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REMARKS ON RANGES OF CHARGES ON σ-FIELDS

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Summary

In this paper we present the following results about ranges of charges on a σ-field $\mathscr{A}$ of subsets of a set X.

(1) For any bounded charge the range is either a finite set or contains a perfect set, contrary to an assertion made by Sobczyk and Hammer [8].

(2) If $\mu_1, \mu_2, \ldots, \mu_n$ are strongly continuous bounded charges on $\mathscr{A}$ then the range of the vector measure $(\mu_1, \mu_2, \ldots, \mu_n)$ is a convex set and need not be closed.

(3) There is a positive bounded charge, on any infinite σ-field, whose range is neither Lebesgue measurable nor has the property of Baire.

1. Notation and definitions

Let $\mathscr{A}$ stand for a σ-field of subsets of a set X. A charge on $\mathscr{A}$ is a finitely additive measure on $\mathscr{A}$. If μ is a charge, μ^+, μ^-, and $|\mu|$ stand for the positive, negative and total variations of μ respectively. For any bounded charge μ, $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. A charge μ is said to be strongly continuous if for any $\varepsilon > 0$ there is a partition $\{A_1, A_2, \ldots, A_n\}$ of X of sets from $\mathscr{A}$ such that $|\mu|(A_i) < \varepsilon$ for all i. If A_n, $n \geq 1$, is a sequence of sets from $\mathscr{A}$ which are pairwise disjoint such that

$$\mu(B) = \sum_{n \geq 1} \mu(B \cap A_n) \quad \text{for all } B \subset \bigcup_{n \geq 1} A_n,\ B \in \mathscr{A}$$

then we say that μ is countably additive across A_n, $n \geq 1$. If $\mu_1, \mu_2, \ldots, \mu_n$ are charges on $\mathscr{A}$, $R(\mu_1, \mu_2, \ldots, \mu_n)$ denotes the range of the vector measure $(\mu_1, \mu_2, \ldots, \mu_n)$ namely $\{(\mu_1(A), \mu_2(A), \ldots, \mu_n(A)); A \in \mathscr{A}\}$.

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2. The range of a bounded charge

The following proposition characterises countable additivity across a sequence.

PROPOSITION. *If μ is a bounded charge on $(X, \mathscr{A})$ and if A_n, $n \geq 1$, is a sequence of pairwise disjoint sets from $\mathscr{A}$ then the following are equivalent.*

(i) *μ is countably additive across A_n, $n \geq 1$.*
(ii) *μ^+ is countably additive across A_n, $n \geq 1$, and μ^- is countably additive across A_n, $n \geq 1$.*
(iii) *$|\mu|$ is countably additive across A_n, $n \geq 1$.*
(iv) $|\mu|(\bigcup_{n\geq 1} A_n) = \sum_{n\geq 1} |\mu|(A_n)$
(v) *$|\mu|(\bigcup_{n\geq m} A_n)$, $m \geq 1$, converges to zero.*

The easy proof of this proposition is omitted. We now present the theorem of this section.

THEOREM 1. *If μ is a bounded charge on $(X, \mathscr{A})$ then $R(\mu)$ is either finite or contains a perfect set. More generally, either every point in $R(\mu)$ is isolated, in which case $R(\mu)$ is finite or every neighborhood of every point in $R(\mu)$ contains a perfect set.*

Proof. If $R(\mu)$ is not finite then clearly $R(|\mu|)$ is not finite. So by a result of Sobczyk and Hammer [8], there is a sequence A_n, $n \geq 1$, of pairwise disjoint sets from $\mathscr{A}$ such that $|\mu|$ is countably additive across A_n, $n \geq 1$, and $|\mu|(A_n) > 0$ for all $n \geq 1$. Let $n_1, n_2, \ldots,$ be an infinite sequence of indices and $B_{n_i} \subset A_{n_i}$, $i \geq 1$, be a sequence of sets from $\mathscr{A}$ such that $\mu(B_{n_i}) > 0$ for all i or $\mu(B_{n_i}) < 0$ for all i. Clearly such $n_1, n_2, \ldots, B_{n_1}, B_{n_2}, \ldots,$ exist. Now, observe that μ is countably additive across B_{n_i}, $i \geq 1$. So,

$$\left\{\sum_{i\in I} \mu(B_{n_i});\quad I \subset \{1, 2, 3, \ldots\}\right\} = \left\{\mu\left(\bigcup_{i\in I} B_{n_i}\right);\quad I \subset \{1, 2, 3, \ldots\}\right\}$$

is a perfect subset of $R(\mu)$. The proof of the rest of the assertion of the theorem easily follows.

Remark 1. Theorem 1 for positive charges was obtained by Sobczyk and and Hammer [8] and they claim that this result cannot be extended to general bounded charges. In fact, our Theorem 1 says that Theorem 3.4 of [8] is not correct.

3. The range of n strongly continuous bounded charges

Sobzyk and Hammer [7] and Maharam [5] have proved that if μ is a strongly continuous positive charge on $(X, \mathscr{A})$ then the range $R(\mu)$ is a closed interval. This result can be extended to finitely many strongly continuous bounded charges as follows.

THEOREM 2. *If* $\mu_1, \mu_2, \ldots, \mu_n$ *are strongly continuous bounded charges on* $(X, \mathscr{A})$ *then* $R(\mu_1, \mu_2, \ldots, \mu_n)$ *is a convex set.*

Proof. We shall imitate the proof of Halmos for the Liapounoff's theorem as presented by Jørsboe [4]. We shall only present a sketch of the proof.

We shall first prove the result for positive strongly continuous bounded charges $\mu_1, \mu_2, \ldots, \mu_n$. The proof is by induction. For the case $n = 1$ see [7] or [5]. Let us assume the result for $n = k$ and prove the result for $n = k + 1$. To show that $R(\mu_1, \mu_2, \ldots, \mu_{k+1})$ is convex it is clearly sufficient to show that $R(\tau_1, \tau_2, \ldots, \tau_{k+1})$ is convex where $\tau_i = \mu_i + \mu_{i+1} + \cdots + \mu_{k+1}$ for $1 \leq i \leq k + 1$. To show that $R(\tau_1, \tau_2, \ldots, \tau_{k+1})$ is convex it is sufficient to show that for any A in $\mathscr{A}$ there exists a set B in $\mathscr{A}$, $B \subset A$, such that $\tau_i(B) = \frac{1}{2}\tau_i(A)$ for $i = 1, 2, \ldots, k + 1$. Let $C \subset A$, $C \in \mathscr{A}$ be a set obtained by the induction hypothesis such that $\tau_i(C) = \frac{1}{2}\tau_i(A)$ for $i = 1, 2, \ldots, k$.

For any set D in $\mathscr{A}$ let $\{D_a\}_{a \in [0,1]}$ be an increasing family of sets in $\mathscr{A}$ such that $\tau_i(D_a) = a \cdot \tau_i(D)$ for $i = 1, 2, \ldots, k$ and $0 \leq a \leq 1$. By the induction hypothesis such a family exists. If we denote by $\{C_a\}_{a \in [0,1]}$ and $\{(A - C)_a\}_{a \in [0,1]}$ such families obtained for C and $A - C$ respectively then

$$\tau_{k+1}(C_a \cup (A - C)_{1-a})$$

is a continuous function of a, since

$$\tau_{k+1}(D_a - D_b) \leq \tau_k(D_a - D_b) \leq |a - b| \tau_k(D)$$

for any D in $\mathscr{A}$ and $0 \leq a, b \leq 1$. This function takes the value $\tau_{k+1}(C)$ at $a = 1$ and the value $\tau_{k+1}(A - C)$ at $a = 0$. Since $\frac{1}{2}\tau_{k+1}(A)$ lies between $\tau_{k+1}(C)$ and $\tau_{k+1}(A - C)$, there is an a_0 such that

$$\tau_{k+1}(C_{a0} \cup (A - C)_{1-a0}) = \tfrac{1}{2}\tau_{k+1}(A)$$

and of course

$$\tau_i(C_{a0} \cup (A - C)_{1-a0}) = a_0 \tau_i(C) + (1 - a_0)\tau_i(A - C) = \tfrac{1}{2}\tau_i(A)$$

for $1 \leq i \leq k$. Hence the result.

For general strongly continuous bounded charges $\mu_1, \mu_2, \ldots, \mu_n$ the convexity of $R(\mu_1, \mu_2, \ldots, \mu_n)$ follows from the convexity of

$$R(\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^-, \ldots, \mu_n^+, \mu_n^-).$$

In the above theorem we have proved only the convexity of the range

$$R(\mu_1, \mu_2, \ldots, \mu_n)$$

for strongly continuous charges. However, $R(\mu_1, \mu_2, \ldots, \mu_n)$ need not be closed in general. If μ is a positive strongly continuous bounded charge then $R(\mu)$ is a closed interval. Beyond this case nothing definite can be said about the closedness of the range. First we give necessary and sufficient conditions for $R(\mu)$ to be closed for a strongly continuous bounded charge μ.

THEOREM 3. *Consider the following conditions:*

(a) *$R(\mu)$ is closed;*
(b) $\mathrm{Sup}_{B \in \mathscr{A}}\ \mu(B) \in R(\mu)$;
(c) $\mathrm{Inf}_{B \in \mathscr{A}}\ \mu(B) \in R(\mu)$;
(d) *μ has a Hahn decomposition: i.e., there exists A_0 in $\mathscr{A}$ such that*

$$\mu^+(A_0) = 0 = \mu^-(X - A_0).$$

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) *for any bounded charge μ. If μ is further strongly continuous then* (d) $\Rightarrow$ (a).

Proof. That (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) is clear. To show (c) $\Rightarrow$ (d) simply observe that if A_0 is a set in $\mathscr{A}$ such that $\mu(A_0) = \mathrm{Inf}_{B \in \mathscr{A}}\ \mu(B)$ then

$$\mu^+(A_0) = 0 = \mu^-(X - A_0).$$

The implication (d) $\Rightarrow$ (c) follows from the observation that

$$\operatorname*{Inf}_{B \in \mathscr{A}}\ \mu(B) = -\mu^-(X) = \mu(A_0).$$

To prove (d) $\Rightarrow$ (a) for a strongly continuous bounded charge μ observe that $R(\mu)$ is an interval because of Theorem 2 and is a closed interval with end points $\mu(A_0)$ and $\mu(X - A_0)$ because of (d).

Now we shall give an example of a strongly continuous bounded charge μ such that $R(\mu)$ is not closed. By Theorem 3, it suffices to construct a strongly continuous bounded charge μ which does not admit a Hahn set. We shall also obtain incidentally two positive strongly continuous charges ν and τ such that $R(\nu, \tau)$ is not closed.

Example 1. Let $\mathscr{A}$ be any infinite σ-field of subsets of a set X. Let ν be any strongly continuous probability charge on $\mathscr{A}$. Such a ν exists by Corollary 4.3 of [1]. By Theorem 2, we obtain a tree

$$\{A_{\delta_1, \delta_2, \dots, \delta_n};\ \delta_1, \delta_2, \dots, \delta_n \text{ a finite sequence of 0's and 1's, } n \geq 1\}$$

having the following properties:

(i) $A_{\delta_1, \delta_2, \dots \delta_n, 0} \cap A_{\delta_1, \delta_2, \dots, \delta_n, 1} = \emptyset$ for any $n \geq 1$ and any finite sequence $\delta_1, \delta_2, \dots, \delta_n$ of 0's and 1's.

(ii) $A_{\delta_1, \delta_2, \dots, \delta, 0} \cup A_{\delta_1, \delta_2, \dots, \delta_n, 1} = A_{\delta_1, \delta_2, \dots, \delta_n}$ for any $n \geq 1$ and any finite sequence $\delta_1, \delta_2, \dots, \delta_n$ of 0's and 1's.

(iii) $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 = X$.

(iv) $\nu(A_{\delta_1, \delta_2, \dots, \delta_n}) = \alpha_1(\delta_1) \cdot \alpha_2(\delta_2), \dots, \alpha_n(\delta_n)$ for any $n \geq 1$ and any sequence $\delta_1, \delta_2, \dots, \delta_n$ of 0's and 1's, where $\alpha_n(0) = 1/n + 1$ and $\alpha_n(1) = n/(n + 1)$ for $n \geq 1$.

Let $\mathcal{N} = [A \in \mathcal{A}: \nu(A) = 0\}$. Look at the quotient Boolean algebra $\mathcal{A}/\mathcal{N}$. For A in $\mathcal{A}$, $[A]$ denotes the equivalence class in $\mathcal{A}/\mathcal{N}$ containing A. Note that $\{[A_{\delta_1, \delta_2, \ldots, \delta_n}]; \delta_1, \delta_2, \ldots, \delta_n$ is a sequence of 0's and 1's and $n \geq 1\}$ is a tree in $\mathcal{A}/\mathcal{N}$. As in the proof of (ii)$\Rightarrow$(iii) of Theorem 4.1 of [1], one can construct a strongly continuous probability charge $\tilde{\tau}$ on $\mathcal{A}/\mathcal{N}$ such that

$$\tilde{\tau}([A_{\delta_1, \delta_2, \ldots, \delta_n}]) = \alpha_1(1 - \delta_1)\alpha_2(1 - \delta_2) \ldots, \alpha_n(1 - \delta_n)$$

for any finite sequence $\delta_1, \delta_2, \ldots, \delta_n$ of 0's and 1's, and any $n \geq 1$. Now we define τ on $\mathcal{A}$ by $\tau(A) = \tilde{\tau}([A])$ for A in $\mathcal{A}$. τ is a strongly continuous probability charge on $\mathcal{A}$ because for any finite sequence $\delta_1, \delta_2, \ldots, \delta_n$ of 0's and 1's,

$$\tau(A_{\delta_1, \delta_2, \ldots, \delta_n}) \leq 1/n$$

for every $n \geq 1$. Observe that $\nu \wedge \tau = 0$ because for any $n \geq 1$,

$$\bigcup A_{\delta_1, \delta_2, \ldots, \delta_{n,0}} \quad \text{and} \quad \bigcup A_{\delta_1, \delta_2, \ldots, \delta_{n,1}},$$

where both the unions are taken over all $\delta_1, \delta_2, \ldots, \delta_n$ in $\{0, 1\}$, are disjoint with union equal to X; the ν-value of the first set and the τ-value of the second set are each equal to $1/(n + 2)$. τ also has the property that if $A \in \mathcal{A}$ and $\nu(A) = 0$, then $\tau(A) = 0$.

Now define μ on $\mathcal{A}$ by $\mu = \nu - \tau$. Since $\nu \wedge \tau = 0$, $\mu^+ = \nu$ and $\mu^- = \tau$. It is clear that ν and τ are distinct and μ is a strongly continuous bounded charge on $\mathcal{A}$. We note that μ does not admit a Hahn set. If μ admits a Hahn set A in $\mathcal{A}$, then $\mu^+(X - A) = \nu(X - A) = 0$. So, $\tau(X - A) = 0$. Also $\mu^-(A) = \tau(A) = 0$. This implies that $\tau(X) = 0$ which is a contradiction. Hence $R(\mu)$ is not a closed interval.

Also $R(\nu, \tau)$ is not a closed set because $(1, 0) \notin R(\nu, \tau)$ but belongs to its closure.

4. Charges whose ranges are not Lebesgue measurable

One of the interesting problems about the range is to determine if the range of every bounded charge is a Borel set. See [5] for some related remarks. In this section we present an example of a charge μ whose range is not even Lebesgue measurable. In view of the Sobczyk–Hammer decomposition Theorem (see [7] and [1]) and since the range of any strongly continuous bounded charge is an interval, it is only expected that our example is a sum of two valued charges. We need some definitions.

DEFINITION. A sequence μ_n, $n \geq 1$, of 0-1 valued charges is said to be a *discrete* sequence if for every n there exists A in $\mathcal{A}$ such that $\mu_n(A) = 1$ and $\mu_m(A) = 0$ if $m \neq n$. A 0-1 valued charge μ_0 is said to be an *accumulation point* of a sequence μ_n, $n \geq 1$, of 0-1 valued charges if $A \in \mathcal{A}$, $\mu_0(A) = 1$

implies that $\mu_n(A) = 1$ for infinitely many indices n. These two notions are just translations of the corresponding notions on the Stone space of $\mathscr{A}$.

THEOREM 4. *Let $\mathscr{A}$ be a σ-field of subsets of a set X and μ_n, $n \geq 0$, be a sequence of* 0-1 *valued charges on $\mathscr{A}$ such that μ_n, $n \geq 1$, is a discrete sequence and μ_0 is an accumulation point of μ_n, $n \geq 1$. Define μ on $\mathscr{A}$ by $\mu = \sum_{n \geq 0} (1/2^{n+1})\mu_n$. Then the range $R(\mu)$ of μ is neither Lebesgue measurable nor has the property of Baire.*

Proof. Let $Z = \{0, 1\}$ and let ν be the measure on the discrete σ-field on Z given by $\nu(\{0\}) = \nu(\{1\}) = 1/2$. Let $C = Z^{\aleph_0}$ equipped with the product σ-field and the product probability measure $\tau = \nu \times \nu \times \nu \times \cdots$ on this σ-field. If we define a function h on C by

$$h(x_1, x_2, \ldots.) = \sum_{n \geq 1} (1/2^n)x_n$$

for $(x_1, x_2, \ldots)$ in C then h has many interesting properties. It is one-to-one except for a countable set of points; it is a homeomorphism except for a countable set of points; $h(A)$ is a Borel subset of $[0, 1]$ if and only if A is a Borel subset of C; h preserves the measure τ and the Lebesgue measure λ on $[0, 1]$; $h(A)$ is Lebesgue measurable if and only if A is τ-measurable; $h(A)$ has the property of Baire in $[0, 1]$ if and only if A has the property of Baire in C.

In view of these properties of h, if we show that

$$F = \{(\mu_0(A), \mu_1(A), \ldots); A \in \mathscr{A}\}$$

is neither τ-measurable nor has the property of Baire in C, then it will follow that $R(\mu)$ is neither Lebesgue measurable nor has the property of Baire in $[0, 1]$. This is because $h(F) = R(\mu)$.

Let $D = \{(\mu_1, (A), \mu_2(A), \ldots); A \in \mathscr{A}, \mu_0(A) = 1\}$. If we show that D is neither τ-measurable nor has the property of Baire in C, then the desired conclusion about F follows.

First we show that D is not τ-measurable. Let us see that D is a tail set. Since μ_n, $n \geq 1$, is a discrete sequence, we can find a sequence A_n, $n \geq 1$, of pairwise disjoint sets in $\mathscr{A}$ such that $\mu_n(A_n) = 1$ for every $n \geq 1$ and $\mu_n(A_m) = 0$ for $m \neq n$. Since μ_0 is an accumulation point of μ_n, $n \geq 1$, μ_0 is distinct from all μ_n, $n \geq 1$. One can assume without loss of generality, that $\mu_0(A_n) = 0$ for every $n \geq 1$. This follows from the fact that if ξ and η are two distinct 0-1 valued charges on $\mathscr{A}$, then there is a set A in $\mathscr{A}$ such that $\xi(A) = 0 = \eta(X - A)$. Let $(x_1, x_2, \ldots) = (\mu_1(A), \mu_2(A), \ldots.) \in D$ for some A in $\mathscr{A}$ with $\mu_0(A) = 1$. Let k be any positive integer and $y_1, y_2, \ldots, y_k$ be any finite sequence of 0's and 1's. Let

$$E_1 = \{1 \leq i \leq k; y_i = 0\} \quad \text{and} \quad E_2 = \{1 \leq i \leq k; y_i = 1\}.$$

Let $B = (A - \bigcup_{i \in E_1} A_i) \cup (\bigcup_{i \in E_2} A_i)$. It is obvious that $\mu_0(B) = 1$ and $\mu_i(B) = y_i$ for $1 \leq i \leq k$. Consequently,

$$(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots) \in D.$$

Hence D is a tail set.

Suppose D is τ-measurable. By the Kolmogorov's zero-one law, $\tau(D) = 0$ or 1. Let us look at the map ψ from C to C defined by

$$\psi(x_1, x_2, \ldots) = (1 - x_1, 1 - x_2, \ldots)$$

for $(x_1, x_2, \ldots)$ in C. We claim that $\psi(D) \cap D = \emptyset$ and $\psi(D) \cup D = C$. Suppose that $\psi(D) \cap D \neq \emptyset$. Let $(x_1, x_2, \ldots) \in \psi(D) \cap D$. Then we can find two sets A and B in $\mathscr{A}$ such that $\mu_0(A) = 1 = \mu_0(B)$ and

$$(x_1, x_2, \ldots) = (\mu_1(A), \mu_2(A), \ldots),$$

$$(1 - x_1, 1 - x_2, \ldots) = (\mu_1(B), \mu_2(B), \ldots).$$

Note that $\mu_0(A \cap B) = 1$ and $(\mu_1(A \cap B), \mu_2(A \cap B), \ldots) = (0, 0, \ldots)$. This is a contradiction to the fact that μ_0 is an accumulation point of μ_n, $n \geq 1$. Therefore $\psi(D) \cap D = \emptyset$. To show that $\psi(D) \cup D = C$, let

$$(x_1, x_2, \ldots) \in C.$$

Let $E = \{n \geq 1;\, x_n = 1\}$ and $A = \bigcup_{n \in E} A_n$. Then

$$(\mu_1(A), \mu_2(A), \ldots) = (x_1, x_2, \ldots).$$

If $\mu_0(A) = 1$, then $(x_1, x_2, \ldots) \in D$. If $\mu_0(A) = 0$, then

$$(x_1, x_2, \ldots) \in \psi(D).$$

This shows that $\psi(D) \cup D = C$. Note that ψ preserves the measure τ. Now, if $\tau(D) = 1$, then $\tau(\psi(D)) = 1$ and consequently, $\tau(C) = 2$ which is a contradiction. If $\tau(D) = 0$ then $\tau(\psi(D)) = 0$ which is again a contradiction since $\tau(C) = 1$. Thus D is not τ-measurable.

To prove that D does not have the property of Baire, one can repeat the above argument and use Oxtoby's category analogue of Kolmogorov's 0-1 law and the Baire Category theorem [6].

Remark 2. On any infinite σ-field it is always possible to obtain μ_n, $n \geq 0$, satisfying the hypothesis of the above theorem.

Remark 3. If μ_n, $n \geq 0$, is a sequence of 0-1 valued charges which contains a subset $\{\mu_{n_0}, \mu_{n_1}, \ldots\}$ such that $\{\mu_{n_1}, \mu_{n_2}, \ldots\}$ is a discrete set and μ_{n_0} is an accumulation point of $\{\mu_{n_1}, \mu_{n_2}, \ldots\}$ then it is possible to show that the range of $\sum_{n \geq 0} (1/2^n)\mu_n$ is not Borel, because one can easily see that the range of $\sum_{i \geq 0} (1/2^{n_i})\mu_{n_i}$ is a continuous image of

$$R\left(\sum_{n \geq 0} (1/2^n)\mu_n\right).$$

But not every sequence μ_n, $n \geq 0$, of 0-1 valued charges need contain a subset with this property. In [2], van Douwen has constructed countable Hausdorff extremally disconnected "Nodec' spaces (i.e., every nowhere dense subset is closed). In a personal communication van Douwen informs me that such spaces can be constructed in βN. If we write such a space as μ_n, $n \geq 0$, then it would be interesting to know whether the range of $\sum_{n\geq 0}(1/2^n)\mu_n$ is Borel or not.

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