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THE EILENBERG–MOORE SPECTRAL SEQUENCE AND THE MOD 2 COHOMOLOGY OF CERTAIN FREE LOOP SPACES

BY

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There has been considerable interest in computing the cohomology of the space $\Lambda(X)$ of free loops on X, at least since the theorem of Gromoll and Meyer, connecting the unboundedness of the Betti numbers $\{\dim H^i(\Lambda X; \mathbf{Q})\}$ with the existence of infinitely many closed geodesics on X when X is a Riemannian manifold. There have been however relatively few explicit calculations. The minimal model theory of Sullivan has been used in the rational case to obtain a few results. However with finite coefficients, aside from [12], which only contains Betti number estimates, there seems nothing known, apart from those facts that are easily knowable. In [9] we observed that the free loop space sits in a fibre square for any connected space X, where Δ is the diagonal map:

$$\mathscr{L}(X)\qquad \begin{array}{ccc} \Lambda(X) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

This observation makes available the Eilenberg–Moore spectral sequence (see for example [8]) as a tool for computing $H^*(\Lambda(X); k)$ for simply connected X. In [9] we dealt with the case where the coefficient field k was of characteristic zero. In this note we take up the case of $k = \mathbf{Z}/2$, and derive the following not so easily knowable result.

THEOREM. *Let X be a simply connected space, and suppose Sq^1 vanishes on $H^*(X; \mathbf{Z}/2)$ and*

$$(*)\qquad H^*(X; \mathbf{Z}/2) \simeq P[x_1, \ldots, x_n]/(x_1^{e_1}, \ldots, x_n^{e_n})$$

where $e_1 \cdots e_n$ is a power of 2, and $P[\ \ ]$ denotes a polynomial algebra. Then the Eilenberg–Moore spectral sequence

$$E_r \Rightarrow H^*(\Lambda(X); \mathbf{Z}/2), \quad E_2 = \mathrm{Tor}^{**}_{H^*(X;\mathbf{Z}/2)\otimes H^*(X;\mathbf{Z}/2)}(H^*(X; \mathbf{Z}/2), H^*(X; \mathbf{Z}/2))$$

collapses.

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An essential step in the proof is the explicit computation of E_2, along the lines of [9; 3.5] (see also Proposition 1 below). A particularly simple example of a space that satisfies (*) are the complex quadrics

$$Q_n := \{[z] \in \mathbf{C}P(n+1) \mid z_0^2 + \cdots + z_{n+1}^2 = 0\}$$

when $n+1$ is a power of 2. Combining the preceding theorem with the known mod 2 cohomology of the quadric and applying Proposition 1 below we have:

COROLLARY. *Let $n+1$ be a power of 2. Then there is a filtration on $H^*(\Lambda Q_n; \mathbf{Z}/2)$ such that*

$$E^\circ H^*(\Lambda Q_n\colon \mathbf{Z}/2) \simeq \frac{P[u, v]}{(u^{m+1}, v^2)} \otimes E[su, sv] \otimes \Gamma[\tau u, \tau v]$$

where $n = 2m+1$, $E[\ \ ]$ an exterior algebra, and $\Gamma[\ \ ]$ a divided power algebra and the degrees of the generators are as follows:

$$\begin{array}{lll} \deg u = 2, & \deg su = 1, & \deg \tau u = 2m, \\ \deg v = 2m+2, & \deg sv = 2m+1, & \deg \tau v = 4m. \end{array}$$

The classes u and v have filtration 0, su, sv filtration -1 and τu, τv filtration -2.

Other examples of spaces that satisfy (*) are $U(n)/SO(n)$, $Sp(n)/SO(n)$, etc.

I want to thank Frank Conolley for suggesting the problem of computing $H^*(\Lambda Q_n; \mathbf{Z}/2)$ as being a natural "test case" for extending the results of [9], for as he pointed out, $H^*(Q_n; \mathbf{Q})$ and $H^*(\mathbf{C}P(n); \mathbf{Q})$ are isomorphic when n is odd, so the rational results of [11] does not apply to deduce anything about closed geodesics on odd quadrics.

The proof of the theorem requires a number of preliminary manoeuvres. We begin by recalling some results from [9]. Let X be a connected topological space, and $\Lambda(X) := X^{S^1}$ the space of free loops on X. There is then the fibre square

$$\mathscr{L}(X) \qquad \begin{array}{ccc} \Lambda(X) & \longrightarrow & X \\ \Big\downarrow p & & \Big\downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

where Δ is the diagonal map, whose fibre is the ordinary loop space $\Omega(X)$ of X. Thus for simply connected X and coefficients in a field k we obtain an Eilenberg–Moore spectral sequence [8], [9]

$$E_r \Rightarrow H^*(\Lambda(X); k), \quad E_2 = \operatorname{Tor}_{H_*(X;k) \otimes H^*(X;k)} (H^*(X; k), H^*(X; k)).$$

For notational simplicity it will be convenient to set

$$T^{*,*}(A^*) := \mathrm{Tor}^{*,*}_{A^*\otimes A^*}(A^*, A^*)$$

for any graded connected algebra A^*.

Convention. For the remainder of this note all cohomology will be taken with $\mathbf{Z}/2$ coefficients, and we write $H^*(\ \ )$ for $H^*(\ \ ; \mathbf{Z}/2)$.

Using the mod 2 analog of [9; Section 3] one can easily prove:

Proposition 1. *Let*

$$A^* \simeq P[x_1, \ldots, x_n]/(x_1^{e_1}, \ldots, x_n^{e_n})$$

where $e := e_1 \cdots e_n$ *is a power of* 2. *Then*

$$\begin{aligned} T^{**}(A^*) &\simeq A^* \otimes \mathrm{Tor}^{**}_{A^*}(\mathbf{Z}/2, \mathbf{Z}/2) \\ &\simeq \frac{P[x_1, \ldots, x_n]}{(x_1^{e_1}, \ldots, x_n^{e_n})} \otimes E[su_1, \ldots, su_n] \otimes \Gamma[\tau u_1, \ldots, \tau u_n] \end{aligned}$$

where $E[\ \ ]$ *is an exterior algebra,* $\Gamma[\ \ ]$ *a divided power algebra and*

$$\deg su_i = (-1, \deg u_i),\ i = 1, \ldots, n; \quad \deg \tau u_i = (-2, e_i),\ i = 1, \ldots, n$$

Proof. For the sake of completeness we sketch a proof based on Hopf algebra considerations. Since e is a power of 2 we may impose a Hopf algebra structure on A^* by declaring the generators to be primitive elements. Then

$$\mathbf{Z}/2 \to A^* \xrightarrow{\Delta} A^* \otimes A^* \xrightarrow{\mu} A^* \to \mathbf{Z}/2$$

is a coexact sequence of algebras, where μ is multiplication and Δ the diagonal map. Moreover, $A^* \otimes A^*$ is free over A^* [6; 4.4] so the change of rings spectral sequence [1; XVI.6.1. (1a)] may be applied. There being no problem with local coefficients we conclude

$$E_r \Rightarrow \mathrm{Tor}_{A^*\otimes A^*}(A^*, A^*),$$

$$E_2^{p,q} \simeq A^* \otimes \mathrm{Tor}^p_{A^*}(\mathbf{Z}/2, A^*) \otimes \mathrm{Tor}^q_{A^*}(\mathbf{Z}/2, \mathbf{Z}/2)$$

whence the spectral sequence collapses to the isomorphism

$$\mathrm{Tor}_{A^*\otimes A^*}(A^*, A^*) \simeq A^* \otimes \mathrm{Tor}_{A^*}(\mathbf{Z}/2, \mathbf{Z}/2).$$

The computation of $\mathrm{Tor}_{A^*}(\mathbf{Z}/2, \mathbf{Z}/2)$ is routine. ∎

Proposition 1 gives us the structure of the E_2 term of the Eilenberg–Moore spectral sequence of the fibre square $\mathscr{L}(X)$ when X satisfies (*).

THEOREM 2. *Let X be a simply connected space such that*

(1) $Sq^1 : H^*(X; \mathbf{Z}/2) \to H^*(X; \mathbf{Z}/2)$ *vanishes*

and

(2) $H^*(X; \mathbf{Z}/2) \simeq P[x_1 \cdots x_n]/(x_1^{e_1}, \ldots, x_n^{e_n})$
where $e_1 \cdots e_n$ *is a power of* 2. *Then the Eilenberg–Moore spectral sequence*

$$E_r \Rightarrow H^*(\Lambda(X)), \quad E_2 = T^{**}(H^*(X))$$

for $\mathscr{L}(X)$ *collapses.*

The proof of Theorem 2 proceeds by comparing $\{E_r(\mathscr{L}(X)), d_r(\mathscr{L}(X))\}$ to the corresponding spectral sequence for certain universal examples E. The universal examples are H-spaces, so the following lemma allows us to reduce the study of $\{E_r(\mathscr{L}(E), d_r(\mathscr{L}(E))\}$ to more familiar Eilenberg–Moore spectral sequence considerations.

LEMMA 3. *Let* X *be an* H*-space. Then* $\Lambda(X)$ *is homotopy equivalent to* $X \times \Omega(X)$. *Moreover, for simply connected* X *the Eilenberg–Moore spectral sequence*

$$\{E_r(\mathscr{L}(X), d_r(\mathscr{L}(X))\}$$

and

$$\{H^*(X) \otimes E_r(X), 1 \otimes d_r(X)\}$$

are isomorphic, where $\{E_r(X), d_r(X)\}$ *is the Eilenberg–Moore spectral sequence of the path-loop fibration*

$$\Omega X \hookrightarrow PX \downarrow X.$$

Proof. Since X is an H-space, so is $\Lambda X = X^{S^1}$. Moreover, seen with this H-space structure, the evaluation map $e\colon \Lambda(X) \downarrow X$ becomes an H-map. Thus

$$\Omega X \hookrightarrow \Lambda X \downarrow^e X$$

becomes a principal bundle with

$$s\colon X \to \Lambda X\colon s(x) = \text{constant loop at } x$$

as cross-section. Hence the multiplication gives a map

$$X \times \Omega X \to \Lambda X$$

which is a homotopy equivalence. Naturality and diagram chasing yields the rest. ∎

Proof of Proposition 2. We proceed by induction to show that $d_r = 0$. The structure of $E_2(\mathscr{L}(X))$ is given in Proposition 1. As an algebra we see that E_2 is generated by classes

$$u_1, \ldots \text{ of filtration zero}, \quad s^{-1}u_1, \ldots \text{ of filtration } -1,$$

$$\gamma_{2^s}(\tau u_1), \ldots \text{ of filtration } 2^{s+1}, \quad s = 0, 1, \ldots.$$

If there is a nonvanishing differential, then it must take a nonzero value on some indecomposable element, and so it suffices from filtration considerations to show that d_r vanishes $\gamma_{2^s}(\tau u_1)$ for all $s \geq 0$, and all $r \geq 2$. To simplify notations we drop subscripts, setting $u = u_i$, $d := \deg u_i$, $e = e_i$, etc., and consider $\gamma_{2^s}(\tau u)$. Let E_m be the stable two stage Postnikov system defined by the fibre square

$$\begin{array}{ccc} E_m & \longrightarrow & L(\mathbf{Z}/2, de) \\ \downarrow & & \downarrow \\ K(\mathbf{Z}/2, d) & \xrightarrow{\lambda} & K(\mathbf{Z}/2, de) \end{array}$$

where

$$\lambda^*(i_{de}) = i_d^e$$

and

$$i_{de} \in H^{de}(K(\mathbf{Z}/2, de)), \quad i_d \in H^d(K(\mathbf{Z}/2, d)$$

are the fundamental classes. The cohomology of E_m can be computed by [7; (2.1) and (2.2]. We get

$$H^*(E_m) = \frac{\mathbf{Z}/2[j, \ldots]}{(j^e, \ldots)} \otimes \text{Poly}$$

where: $j := \pi^* i_d$, and Poly is a certain polynomial algebra (see also [5]). Standard Hopf algebra considerations applied to the Eilenberg–Moore spectral sequence of the fibration

$$\Omega E_m \hookrightarrow PE_m \downarrow E_m$$

show that $d_2 = 0$ (the usual argument that d_2 of an indecomposable of minimal degree $\neq 0$ implies d_2 (there on) is primitive, and an inspection of primitives). Thus in the Eilenberg–Moore spectral sequence of the fibre square

$$\mathscr{L}(E_m) \quad \begin{array}{ccc} \Lambda E_m & \longrightarrow & E_m \\ \downarrow & & \downarrow \\ E_m & \longrightarrow & E_m \times E_m \end{array}$$

one sees $d_2(\mathscr{L}(E_m)) = 0$ by noting that E_m is an H space and applying Lemma 3. Let

$$f: X \to K(\mathbf{Z}/2, d)$$

represent u, i.e., $f^*i_d = u \in H^2(X)$. Since $u^e = 0$ there is a lift ϕ in the indicated diagram

$$\begin{tikzcd} & E_m \arrow[d] \\ X \arrow[ur, dotted, "\phi"] \arrow[r] & K(\mathbf{Z}/2, d) \end{tikzcd}$$

such that $\phi^*j = u$. The map ϕ defines a map of fibre squares

$$\mathscr{L}(\phi)\colon \mathscr{L}(X) \to \mathscr{L}(E_m)$$

and one easily sees that

$$\mathscr{L}(\phi)^*(\gamma_{2^s}(\tau j)) = \gamma_{2^s}(\tau u).$$

Since d_2 vanishes on $\gamma_{2^s}(\tau j)$ it follows that d_2 vanishes on $\gamma_{2^s}(\tau u)$ by naturality.

An induction is thus started. Assume now that we have shown $d_r = 0$ for $r = 2, \ldots, 2^k - 2$ and consider the inductive step. We replace the two stage Postnikov systems by the stable $k + 1$ stage Postnikov system

$$(\mathscr{P}) \qquad \begin{array}{c} P(\ \ ) \\ \downarrow \\ \vdots \\ \downarrow \\ E_m =: P(1) \\ \downarrow \\ P(0) = K(\mathbf{Z}/2, d) \end{array}$$

of [4; Theorem 4.2, $p = 2$, $s = 1$]. The essential features of this example for our purposes are as follows. Let $j \in H^2(P(k))$ be the image of the fundamental class $i_2 \in H^2(P(0))$. Then:

(i) $j^e = 0$.

(ii) The first nonzero differential in the Eilenberg–Moore spectral sequence of the fibration $\Omega P(k) \hookrightarrow PP(k) \to P(k)$ defined on a class $\gamma_{2^s}(\tau j)$ is $d_{2^{k+1}-1}$ (4; (6.4)].

Let $f\colon X \to P(0) = K(\mathbf{Z}/2, d)$ represent u, i.e., $f^*(i_d) = u$. Then in the diagram

$$\begin{tikzcd} & P(k) \arrow[d] \\ X \arrow[ur, dotted, "\phi"] \arrow[r, "f"'] & P(0) \end{tikzcd}$$

we can find a lift ϕ because the successive k-invariants used to construct the tower ($\mathscr{P}$) all begin with a $\beta = Sq^1$ [4; Theorem 5.2 (1)], and $\beta \equiv 0$, $H^*(X) \circlearrowleft$. By commutativity, $\phi^*(j) = u$. The map ϕ induces a map of fibre squares

$$\mathscr{L}(\phi)\colon \mathscr{L}(X) \to \mathscr{L}(P(k))$$

and hence a map of Eilenberg–Moore spectral sequences. From property (i) we see that

$$\mathscr{L}(\phi)^*(\gamma_t(\tau j)) = \gamma_t(\tau u), \quad t = 0, 1, \ldots.$$

Since $P(k)$ is an H-space, it follows from (ii) and Lemma 3 that $d_r(\gamma_t(\tau j)) = 0$ for $r = 2, \ldots, 2^{k+1} - 2$ and $t = 0, 1, \ldots$. By naturality we conclude that

$$d_r(\mathscr{L}(X))(\gamma_t(\tau u)) = 0 \quad \text{for } r = 2, \ldots, 2^{k+1} - 2.$$

This completes the inductive step and hence the proof of Proposition 2. ∎

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