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# THE DISTRIBUTION OF VALUES OF AN INNER FUNCTION

BY

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The purpose of this note is to show that there is a theory of the distribution of values for an inner function that is analogous to some parts of the value distribution theory for meromorphic functions.

**1.** A bounded holomorphic in the unit disc $U$ whose radial limits have modulus 1 almost everywhere is called an inner function, see [7], Chapter 17, for details on the structure of inner functions. If $\phi$ is an inner function and $\alpha \in U$, then $\phi_\alpha$ will denote the inner function $(\phi - \alpha)/(1 - \bar{\alpha}\phi)$. We let $n(r, \alpha)$ denote the number of zeros of $\phi_\alpha$ whose moduli are at most $r$ and define

$$v(r, \alpha) = \int_r^1 \frac{n(t, \alpha)}{t}\, dt.$$

Following the notation of O. Frostman [4], we let $\delta(\alpha)$ be the total mass of the singular measure, $\sigma_\alpha$, associated to the inner function $\phi_\alpha$, and

$$L(r, \alpha) = -\frac{1}{2\pi} \int_0^{2\pi} \log |\phi_\alpha(re^{i\theta})|\, d\theta.$$

The quantity

$$\frac{1}{2\pi} \int_0^{2\pi} (1 - |\phi(re^{i\theta})|^2)\, d\theta$$

will be denoted by $\Delta(r)$. It is a simple consequence of Jensen's formula that

$$L(r, \alpha) = v(r, \alpha) + \delta(\alpha).$$

We may say that $v(r, \alpha)$ is a measure of the number of zeros of $\phi_\alpha$ in

$$\{z: r < |z| < 1\}.$$

Since $\phi_\alpha$ has a radial limit equal to 0 almost everywhere with respect to $\sigma_\alpha$, we may say that $\delta(\alpha)$ measures the number of zeros of $\phi_\alpha$ on the unit circle. In other words $L(r, \alpha)$ measures the number of zeros of $\phi_\alpha$ in

$$\{z: r < |z| \leq 1\}.$$

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Next we recall some notions from potential theory. If $\mu$ is a positive Borel measure of total mass 1 in $U$ then define

$$\hat{\mu}(z) = \int \log \left| \frac{1 - \bar{\xi} z}{\xi - z} \right| d\mu(\xi) \quad \text{and} \quad V_\mu = \sup_{z \in U} \hat{\mu}(z).$$

If $K \subseteq U$ is compact we let $V_K = \inf \{V_\mu \colon \operatorname{supp} \mu \subseteq K\}$, and if $E \subseteq U$ we let $V_E = \inf \{V_K \colon K \subseteq E,\ K \text{ compact}\}$. The inner capacity of $E$ is defined to be $\gamma(E) = e^{-V_E}$. The set function $\gamma$ is monotone and we have the subadditivity property: if $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\frac{1}{V_E} \leq \sum_{n=1}^{\infty} \frac{1}{V_{E_n}}.$$

We refer to [8], Chapter III, for details.

In this note it is shown that the distribution of values of $\phi$ is determined by the quantity $\Delta(r)$ in the following sense:

$$\text{(i)} \qquad 0 < \varliminf_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} \quad \text{for all } \alpha \in U,$$

and

$$\text{(ii)} \qquad \varlimsup_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} < \infty \quad \text{for all } \alpha \in U,$$

with the exception of a set of capacity 0.

We may say that the exceptional values of $\phi$ are the ones that are taken too often. This can happen in two ways, either $\delta(\alpha) \neq 0$, or $\delta(\alpha) = 0$ and

$$\varlimsup_{r \to 1} \frac{v(r, \alpha)}{\Delta(r)} = \infty.$$

We show by example that the second possibility can occur. Since

$$\varliminf_{r \to 1} \Delta(r) = 0,$$

one consequence of having

$$\varlimsup \frac{L(r, \alpha)}{\Delta(r)} < \infty$$

with the exception of a set of capacity 0, is that $\delta(\alpha) = 0$ with the exception of a set of capacity 0. This is, of course, the well known theorem of Frostman [4].

It is probably not true that, for every inner function $\phi$, we have

$$\varlimsup_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)} < \infty,$$

with the exception of a set of capacity 0, but we can find no counterexample. We *can* show that our results are close to sharp, in that

$$\lim_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} = 0$$

with the exception of a set of capacity 0, if $\lambda$ is any positive decreasing function on (0, 1) such that

$$\int_0^1 \frac{dt}{t\lambda(t)} < \infty.$$

Finally we show that this result can be somewhat improved if we allow a slightly larger exceptional set. To do this we need to develop a relation between capacity and some Hausdorff-like set functions that may not have been observed before.

**2.** The notion that $L(r, \alpha)$ is in some way dominated from above by $\Delta(r)$ is suggested by the proof of Theorem 4.3 of [3]. The proof of part (ii) of the following theorem is a modification of that proof.

THEOREM 1.

(i) $$\frac{L(r, \alpha)}{\Delta(r)} \geq \frac{1 - |\alpha|}{4} \quad \textit{for all } \alpha \in U \textit{ and all } r,\ 0 < r < 1.$$

(ii) *If* $0 < \rho < 1$, *and if* $\mu$ *is a distribution of the unit mass on* $\{z\colon |z| \leq \rho\}$ *then*

$$\int L(r, \alpha)\, d\mu(\alpha) \leq \frac{V_\mu}{\rho(1-\rho)} \Delta(r).$$

*Proof.* To prove (i) we start with the inequality $1 - x \leq -\log x$, valid for $0 < x < 1$. We obtain

$$1 - |\phi_\alpha(re^{i\theta})|^2 \leq -2 \log |\phi_\alpha(re^{i\theta})|.$$

We also have the following identity (see [6], for example):

$$1 - |\phi_\alpha(re^{i\theta})|^2 = \frac{1 - |\alpha|^2}{|1 - \bar{\alpha}\phi(re^{i\theta})|^2} (1 - |\phi(re^{i\theta})|^2).$$

We may conclude that

$$\begin{aligned}1 - |\phi(re^{i\theta})|^2 &\leq \frac{(1+|\alpha|)^2}{1-|\alpha|^2}(1-|\phi_\alpha(re^{i\phi})|^2)\\ &\leq \frac{2}{1-|\alpha|}(-2\log|\phi_\alpha(re^{i\theta})|)\\ &= \frac{-4}{1-|\alpha|}\log|\phi_\alpha(re^{i\theta})|.\end{aligned}$$

Integrating on $\theta$, we get $\Delta(r) \leq \frac{4}{1-|\alpha|} L(r, \alpha)$, which gives us (i).

To prove (ii) we use Fubini's theorem to see that

$$\int L(r, \alpha)\, d\mu(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \hat{\mu}(\phi(re^{i\theta}))\, d\theta.$$

Next we write

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{\mu}(\phi(re^{i\theta}))\, d\theta = \frac{1}{2\pi}\int_{E_r} \hat{\mu}(\phi(re^{i\theta}))\, d\theta + \frac{1}{2\pi}\int_{E_r'} \hat{\mu}(\phi(re^{i\theta}))\, d\phi,$$

where

$$E_r = \{\theta: 0 \leq \theta \leq 2\pi, |\phi(re^{i\theta})| \leq \rho\}$$

and

$$E_r' = \{\theta: 0 \leq \theta \leq 2\pi, |\phi(re^{i\theta})| > \rho\}.$$

From the definition of $V_\mu$ we see that

$$\frac{1}{2\pi}\int_{E_r} \hat{\mu}(\phi(re^{i\theta}))\, d\theta \leq V_\mu \frac{1}{2\pi}\int_{E_r} d\theta.$$

To deal with the integral over $E_r'$ we note that $\hat{\mu}$ is harmonic in

$$\{z: \rho < |z| < 1/\rho\}$$

and $\hat{\mu}(z) = 0$ if $|z| = 1$. It follows that the function

$$\hat{\mu}(z) - V_\mu \frac{\log|z|}{\log \rho}$$

is harmonic in the annulus $A = \{z: \rho < |z| < 1\}$ and has a non-positive upper limit at each point of the boundary of $A$. We conclude from the maximum principle that

$$\hat{\mu}(z) \leq V_\mu \frac{\log|z|}{\log \rho} \quad \text{if } z \in A.$$

In particular, if $\theta \in E_r'$ then

$$\hat{\mu}(\phi(re^{i\theta})) \leq V_\mu \frac{\log |\phi(re^{i\theta})|}{\log \rho}.$$

Using the inequalities $1 - x \leq -\log x \leq (1-x)/x$, valid if $0 < x < 1$, we see that

$$\frac{\log |\phi(re^{i\phi})|}{\log \rho} = \frac{-\log |\phi(re^{i\theta})|}{-\log \rho} \leq \frac{1 - |\phi(re^{i\theta})|}{(1-\rho)|\phi(re^{i\theta})|} \leq \frac{1 - |\phi(re^{i\theta})|^2}{(1-\rho)\rho}.$$

So we see that if $\theta \in E_r'$, then

$$\hat{\mu}(\phi(re^{i\theta})) \leq \frac{V_\mu}{\rho(1-\rho)} (1 - |\phi(re^{i\theta})|^2.$$

We can conclude that

$$\begin{aligned}
\int L(r, \alpha)\, d\mu(\alpha) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{\mu}(\phi(re^{i\theta}))\, d\theta \\
&\leq V_\mu \frac{1}{2\pi} \int_{E_r} d\theta + \frac{V_\mu}{\rho(1-\rho)} \frac{1}{2\pi} \int_{E_r'} (1 - |\phi(re^{i\theta})|^2)\, d\theta \\
&= \frac{V_\mu}{\rho(1-\rho)} \left[\frac{\rho}{2\pi} \int_{E_r} (1-\rho)\, d\theta + \frac{1}{2\pi} \int_{E_r'} (1 - |\phi(re^{i\theta})|^2)\, d\theta\right] \\
&\leq \frac{V_\mu}{\rho(1-\rho)} \left[\frac{1}{2\pi} \int_{E_r} (1-\rho^2)\, d\theta + \frac{1}{2\pi} \int_{E_r'} (1 - |\phi(re^{i\theta})|^2)\, d\theta\right] \\
&\leq \frac{V_\mu}{\rho(1-\rho)} \frac{1}{2\pi} \int_0^{2\pi} (1 - |\phi(re^{i\theta})|^2\, d\theta \\
&= \frac{V_\mu}{\rho(1-\rho)} \Delta(r),
\end{aligned}$$

since $1 - \rho^2 \leq 1 - |\phi(re^{i\theta})|^2$ if $\theta \in E_r$. This completes the proof.

COROLLARY.

(i) $$\varliminf_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} > 0 \quad \textit{for all } \alpha \in U.$$

(ii) $$\varliminf_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} < \infty$$

*with the exception of a set of capacity* 0.

*Proof.* Part (i) is clear. Part (ii) follows from a well known argument. Since the union of a countable number of sets of capacity 0 has capacity 0 it is enough to show that for each $\rho$, $0 < \rho < 1$,

$$E = \left\{\alpha : |\alpha| \leq \rho \quad \text{and} \quad \varlimsup_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)} = \infty\right\}$$

has capacity 0. If $\mu$ is a distribution of the unit mass with support in $E$, then by (ii) of the theorem we have

$$\int \frac{L(r, \alpha)}{\Delta(r)}\, d\mu(\alpha) \leq \frac{V_\mu}{\rho(1-\rho)}.$$

We conclude from Fatou's lemma that

$$\infty = \int \varliminf_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)}\, d\mu(\alpha) \leq \frac{V_\mu}{\rho(1-\rho)}$$

and hence that $V_\mu = \infty$. This implies that $\gamma(E) = 0$.

If we let

$$E(\phi) = \{\alpha \in U : \delta(\alpha) \neq 0\}$$

and

$$E_1(\phi) = \left\{\alpha \in U : \varlimsup_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)} = \infty\right\},$$

then of course $E(\phi) \subseteq E_1(\phi)$ and so Theorem 1 (ii) may be regarded as a generalization of Frostman's Theorem [4]. To show that it is a true generalization we give an example to show that $E(\phi)$ and $E_1(\phi)$ are not always the same.

THEOREM 2. *Suppose that $B$ is a Blaschke product whose zeros lie on* (0, 1). *Then*

(i) $E(B) = \emptyset$, *and*
(ii) $\Delta(r) = O(\sqrt{1 - r})$.

*Proof.* We assume $B$ has infinitely many zeros. If $\alpha \in U$, $\alpha \neq 0$, and $B_\alpha$ were not a Blaschke product then $B_\alpha$ would have a radial limit equal to 0 somewhere. That is to say that $B$ would have radial limit equal to $\alpha$ at $e^{i\theta}$ for some $\theta$, $0 \leq \theta \leq 2\pi$. If $e^{i\theta} \neq 1$, then $B$ has a radial limit of modulus 1 at $e^{i\theta}$. If $B$ has a radial limit at 1, that limit must be 0 since $B$ has infinitely many zeros on (0, 1). This proves part (i). Part (ii) is proved by Carleson in [3], page 48, see also [1], Theorem 7, with $\beta = 1$.

Now, to get an example of a Blaschke product $B$ with $E(B) \neq E_1(B)$, let $B$ have the zeros $a_k = 1 - k^{-\alpha}$, $1 < \alpha < 2$. By Theorem 2, (i), $E(B) = \phi$. It is easy to calculate that

$$L(r, 0) = \int_r^1 \frac{n(t, 0)}{t}\, dt > \varepsilon(1 - r)^{(\alpha - 1)/\alpha},$$

for some $\varepsilon > 0$, and hence that

$$\frac{L(r, 0)}{\Delta(r)} \geq \delta(1 - r)^{(\alpha - 2)/2\alpha},$$

for some $\delta > 0$. It follows that $0 \in E_1(B)$.

**3.** Next we want to exploit the inequality in Theorem 1 (ii) to get some information about

$$\overline{\lim_{r \to 1}} \frac{L(r, \alpha)}{\Delta(r)}.$$

The method we use is analogous to one used in the theory of meromorphic functions by J. E. Littlewood [5], and refined by L. Ahlfors [2].

THEOREM 3. *Suppose $\lambda$ is a positive decreasing function on* (0, 1) *such that*

$$\int_0^1 \frac{dt}{t\lambda(t)} < \infty.$$

*Then for any inner function $\phi$, we have*

$$\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} = 0,$$

*with the exception of a set of capacity* 0.

*Proof.* Given $\rho$, $0 < \rho < 1$, it is enough to show that

$$\left\{\alpha\colon |\alpha| \leq \rho,\ \overline{\lim_{r \to 1}} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} > 0\right\}$$

has capacity 0. To show this it is enough to show that

$$\left\{\alpha\colon |\alpha| \leq \rho,\ \overline{\lim_{r \to 1}} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} > 2\right\}$$

has capacity 0, because once this is done we may replace $\lambda$ by $\varepsilon\lambda$, $\varepsilon > 0$. This being said, for all sufficiently large $n$ we may choose $r_n$ such that $\Delta(r_n) = 2^{-n}$ and let

$$E_n = \left\{\alpha\colon |\alpha| \leq \rho,\ \frac{L(r_n, \alpha)}{\Delta(r_n)} \geq \lambda(2^{-n})\right\}.$$

If $\mu$ is a distribution of the unit mass with support in $E_n$ we see that

$$\lambda(2^{-n}) = \int \lambda(2^{-n})\, d\mu \leq \int \frac{L(r_n, \alpha)}{\Delta(r_n)}\, d\mu(\alpha) \leq \frac{V_\mu}{\rho(1-\rho)}.$$

It follows that $V_n = V_{E_n} \geq \lambda(2^{-n})\rho(1-\rho)$ and hence that

$$\begin{aligned} \frac{1}{V_{n+1}} &\leq \frac{1}{\rho(1-\rho)} \frac{1}{\lambda(2^{-(n+1)})} \\ &= \frac{1}{\rho(1-\rho)\log 2} \cdot \frac{1}{\lambda(2^{-(n+1)})} \int_{2^{-(n+1)}}^{2^{-n}} \frac{1}{t}\, dt \\ &\leq \frac{1}{\rho(1-\rho)\log 2} \int_{2^{-(n+1)}}^{2^{-n}} \frac{1}{t\lambda(t)}\, dt, \end{aligned}$$

since $\lambda$ is decreasing. It now follows from the hypothesis on $\lambda$ that

$$\sum_{n=1}^{\infty} \frac{1}{V_n} < \infty.$$

And from this it follows that $E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ has capacity 0. Now if $\alpha \notin E$, $|\alpha| \leq \rho$, then $\alpha \notin \bigcup_{k=n}^{\infty} E_k$ for some $n$, and hence

$$\frac{L(r_k, \alpha)}{\Delta(r_k)} \leq \lambda(2^{-k})$$

for all $k \geq n$. Now fix $k \geq n$ and take $r$, $r_k \leq r \leq r_{k+1}$; then

$$\frac{L(r, \alpha)}{\Delta(r)} \leq \frac{L(r_k, \alpha)}{\Delta(r_{k+1})} = \frac{2L(r_k, \alpha)}{\Delta(r_k)} \leq 2\lambda(2^{-k}) = 2\lambda(\Delta(r_k)) \leq 2\lambda(\Delta(r)).$$

In other words, if $|\alpha| \leq \rho$, $\alpha \notin E$ then there is an $n$ such that

$$\frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} \leq 2 \quad \text{for } r \geq r_n.$$

In particular

$$\overline{\lim_{r \to 1}} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} \leq 2.$$

*Remark.* We note that the function

$$\lambda(t) = \left(\log \frac{1}{t}\right)^{\alpha}$$

satisfies the hypotheses of Theorem 3, for $\alpha > 1$.

**4.** Next we show that if we allow a slightly larger exceptional set we get a somewhat better result. We recall some notions from the theory of Hausdorff measures. A positive increasing function, $h$, defined on $(0, \infty)$ such that $\lim_{r \to 0} h(r) = 0$ is called a measure function. If $\alpha \in \mathbf{C}$ and $r \geq 0$, $\Delta(a, r)$ will denote $\{z : |z - a| < r\}$. We have the set function

$$M_h(E) = \inf \{\Sigma h(r_k) : E \subseteq \cup \Delta(a_k, r_k)\}.$$

The set function $M_h$ is monotone and subadditive. For $\varepsilon > 0$ there is the set function

$$M_h^{\varepsilon}(E) = \inf \{\Sigma h(r_k) : E \subseteq \cup \Delta(a_k, r_k), r_k \leq \varepsilon\}$$

and finally,

$$\Lambda_h(E) = \lim_{\varepsilon \to 0} M_h^{\varepsilon}(E).$$

The set function $\Lambda_h$ is actually a measure on the Borel sets, also $M_h$ and $\Lambda_h$ have the same null sets. In [4], Frostman has shown that if the measure function $h$ satisfies

$$\int_0^1 \frac{h(t)}{t}\, dt < \infty$$

then for any Borel set $E$ such that $\gamma(E) = 0$ we must have $\Lambda_h(E) = 0$. This cannot be a consequence of an inequality involving $\gamma$ and $\Lambda_h$ because $\gamma$ is finite on bounded sets but $\Lambda_h$ is in general infinite. We will show that under some additional mild assumptions on $h$ there is a general inequality between $M_h$ and $\gamma$. We will assume that the measure function $h$ is continuous and that $h(r)/r^2$ is decreasing, and that

$$\int_0^1 \frac{h(t)}{t}\, dt < \infty.$$

We define

$$\bar{h}(\varepsilon) = \int_0^{\varepsilon} \frac{h(t)}{t}\, dt.$$

*Lemma.* (i) *There is a constant $C$ such that for every compact set $K \subseteq U$ we have $M_h(K) \leq C\bar{h}(\gamma(K))$.*

(ii) *If there is a constant $C_0$ such that*

$$h(t) \leq C_0\, h(\tfrac{1}{2}te^{-\bar{h}(t)/h(t)}) \tag{*}$$

*then $M_h(K) \leq C_1 h(\gamma(K))$ for some constant $C_1$, independent of $K$.*

*Remarks.* It follows from the assumption that $h(r)/r^2$ is decreasing that

$$h(cr) \leq c^2 h(r) \quad \text{for any } c \geq 1.$$

It then follows that (*) holds any time that $\bar{h}(t) \leq ch(t)$ for some constant $c$. This is the case for example if $h(r) = r^\alpha$, $0 < \alpha < 2$. If we take

$$h(r) = \left(\log \frac{1}{r}\right)^{-\alpha} \quad \text{with } \alpha > 1$$

then (*) is still true but it is no longer true that $\bar{h}(r) \leq ch(r)$ for some constant $c$. Condition (*) fails for

$$h(r) = \left[\log \frac{1}{r}\left(\log\log \frac{1}{r}\right)^{\alpha}\right]^{-1}, \quad \alpha > 1.$$

*Proof of lemma.* The proof is a modification of the proof of Frostman [4] and depends on his basic result that says that there is a constant $a > 0$, independent of $h$, such that if $K \subseteq U$ is compact, there is a positive Borel measure $\mu$ on $K$ such that $\mu(\Delta(z, r)) \leq h(r)$ for all $z \in \mathbf{C}$ and $r \geq 0$ and $\mu(K) \geq aM_h(K)$. We calculate

$$\hat{\mu}(z) = \int \log \left|\frac{1 - \bar{\xi}z}{\xi - z}\right| d\mu(\xi) \leq \int_0^R \log \frac{1}{r}\, d\Omega(r) + \mu(K) \log 2,$$

where $R$ is chosen so that $\Delta(z, R) \supseteq K$, and $\Omega(r) = \mu(\Delta(z, r))$. After integrating by parts we find that

$$\begin{aligned}
\hat{\mu}(z) &\leq \Omega(R) \log \frac{1}{R} + \int_0^R \frac{\Omega(r)}{r}\, dr + \mu(K) \log 2 \\
&\leq \mu(K) \log \frac{1}{R} + \int_0^\varepsilon \frac{h(r)}{r}\, dr + \mu(K)[\log R - \log \varepsilon] + \mu(K) \log 2 \\
&= \bar{h}(\varepsilon) + \mu(K) \log \frac{1}{\varepsilon} + \mu(K) \log 2,
\end{aligned}$$

for any $\varepsilon > 0$. Now the measure $\nu = \mu/\mu(K)$ is a distribution of the unit mass on $K$ and

$$\hat{\nu}(z) \leq \frac{\bar{h}(\varepsilon)}{\mu(K)} + \log \frac{1}{\varepsilon} + \log 2.$$

To prove (i) we just choose $\varepsilon = \bar{h}^{-1}(\mu(K))$ and get

$$\hat{v}(z) \leq C + \log \frac{1}{\bar{h}^{-1}(\mu(K))};$$

this means that

$$V_K \leq C + \log \frac{1}{\bar{h}^{-1}(\mu(K))},$$

so

$$\mu(K) \leq \bar{h}(e^c \gamma(K)) \leq e^{2c} \bar{h}(\gamma(K)).$$

Since $\mu(K) \geq a M_h(K)$, the proof of (i) is complete.

To prove (ii) we return to the inequality

$$\hat{v}(z) \leq \frac{\bar{h}(\varepsilon)}{\mu(K)} + \log \frac{1}{\varepsilon} + \log 2.$$

This time we let $\varepsilon = h^{-1}(\mu(K))$. Now we check that (*) yields

$$\frac{\bar{h}(h^{-1}(\mu(K)))}{\mu(K)} + \log \frac{1}{h^{-1}(\mu(K))} + \log 2 \leq \log \frac{1}{h^{-1}(\mu(K)/c)}.$$

We conclude that

$$V_K \leq -\log h^{-1}\left(\frac{\mu(K)}{c}\right),$$

and hence

$$M_h(K) \leq \frac{1}{a} \mu(K) \leq \frac{c}{a} h(\gamma(K)).$$

COROLLARY. *Let $\mathcal{O} \subseteq U$ be open. Then*

(i) $M_h(\mathcal{O}) \leq c\bar{h}(\gamma(\mathcal{O}))$,

*and*

(ii) *if* (*) *holds then* $M_h(\mathcal{O}) \leq ch(\gamma(\mathcal{O}))$.

*Proof.* From (i) of the lemma we conclude that

$$\sup \{M_h(K): K \subseteq \mathcal{O}, K \text{ is compact}\} \leq c\bar{h}(\gamma(\mathcal{O})).$$

But Carleson has shown [3] that

$$M_h(\mathcal{O}) \leq 24 \sup \{M_h(K): K \subseteq \mathcal{O}, K \text{ compact}\}.$$

This proves (i) of the corollary; (ii) is proved in the same way.

THEOREM 4. *Suppose $h$ is a measure function and $\lambda$ is a positive decreasing function on* (0, 1) *such that*

$$\int_0^1 \frac{\bar{h}(e^{-\lambda(t)})}{t}\, dt < \infty.$$

*Then there is a set $E \subseteq U$, such that $M_h(E) = 0$ and*

$$\overline{\lim_{r\to 1}} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} < \infty \quad \textit{for all } \alpha \notin E.$$

*Proof.* Since $M_h$ is subadditive it is enough to show that for each $\rho$, $0 < \rho < 1$,

$$\left\{\alpha\colon |\alpha| \le \rho, \lim_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r)} = \infty\right\}$$

is a null set for $M_h$. Fix such a $\rho$ and choose $r_n$ such that $\Delta(r_n) = 2^{-n}$ and let

$$\mathcal{O}_n = \left\{\alpha\colon |\alpha| < \rho, \frac{L(r_n, \alpha)}{\Delta(r_n)} > \frac{\lambda(\Delta(r_n))}{\rho(1-\rho)}\right\}.$$

Then $\mathcal{O}_n$ is an open set and if $\mu$ is any distribution of the unit mass with support in $\mathcal{O}_n$ we have from Theorem 1 (ii), $\lambda(\Delta(r_n)) \le V_\mu$ and hence

$$\lambda(\Delta(r_n)) \le V_{\mathcal{O}_n}.$$

From the corollary we see that

$$M_h(\mathcal{O}_n) \le c\bar{h}(\exp(-V_{\mathcal{O}_n})) \le c\bar{h}(\exp(-\lambda(\Delta(r_n)))).$$

We conclude as before that

$$M_h(\mathcal{O}_{n+1}) \le \frac{c}{\log 2} \int_{2^{-(n+1)}}^{2^{-n}} \bar{h}(e^{-\lambda(t)})\, \frac{dt}{t}$$

and hence that $\Sigma M_h(\mathcal{O}_n) < \infty$. Since $M_h$ is monotone and subadditive we see that $M_h(E) = 0$, where $E = \bigcap_k \bigcup_{n\ge k} \mathcal{O}_n$. As before we conclude that if $\alpha \notin E$, $|\alpha| \le \rho$, then

$$\overline{\lim} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} \le \frac{2}{\rho(1-\rho)}.$$

*Remark.* Of course if condition (*) of the lemma holds, then the hypothesis of Theorem 4 may be weakened to read

$$\int_0^1 \frac{h(e^{-\lambda(t)})}{t}\, dt < \infty.$$

COROLLARY. *Let $M_\beta$ be the set function associated to the measure function $h(r) = r^\beta$, $0 < \beta \le 2$. Suppose $\lambda$ is a positive decreasing function on* (0, 1) *such that*

$$\int_0^1 \frac{e^{-c\lambda(t)}}{t}\, dt < \infty \quad \textit{for some constant } C.$$

*Then there is a set $E \subseteq U$ such that $M_\beta(E) = 0$ for all $\beta$, $0 < \beta \le 2$, and*

$$\varlimsup_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} < \infty \quad \textit{if } \alpha \notin E.$$

(Note that $\lambda(t) = \log\log(1/t)$ will work.)

*Proof.* Fix $\beta$, $0 < \beta \le 2$, and let $\Lambda(t) = c\lambda(t)/\beta$. Then

$$\int_0^1 \frac{[e^{-\Lambda(t)}]^\beta}{t}\, dt < \infty$$

and by Theorem 4 there is a set $E_\beta$ with $M_\beta(E_\beta) = 0$ such that

$$\lim_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)\Lambda(\Delta(r))} < \infty \quad \text{for } \alpha \notin E_\beta.$$

Of course this means that

$$\varlimsup_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} < \infty \quad \text{for } \alpha \notin E_\beta.$$

Now choose $\beta_n \searrow 0$ and define

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k\ge n} E_{\beta_k}.$$

If $\alpha \notin E$ then $\alpha \notin \bigcup_{k\ge n} E_{\beta_k}$ for some $n$. In particular $\alpha \notin E_{\beta_n}$ and hence

$$\varlimsup_{r\to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} < \infty.$$

Fix $\beta$, $0 < \beta \le 2$; then $\beta > \beta_n$ for some $n$. Now

$$E \subseteq \bigcup_{k\ge n} E_{\beta_k}, \quad \text{so} \quad M_\beta(E) \le \sum_{k\ge n} M_\beta(E_{\beta_k}).$$

But clearly $M_\beta(E_{\beta_k}) \le M_{\beta_k}(E_{\beta_k}) = 0$ because $\beta > \beta_k$; that is, $M_\beta(E) = 0$.

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