THE MOD 2 COHOMOLOGY OF CERTAIN FREE LOOP SPACES

BY

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There has been considerable interest in computing the cohomology of the space  $\Lambda(X)$  of free loops on X, at least since the theorem of Gromoll and Meyer, connecting the unboundedness of the Betti numbers  $\{\dim H^i(\Lambda X; \mathbf{Q})\}$  with the existence of infinitely many closed geodesics on X when X is a Riemannian manifold. There have been however relatively few explicit calculations. The minimal model theory of Sullivan has been used in the rational case to obtain a few results. However with finite coefficients, aside from [12], which only contains Betti number estimates, there seems nothing known, apart from those facts that are easily knowable. In [9] we observed that the free loop space sits in a fibre square for any connected space X, where  $\Delta$  is the diagonal map:



This observation makes available the Eilenberg-Moore spectral sequence (see for example [8]) as a tool for computing  $H^*(\Lambda(X); k)$  for simply connected X. In [9] we dealt with the case where the coefficient field k was of characteristic zero. In this note we take up the case of  $k = \mathbb{Z}/2$ , and derive the following not so easily knowable result.

THEOREM. Let X be a simply connected space, and suppose  $Sq^1$  vanishes on  $H^*(X; \mathbb{Z}/2)$  and

(\*) 
$$H^*(X; \mathbb{Z}/2) \simeq P[x_1, ..., x_n]/(x_1^{e_1}, ..., x_n^{e_n})$$

where  $e_1 \cdots e_n$  is a power of 2, and  $P[\ ]$  denotes a polynomial algebra. Then the Eilenberg-Moore spectral sequence

$$E_r \Rightarrow H^*(\Lambda(X); \, \mathbb{Z}/2), \quad E_2 = \mathrm{Tor}_{H^*(X; \, \mathbb{Z}/2) \otimes H^*(X; \, \mathbb{Z}/2)}^{**} \, (H^*(X; \, \mathbb{Z}/2), \, H^*(X; \, \mathbb{Z}/2))$$
 collapses.

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An essential step in the proof is the explicit computation of  $E_2$ , along the lines of [9; 3.5] (see also Proposition 1 below). A particularly simple example of a space that satisfies (\*) are the complex quadrics

$$Q_n := \{ [z] \in \mathbb{C}P(n+1) | z_0^2 + \dots + z_{n+1}^2 = 0 \}$$

when n + 1 is a power of 2. Combining the preceding theorem with the known mod 2 cohomology of the quadric and applying Proposition 1 below we have:

COROLLARY. Let n+1 be a power of 2. Then there is a filtration on  $H^*(\Lambda Q_n; \mathbb{Z}/2)$  such that

$$E^{\circ}H^*(\Lambda Q_n: \mathbf{Z}/2) \simeq \frac{P[u, v]}{(u^{m+1}, v^2)} \otimes E[su, sv] \otimes \Gamma[\tau u, \tau v]$$

where n = 2m + 1,  $E[\ ]$  an exterior algebra, and  $\Gamma[\ ]$  a divided power algebra and the degrees of the generators are as follows:

deg 
$$u = 2$$
, deg  $su = 1$ , deg  $\tau u = 2m$ ,  
deg  $v = 2m + 2$ , deg  $sv = 2m + 1$ , deg  $\tau v = 4m$ .

The classes u and v have filtration 0, su, sv filtration -1 and  $\tau u$ ,  $\tau v$  filtration -2.

Other examples of spaces that satisfy (\*) are U(n)/SO(n), Sp(n)/SO(n), etc.

I want to thank Frank Conolley for suggesting the problem of computing  $H^*(\Lambda Q_n; \mathbb{Z}/2)$  as being a natural "test case" for extending the results of [9], for as he pointed out,  $H^*(Q_n; \mathbb{Q})$  and  $H^*(\mathbb{C}P(n); \mathbb{Q})$  are isomorphic when n is odd, so the rational results of [11] does not apply to deduce anything about closed geodesics on odd quadrics.

The proof of the theorem requires a number of preliminary manoeuvres. We begin by recalling some results from [9]. Let X be a connected topological space, and  $\Lambda(X) := X^{S^1}$  the space of free loops on X. There is then the fibre square

$$\mathcal{L}(X) \xrightarrow{\Lambda(X)} X$$

$$\downarrow^{p} \qquad \downarrow^{\Delta}$$

$$X \xrightarrow{\Delta} X \times X$$

where  $\Delta$  is the diagonal map, whose fibre is the ordinary loop space  $\Omega(X)$  of X. Thus for simply connected X and coefficients in a field k we obtain an Eilenberg-Moore spectral sequence [8], [9]

$$E_r \Rightarrow H^*(\Lambda(X); k), \quad E_2 = \mathrm{Tor}_{H_*(X;k) \otimes H^*(X;k)} (H^*(X; k), H^*(X; k)).$$

For notational simplicity it will be convenient to set

$$T^{*,*}(A^*) := \operatorname{Tor}_{A^* \otimes A^*}^{*,*} (A^*, A^*)$$

for any graded connected algebra A\*.

Convention. For the remainder of this note all cohomology will be taken with  $\mathbb{Z}/2$  coefficients, and we write  $H^*(\ )$  for  $H^*(\ ; \mathbb{Z}/2)$ .

Using the mod 2 analog of [9; Section 3] one can easily prove:

Proposition 1. Let

$$A^* \simeq P[x_1, \ldots, x_n]/(x_1^{e_1}, \ldots, x_n^{e_n})$$

where  $e := e_1 \cdot \cdot \cdot \cdot e_n$  is a power of 2. Then

$$T^{**}(A^*) \simeq A^* \otimes \operatorname{Tor}_{A^*}^{**}(\mathbb{Z}/2, \mathbb{Z}/2)$$

$$\simeq \frac{P[x_1, \dots, x_n]}{(x_1^{e_1}, \dots, x_n^{e_n})} \otimes E[su_1, \dots, su_n] \otimes \Gamma[\tau u_1, \dots, \tau u_n]$$

where  $E[\ ]$  is an exterior algebra,  $\Gamma[\ ]$  a divided power algebra and

$$\deg su_i = (-1, \deg u_i), i = 1, ..., n; \deg \tau u_i = (-2, e_i), i = 1, ..., n$$

*Proof.* For the sake of completeness we sketch a proof based on Hopf algebra considerations. Since e is a power of 2 we may impose a Hopf algebra structure on  $A^*$  by declaring the generators to be primitive elements. Then

$$\mathbb{Z}/2 \to A^* \xrightarrow{\Delta} A^* \otimes A^* \xrightarrow{\mu} A^* \to \mathbb{Z}/2$$

is a coexact sequence of algebras, where  $\mu$  is multiplication and  $\Delta$  the diagonal map. Moreover,  $A^* \otimes A^*$  is free over  $A^*$  [6; 4.4] so the change of rings spectral sequence [1; XVI.6.1. (1a)] may be applied. There being no problem with local coefficients we conclude

$$\begin{split} E_r \Rightarrow \mathrm{Tor}_{A*\otimes A*}(A^*,\ A^*), \\ E_2^{p,q} \simeq A^* \otimes \mathrm{Tor}_{A*}^p(\mathbb{Z}/2,\ A^*) \otimes \mathrm{Tor}_{A*}^q(\mathbb{Z}/2,\ \mathbb{Z}/2) \end{split}$$

whence the spectral sequence collapses to the isomorphism

$$\operatorname{Tor}_{A^*\otimes A^*}(A^*, A^*) \simeq A^* \otimes \operatorname{Tor}_{A^*}(\mathbb{Z}/2, \mathbb{Z}/2).$$

The computation of  $Tor_{A*}(\mathbb{Z}/2, \mathbb{Z}/2)$  is routine.

Proposition 1 gives us the structure of the  $E_2$  term of the Eilenberg-Moore spectral sequence of the fibre square  $\mathcal{L}(X)$  when X satisfies (\*).

THEOREM 2. Let X be a simply connected space such that

(1) 
$$Sq^1: H^*(X; \mathbb{Z}/2) \rightarrow H^*(X; \mathbb{Z}/2)$$
 vanishes

and

(2)  $H^*(X; \mathbb{Z}/2) \simeq P[x_1 \cdots x_n]/(x_1^{e_1}, \dots, x_n^{e_n})$  where  $e_1 \cdots e_n$  is a power of 2. Then the Eilenberg-Moore spectral sequence

$$E_r \Rightarrow H^*(\Lambda(X)), \quad E_2 = T^{**}(H^*(X))$$

for  $\mathcal{L}(X)$  collapses.

The proof of Theorem 2 proceeds by comparing  $\{E_r(\mathcal{L}(X)), d_r(\mathcal{L}(X))\}$  to the corresponding spectral sequence for certain universal examples E. The universal examples are H-spaces, so the following lemma allows us to reduce the study of  $\{E_r(\mathcal{L}(E), d_r(\mathcal{L}(E))\}$  to more familiar Eilenberg-Moore spectral sequence considerations.

LEMMA 3. Let X be an H-space. Then  $\Lambda(X)$  is homotopy equivalent to  $X \times \Omega(X)$ . Moreover, for simply connected X the Eilenberg-Moore spectral sequence

$$\{E_r(\mathcal{L}(X), d_r(\mathcal{L}(X))\}$$

and

$$\{H^*(X) \otimes E_r(X), 1 \otimes d_r(X)\}$$

are isomorphic, where  $\{E_r(X), d_r(X)\}$  is the Eilenberg-Moore spectral sequence of the path-loop fibration

$$\Omega X \subseteq PX \downarrow X$$
.

*Proof.* Since X is an H-space, so is  $\Lambda X = X^{S^1}$ . Moreover, seen with this H-space structure, the evaluation map  $e: \Lambda(X) \downarrow X$  becomes an H-map. Thus

$$\Omega X \subseteq \Lambda X \downarrow^e X$$

becomes a principal bundle with

$$s: X \to \Lambda X: s(x) = \text{constant loop at } x$$

as cross-section. Hence the multiplication gives a map

$$X \times \Omega X \rightarrow \Lambda X$$

which is a homotopy equivalence. Naturality and diagram chasing yields the rest.

**Proof of Proposition 2.** We proceed by induction to show that  $d_r = 0$ . The structure of  $E_2(\mathcal{L}(X))$  is given in Proposition 1. As an algebra we see that  $E_2$  is generated by classes

$$u_1, \ldots$$
 of filtration zero,  $s^{-1}u_1, \ldots$  of filtration  $-1$ ,  $\gamma_{2s}(\tau u_1), \ldots$  of filtration  $2^{s+1}, s=0, 1, \ldots$ 

If there is a nonvanishing differential, then it must take a nonzero value on some indecomposable element, and so it suffices from filtration considerations to show that  $d_r$  vanishes  $\gamma_{2s}(\tau u_1)$  for all  $s \ge 0$ , and all  $r \ge 2$ . To simplify notations we drop subscripts, setting  $u = u_i$ ,  $d := \deg u_i$ ,  $e = e_i$ , etc., and consider  $\gamma_{2s}(\tau u)$ . Let  $E_m$  be the stable two stage Postnikov system defined by the fibre square

$$E_{m} \xrightarrow{} L(\mathbb{Z}/2, de)$$

$$\downarrow \qquad \qquad \downarrow$$

$$K(\mathbb{Z}/2, d) \xrightarrow{\lambda} K(\mathbb{Z}/2, de)$$

where

$$\lambda^*(i_{de}) = i_d^e$$

and

$$i_{de} \in H^{de}(K(\mathbb{Z}/2, de)), \quad i_d \in H^d(K(\mathbb{Z}/2, d))$$

are the fundamental classes. The cohomology of  $E_m$  can be computed by [7; (2.1) and (2.2]. We get

$$H^*(E_m) = \frac{\mathbb{Z}/2[j,\ldots]}{(j^e,\ldots)} \otimes \text{Poly}$$

where:  $j := \pi^* i_d$ , and Poly is a certain polynomial algebra (see also [5]). Standard Hopf algebra considerations applied to the Eilenberg-Moore spectral sequence of the fibration

$$\Omega E_m \subseteq PE_m \downarrow E_m$$

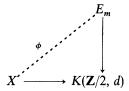
show that  $d_2 = 0$  (the usual argument that  $d_2$  of an indecomposable of minimal degree  $\neq 0$  implies  $d_2$  (there on) is primitive, and an inspection of primitives). Thus in the Eilenberg-Moore spectral sequence of the fibre square

$$\mathcal{L}(E_m) \qquad \begin{matrix} \Lambda E_m & \longrightarrow & E_m \\ \downarrow & & \downarrow \\ E_m & \longrightarrow & E_m \times E_n \end{matrix}$$

one sees  $d_2(\mathcal{L}(E_m)) = 0$  by noting that  $E_m$  is an H space and applying Lemma 3. Let

$$f: X \to K(\mathbb{Z}/2, d)$$

represent u, i.e.,  $f * i_d = u \in H^2(X)$ . Since  $u^e = 0$  there is a lift  $\phi$  in the indicated diagram



such that  $\phi^*j = u$ . The map  $\phi$  defines a map of fibre squares

$$\mathscr{L}(\phi) \colon \mathscr{L}(X) \to \mathscr{L}(E_m)$$

and one easily sees that

$$\mathscr{L}(\phi)^*(\gamma_{2s}(\tau j)) = \gamma_{2s}(\tau u).$$

Since  $d_2$  vanishes on  $\gamma_{2s}(\tau j)$  it follows that  $d_2$  vanishes on  $\gamma_{2s}(\tau u)$  by naturality. An induction is thus started. Assume now that we have shown  $d_r = 0$  for  $r = 2, \ldots, 2^k - 2$  and consider the inductive step. We replace the two stage Postnikov systems by the stable k + 1 stage Postnikov system

$$P( )$$

$$\downarrow$$

$$E_{m} = : P(1)$$

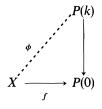
$$\downarrow$$

$$P(0) = K(\mathbb{Z}/2, d)$$

of [4; Theorem 4.2, p=2, s=1]. The essential features of this example for our purposes are as follows. Let  $j \in H^2(P(k))$  be the image of the fundamental class  $i_2 \in H^2(P(0))$ . Then:

- (i)  $j^e = 0$ .
- (ii) The first nonzero differential in the Eilenberg-Moore spectral sequence of the fibration  $\Omega P(k) \subseteq PP(k) \to P(k)$  defined on a class  $\gamma_{2s}(\tau j)$  is  $d_{2^{k+1}-1}$  (4; (6.4)].

Let  $f: X \to P(0) = K(\mathbb{Z}/2, d)$  represent u, i.e.,  $f^*(i_d) = u$ . Then in the diagram



we can find a lift  $\phi$  because the successive k-invariants used to construct the tower ( $\mathcal{P}$ ) all begin with a  $\beta = Sq^1$  [4; Theorem 5.2 (1)], and  $\beta \equiv 0$ ,  $H^*(X) \subseteq 0$ . By commutativity,  $\phi^*(j) = u$ . The map  $\phi$  induces a map of fibre squares

$$\mathcal{L}(\phi) \colon \mathcal{L}(X) \to \mathcal{L}(P(k))$$

and hence a map of Eilenberg-Moore spectral sequences. From property (i) we see that

$$\mathscr{L}(\phi)^*(\gamma_t(\tau j) = \gamma_t(\tau u), \quad t = 0, 1, \ldots$$

Since P(k) is an H-space, it follows from (ii) and Lemma 3 that  $d_r(\gamma_t(\tau j)) = 0$  for  $r = 2, ..., 2^{k+1} - 2$  and t = 0, 1, ... By naturality we conclude that

$$d_r(\mathcal{L}(X))(\gamma_t(\tau u)) = 0$$
 for  $r = 2, ..., 2^{k+1} - 2$ .

This completes the inductive step and hence the proof of Proposition 2.

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