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# HANKEL OPERATORS AND BERGMAN PROJECTIONS ON HARDY SPACES

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## 1. Introduction and statement of results

Let $D$ denote the unit disk in the complex plane and $T$ its boundary, the unit circle. For $0 < p \leq \infty$, $H^p$ will be the usual Hardy space of functions holomorphic on the unit disk; i.e., a holomorphic function $f$ belongs to $H^p$ if its non-tangential maximal function belongs to $L^p(T)$. (See the definition given by equation (7) below.) Let $P_+$ denote the "Szegö projection" given by

$$P_+g(z) = \frac{1}{2\pi}\int_T \frac{g(\eta)}{1-\bar{\eta}z}|d\eta|,$$

for a function $g \in L^1(T)$. If $u$ is an $L^1$ function on $T$ and $f$ is a function in $H^\infty$ then the Toeplitz operator with symbol $u$ is defined by the formula $T_u f(z) = P_+uf(z)$. The Hankel operator $H_u$ with symbol $u$ may be defined to be the boundary distribution of the function

$$u(z)f(z) - P_+uf(z).$$

In particular, if $u \in L^2(T)$ and $f \in H^\infty$ then $H_u f$ is the function defined by the formula

$$H_u f(e^{i\theta}) = u(e^{i\theta})f(e^{i\theta}) - P_+uf(e^{i\theta}).$$

It is well known that the operator $T_u$ extends to be a bounded operator on $H^p$ for $1 < p < \infty$ to $H^p$ if and only if $u$ is a bounded function. On the other hand, if $u \in L^2(T)$ then the Hankel operator $H_u$ extends to be a bounded operator on $H^p$ for $1 < p < \infty$ to $L^p$ if and only if $u = g_1 + \bar{g}_2$ where $g_1 \in H^2$ and $g_2 \in BMOA$. See [CRW] or [P].

Here, $BMOA$ is the space of holomorphic functions of bounded mean oscillation on the unit disk. See [G] for the definition of $BMOA$ and its various characterizations in terms of Carleson measures, as well as (1) and (4) below.

In [C2], some operators on Hardy spaces which are analogues of the operator $T_u$ were obtained by using the projections associated with the weighted Bergman spaces $L_a^2(dm_s)$. For $s > 0$ let $z = re^{i\theta}$ be a point in the unit disk and let $dm_s$ be the measure $dm_s(z) = \frac{s}{2\pi i}(1-r^2)^{s-1}d\bar{z}\wedge dz$. $L^p(dm_s)$ will denote the Lebesgue space

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of measurable functions defined on the unit disk integrable with respect to $dm_s$ and $L_a^p(dm_s)$ will denote the Bergman space of holomorphic functions in $L^p(dm_s)$. The orthogonal projection of $L^2(dm_s)$ to $L_a^2(dm_s)$ is given by the formula

$$P_s g(z) = \frac{s}{2\pi i}\int_D g(\zeta)\frac{(1-|\zeta|^2)^{s-1}}{(1-\bar{\zeta}z)^{s+1}}d\bar{\zeta}\wedge d\zeta.$$

Notice that if $F$ is continuous on the closed disk then

$$\lim_{s\to 0} P_s F(z) = P_+ F(z),$$

so the Szegö projection $P_+$ can be considered as the limiting case of the projections $P_s$.

If $u$ is any function in $L^1(dm_s)$ and if $f$ is a function in $H^\infty$, then the operator $T_u^s f$ is defined to be

$$T_u^s f(z) = \frac{s}{2\pi i}\int_D u(\zeta)f(\zeta)\frac{(1-|\zeta|^2)^{s-1}}{(1-\bar{\zeta}z)^{s+1}}d\bar{\zeta}\wedge d\zeta.$$

The following result was obtained in [C2].

THEOREM A. *Let $s > 0$, $1 \le p < \infty$, and suppose $u = h + G\mu$ is in $L^1(dm_s)$ where $h$ is harmonic and $G\mu$ is the Green potential of a non–negative measure $d\mu$. Then $T_u^s$ is a bounded operator from $H^p$ to $H^p$ if and only if $h$ is a bounded harmonic function and $(1-|z|)d\mu(z)$ is a Carleson measure.*

Here, the Green potential refers to the Green's function of the unit disk; see [C2].

In this paper we study operators on $H^p$ which may be regarded as analogues of the Hankel operator $H_u$ obtained by using the weighted Bergman projections $P_s$, $s > 0$, in place of $P_+$. Thus, for $s > 0$, $z \in D$, and $f \in H^\infty$ define

$$H_u^s f(z) = u(z)f(z) - P_s(uf)(z).$$

For a function $v$ defined on $D$ and $0 < r < 1$, let $v_r$ be the function defined on $T$ given by $v_r(e^{i\theta}) = v(re^{i\theta})$. This definition will give a well defined measurable function in each context we apply it. We shall say that $H_u^s f$ has boundary distribution in $L^p(T)$ provided that

$$\lim_{r\to 1}(H_u^s f)_r$$

exists in $L^p(T)$.

The operator $H_u^s$ is interpreted as taking a function $f$ to the boundary distribution of $H_u^s f$.

Our main result is the following theorem.

THEOREM 1. *Let $1 \leq p < \infty$ and $s > 0$. Suppose $u = \bar{\phi}$ where $\phi$ is in $H^1$. Then $H_u^s$ extends to a bounded operator from $H^p$ to $L^p(T)$ if and only if $\phi \in BMOA$.*

Note that if $s > 0$ and $\phi \in BMOA$ then the operator $H_{\bar{\phi}}^s$ is bounded from $H^1$ to $L^1$ if $\phi \in BMOA$ but this is not true in general for the operator $H_{\bar{\phi}}$.

The dual of this result is also of interest. Suppose $s > 0$. Let $C_s f$ the operator defined by

$$C_s f(\zeta) = \frac{1}{2\pi} \int_T \frac{f(z)}{(1 - \bar{z}\zeta)^{1+s}} |dz|.$$

If $f$ is holomorphic with power series representation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ then we expect $C_s f$ to behave like $R^s f$, the fractional derivative of $f$ of order $s$ given by the formula

$$R^s f(z) = \sum_{n=0}^{\infty} (n+1)^s \hat{f}(n) z^n.$$

For a function $\phi$ let $M_\phi$ denote the multiplication operator $M_\phi f = \phi f$. Finally, let $H_{-s}^p$ (resp. $BMOA_{-s}$) be the space of holomorphic functions $f$ whose fractional integral of order $s$, $I^s f$, is in $H^p$ (resp. $BMOA_{-s}$). Here, $I^s$ is the inverse of the fractional derivative operator $R^s$ and is given by the formula

$$I^s f(z) = \frac{1}{\Gamma(s)} \int_0^1 \left( \log \frac{1}{t} \right)^{s-1} f(tz) dt.$$

In the rest of the paper we will not distinguish between a bounded operator and an operator which extends to be a bounded operator.

THEOREM 2. *Let $1 < p < \infty$ and $s > 0$. Suppose $\phi \in H^1$. Then the following conditions are equivalent.*

1. *The operator $[M_\phi, C_s]$ is bounded from $L^p$ to $H_{-s}^p$.*
2. *The operator $[M_\phi, C_s]$ is bounded from $L^\infty$ to $BMOA_{-s}$.*
3. *The function $\phi$ belongs to BMOA.*

For a smooth function $g$ on $T$, the commutator $[M_\phi, C_s]$ is given by the formula

$$[M_\phi, C_s] g(\zeta) = \phi(\zeta) C_s g(\zeta) - C_s(\phi g)(\zeta),$$

for $|\zeta| < 1$.

Results by other authors concerning commutators of the form $[M_\phi, K]$ where $K$ is a singular integral operator or a Riesz potential are discussed in [T], chapter XVI. The methods employed by these authors do not seem to yield Theorem 2.

If $g \in H^p$ then $[M_\phi, C_1] g(\zeta) = \phi'(\zeta) \zeta g(\zeta)$. Thus Theorem 2 implies that if $1 < p < \infty$ and $g \in H^p$ and $\phi \in BMOA$, then a function of the form $g\phi'$ is the derivative of an $H^p$ function. In fact this is true for all $0 < p < \infty$. (See [C4] for a converse to this fact.) This result was proved in [C2] and a stronger result is used in the proof of Theorem 1 and 2; see Theorem B below. It can be shown that if $\phi \in BMOA$ then the operator $[M_\phi, C_s]$ is bounded from $H^p$ to $H^p_{-s}$ for all $0 < p < \infty$ whenever $s$ is a positive integer. We do not know, however, if the condition $\phi \in BMOA$ implies that the operator $[M_\phi, C_s]$ is bounded from $H^p$ to $H^p_{-s}$ for $0 < p \leq 1$ in the general case of a non-integral s.

Since the commutator $[M_\phi, C_1]$ is essentially multiplication by $\phi'$, it is natural to ask about the holomorphic multipliers of $H^p$ into $H^p_{-s}$. We have the following result.

THEOREM 3. *Let $0 < p < \infty$ and suppose $s > 0$. Let $\Phi$ be holomorphic on the unit disk. Then $M_\Phi$ defines a bounded operator from $H^p$ to $H^p_{-s}$ if and only if $\Phi \in BMOA_{-s}$.*

Our proof of Theorem 1 leads to an interesting formula for the difference operator $T^{s+1}_{\bar{\phi}} f - T^s_{\bar{\phi}} f$. In what follows we will let $T^0_u$ be the usual Toeplitz operator, $T_u$, and $H^0_u$ be the usual Hankel operator, $H_u$.

THEOREM 4. *Let $\phi \in H^1$. Suppose $1 < p < \infty$. Then $T^1_{\bar{\phi}} - T^0_{\bar{\phi}}$ is a bounded operator from $H^p$ to $H^p$ if and only if $\phi \in BMOA$.*

In [C2] it was shown that the Toeplitz operator $T^s_{\bar{\phi}}$ is bounded from $H^p$ to $H^p$ whenever $\phi \in H^\infty$ and $(s+1)p - 1 > 0$. It follows that there is no extension of Theorem 4 to Hardy spaces $H^p$ for $p \leq 1$ since in general, the operator $T^0_{\bar{\phi}}$ is not bounded from $H^1$ to $H^1$ for $\phi \in H^\infty$; see [S]. On the other hand, given the results in [C2], it is natural to expect there to be some type of theorem for $H^p$ when $0 < p \leq 1$. In this direction we have the following result.

THEOREM 5. *Let $s > 0$ and suppose $(s+1)p - 1 > 0$. Suppose $\phi$ is holomorphic in $D$ and $\phi \in L^1(dm_s)$. Then $T^{s+1}_{\bar{\phi}} - T^s_{\bar{\phi}}$ is a bounded operator from $H^p$ to $H^p$ if and only if $|\phi'(\zeta)| = O((1 - |\zeta|)^{-1})$.*

We also consider the operator $H^s_u$ for some symbols $u$ that are not necessarily antiholomorphic. Recall that $G\mu$ denotes the Green potential (with respect to the Green's function of the unit disk) of the measure $\mu$.

THEOREM 6. *Let $s > 0$ and $1 \leq p < \infty$. Suppose $u = h + G\mu \in L^1(dm_s)$ where $h$ is a harmonic function and $\mu$ is a non-negative measure on $D$. Then $H^s_u$ is a bounded operator from $H^p$ to $L^p$ if and only if $h = g + \bar{\phi}$ where $\phi \in BMOA$ and $(1 - |z|)d\mu(z)$ is a Carleson measure.*

## 2. Background and preparation for proofs of Theorem 1–6

We will adopt the following conventions. The notation $A \doteq B$ means that there is a constant $C$ such that $C^{-1}B \leq AC \leq CB$. The letter $C$ will be used to denote various numerical constants that change in value depending on the context.

It will be convenient to characterize the functions in $H^p$, $H^p_{-s}$, $BMOA$, and $BMOA_{-s}$ in terms of tent spaces.

For $\eta \in T$, let $\Gamma(\eta)$ be the approach region

$$\Gamma(\eta) = \{\zeta\colon\ |1 - \bar{\zeta}\eta| < 1 - |\zeta|^2\},$$

contained in $D$. If $u$ is a function defined on $D$ then we say $u \in T_2^p$ if

$$\|u\|^p_{T_2^p} = \int_T \left(\frac{1}{2\pi i}\int_{\Gamma(\eta)} |u(z)|^2 \frac{d\bar{z} \wedge dz}{(1 - |z|)^2}\right)^{p/2} |d\eta| < \infty.$$

If $I$ is a subarc of the circle $T$ then let $|I|$ equal the length of $I$ and let $T(I)$ be the "tent" over $I$; see [CMS]. If $u$ is a function defined on $D$ then we say $u \in T_2^\infty$ if

$$\|u\|^2_{T_2^\infty} = \sup_I \frac{1}{|I|}\left(\frac{1}{2\pi i}\int_{T(I)} |u(z)|^2 \frac{d\bar{z} \wedge dz}{1 - |z|}\right) < \infty. \tag{1}$$

Thus $u \in T_2^\infty$ if and only if $-i|u|^2 \frac{d\bar{z}\wedge dz}{1-|z|}$ is a Carleson measure. We will need the fact that if $1 < p < \infty$, then the dual space of $T_2^p$ is $T_2^{p'}$ with the pairing

$$(u, v) = \frac{1}{2\pi i}\int_D u(\zeta)v(\zeta)\frac{d\bar{\zeta} \wedge d\zeta}{1 - |\zeta|} \tag{2}$$

and the same pairing gives the duality between the spaces $T_2^1$ and $T_2^\infty$. See [CMS].

Recall if $0 < p < \infty$ then a necessary and sufficient condition for a function $f$ holomorphic in the disk to belong to the Hardy space $H^p$ is that for each $k > 0$ the function $(1 - |z|)^k D^k f$ belongs to the tent space $T_2^p$; see [AB]. Let $\rho$ be the function given by $\rho(z) = (1 - |z|^2)$. Then for a fixed $k$ we have the equivalence of norms

$$\|f\|_{H^p} \doteq \|\rho^k R^k\|_{T_2^p}. \tag{3}$$

Similarly, a necessary and sufficient condition that a holomorphic function $g$ belong to $H^p_{-s}$ is that $\rho^s g \in T_2^p$ and we have equivalence of norms

$$\|g\|_{H^p_{-s}} \doteq \|\rho^s g\|_{T_2^p}. \tag{4}$$

The spaces $BMOA$ and $BMOA_{-s}$ may be be characterized in terms of the spaces $T_2^\infty$. We have the following equivalences of norms (see [J]):

$$\|g\|_{BMOA} \doteq \|\rho^k R^k g\|_{T_2^\infty} \tag{5}$$

and

$$\|g\|_{BMOA_{-s}} \doteq \|\rho^s g\|_{T_2^\infty}. \tag{6}$$

Let $Nu(z)$ be the non-tangential maximal function

$$Nu(\eta) = \sup_{\zeta \in \Gamma(\eta)} |u(\zeta)|. \tag{7}$$

If $u$ is a continuous function on $D$ then we say $u \in T_\infty^p$ if

$$\|u\|_{T_\infty^p}^p = \int_T (Nu(\eta))^p |d\eta| < \infty.$$

We will need the following result from [C2].

THEOREM B. *Let $0 < p < \infty$ and suppose $u \in T_\infty^p$. Suppose that $h > 0$ is a function defined on $D$ and that $h \in T_2^\infty$. Then $hu \in T_2^p$ and $\|hu\|_{T_2^p} \le C(p)\|h\|_{T_2^\infty}\|u\|_{T_\infty^p}$.*

We will also need some recent results of Ahern, Cascante and Ortega [ACO] concerning invariant Poisson integrals and tent spaces. Although the main point of [ACO] was to get results for functions defined on the unit ball in $C^n$ where $n > 1$, it is the results for the case $n = 1$ that we need here. If $1 + \alpha + \beta > 0$ and neither $1 + \alpha$ or $1 + \beta$ is a negative integer then define

$$P_{\alpha,\beta}(\zeta, z) = c_{\alpha,\beta} \frac{(1 - |\zeta|^2)^{1+\alpha+\beta}}{(1 - \zeta\bar{z})^{1+\alpha}(1 - \bar{\zeta}z)^{1+\beta}},$$

where $c_{\alpha,\beta} = \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}$. For $g \in L^1(T)$ define

$$P_{\alpha,\beta}g(\zeta) = \frac{1}{2\pi}\int_T g(z)P_{\alpha,\beta}(\zeta, z)|dz|.$$

Let $Df(\zeta) = \frac{\partial f}{\partial \zeta}(\zeta)$ and $\bar{D}f(\zeta) = \frac{\partial f}{\partial \bar{\zeta}}(\zeta)$.
The following result is proved in [ACO].

THEOREM C. *Let $1 < p < \infty$ and suppose $g \in L^1(T)$. Then there are constants $C(p, \alpha, \beta)$ and $C(\alpha, \beta)$ such that the following inequalities hold.*
1. $\|\rho|DP_{\alpha,\beta}g| + \rho|\bar{D}P_{\alpha,\beta}g|\|_{T_2^p} \le C(p, \alpha, \beta)\|g\|_{L^p}$.
2. $\|P_{\alpha,\beta}g\|_{T_\infty^p} \le C(p, \alpha, \beta)\|g\|_{L^p}$.
3. $\|\rho|DP_{\alpha,\beta}g| + \rho|\bar{D}P_{\alpha,\beta}g|\|_{T_2^\infty} \le C(\alpha, \beta)\|g\|_{BMO}$.

Finally, we will need the following result proved in [C1] in a more general form.

THEOREM D. *Let $s > 0$ and $0 < p < \infty$. Suppose $K(z, \zeta)$ is a kernel of the form*

$$K(z,\zeta) = \Psi(z,\zeta)\frac{(1-|z|)(1-|\zeta|)^{s-1}}{(1-\bar{\zeta}z)^{s+2}}$$

*where $\Psi$ is a $C^\infty$ function defined on $\mathbf{C}^2$. Then the operator defined by*

$$Ku(z) = \int_D u(\zeta)K(z,\zeta)dm(\zeta)$$

*maps $T_2^p$ to itself.*

## 3. Proofs of Theorems 1–6

We start by proving Theorems 1 and 2. Our first goal is to establish the duality between the operators $H^s_{\bar{\phi}}$ and $[M_\phi, C_s]$. This is done in Lemmas 5 and 6 below. Lemmas 1–4 will give us some necessary machinery.

The following lemma follows easily from Stokes' theorem and the fact that

$$D(1-|\zeta|^2)^s = -s\bar{\zeta}(1-|\zeta|^2)^{s-1}.$$

We will henceforth use the term "smooth" to mean infinitely differentiable.

LEMMA 1. *Suppose that $f$ is holomorphic in a neighborhood of the closed disk and that $\phi \in H^1$. Then:*

1. *$P_s\bar{\phi}f(z)$ extends to be $C^\infty$ in the closed disk.*
2. *$\lim_{r\to 1} \|P_s\bar{\phi}f - P_s\bar{\phi}_r f\|_{H^\infty} = 0$.*

*Proof.* Let $D' = \{\zeta : \frac{1}{2} \le |\zeta| < 1\}$. To see the first statement, we just need to notice that for any $k > 0$, Stokes' theorem gives

$$P_s\bar{\phi}f(z) = \frac{C}{2\pi i}\int_{D'} \bar{\phi}(\zeta)D^k f(\zeta)\frac{(1-|\zeta|^2)^{k+s-1}}{(1-z\bar{\zeta})^{s+1}}\frac{d\bar{\zeta}\wedge d\zeta}{\bar{\zeta}^k} + E(z)$$

where $E(z)$ is a sum of terms which are all smooth on the closure of $D'$. The second statement is proved in a similar manner. □

Lemma 2 also follows from Stokes' theorem.

LEMMA 2. *Let $\phi \in H^1$ and let $f$ be a function holomorphic in a neighborhood of the closed disk. Then*

$$H^s_{\bar{\phi}}f(z) = \frac{1}{2\pi i}\int_D Df(\zeta)\frac{\bar{\phi}(z)-\bar{\phi}(\zeta)}{(1-\bar{\zeta}z)^{1+s}}(1-|\zeta|^2)^s\frac{d\bar{\zeta}\wedge d\zeta}{\bar{\zeta}} + f(0)(\bar{\phi}(z)-\bar{\phi}(0)).$$

Let

$$J_{\bar{\phi}}^{s} f(z) = H_{\bar{\phi}}^{s} f(z) - f(0)(\bar{\phi}(z) - \bar{\phi}(0)).$$

It follows that $J_{\bar{\phi}}^{s} f = H_{\bar{\phi}}^{s} f$ if $f(0) = 0$.

For two functions $f$ and $g$ defined on $T$ let $< f, g >$ be the usual pairing

$$\langle f, g\rangle = \frac{1}{2\pi} \int_T f(z)\bar{g}(z)\, |dz|.$$

LEMMA 3. *Suppose $\phi \in H^1$. Let $f$ be holomorphic in a neighborhood of the closed disk and suppose $g$ is a smooth function on $T$. Then*

$$\langle J_{\bar{\phi}}^{s} f, g\rangle = \frac{-1}{4\pi^2 i} \int_D Df(\zeta)\overline{[M_\phi, C_s]g}(\zeta)(1 - |\zeta|^2)^s \frac{d\bar{\zeta} \wedge d\zeta}{\bar{\zeta}}.$$

*Proof.* Suppose that $f$ is holomorphic on a neighborhood of the closed disk and that $g$ is a smooth function on $T$. Then

$$\langle J_{\bar{\phi}}^{s} f, g\rangle = \frac{1}{4\pi^2 i} \int_T \int_D Df(\zeta) \frac{\bar{\phi}(z) - \bar{\phi}(\zeta)}{(1 - \bar{\zeta} z)^{1+s}} (1 - |\zeta|^2)^s \frac{d\bar{\zeta} \wedge d\zeta}{\bar{\zeta}} \bar{g}(z)|dz|. \tag{8}$$

Since $f$ and $g$ are smooth functions and

$$\left| \frac{(1 - |\zeta|^2)^s}{(1 - \bar{\zeta} z)^{1+s}} \right| \le C \left| \frac{1}{1 - \bar{\zeta} z} \right|,$$

the hypothesis $\phi \in H^1$ shows that the absolute value of the integrand in (8) is integrable with respect to $d\bar{\zeta} \wedge d\zeta\, |dz|$. We may interchange the order of integration to get the desired formula. □

LEMMA 4. *Let $s > 0$, $\phi \in H^1$, and $1 < p < \infty$. Suppose $J_{\bar{\phi}}^{s}$ is bounded from $H^p$ to $L^p$. Then $\phi \in BMOA$.*

*Proof.* Let $f$ be holomorphic in a neighborhood of the closed disk and suppose $f(0) = 0$. Then $J_{\bar{\phi}}^{s} f = H_{\bar{\phi}}^{s}$. Set $g = H_{\bar{\phi}}^{s} f$. Let $P_- = I - P_+$ where $I$ is the identity. Observe that

$$g_r = \bar{\phi}_r f_r - (P_s \bar{\phi} f)_r$$

and $P_-(P_s\bar{\phi} f)_r = 0$. It follows that

$$\begin{aligned} P_-(g) &= \lim_{r\to 1} P_-(g_r) \\ &= \lim_{r\to 1} P_-(\bar{\phi}_r f_r) \\ &= P_-(\bar{\phi} f). \end{aligned}$$

Thus

$$\|P_-(\bar{\phi} f)\|_{L^p} = \|P_-(g)\|_{L^p} \le C\|g\|_{L^p} \le C\|f\|_{L^p}.$$

Thus $\bar{\phi}$ is the antiholomorphic symbol of a bounded Hankel operator and therefore $\phi \in BMOA$. This completes the proof. □

In the sequel, if $1 \le p < \infty$ then $p'$ will denote the conjugate index: $p' = \frac{p}{p-1}$ if $1 < p < \infty$ and $p' = \infty$ if $p = 1$.

LEMMA 5. *Let $\phi \in H^1$ and $s > 0$. Suppose $[M_\phi, C_s]$ is bounded from $L^{p'}$ to $H^{p'}$ for some $p'$ such that $1 < p' < \infty$ or bounded from $L^\infty$ to $BMOA_{-s}$ if $p' = \infty$. Then*:

1. *$J^s_{\bar{\phi}}$ is bounded from $H^p$ to $L^p$.*
2. *$\phi \in BMOA$.*

*Proof.* Suppose $1 < p' < \infty$. Let $f$ be holomorphic on a neighborhood of the closed disk and suppose $g$ is a smooth function on $T$. If $[M_\phi, C_s]$ is a bounded operator from $L^{p'}$ to $H^{p'}_{-s}$ then it follows from Lemma 3 and the tent space characterizations of $H^p$ and $H^{p'}_{-s}$ given by (3) and (4) and the duality between $T^p_2$ and $T^{p'}_2$ given by (2) that

$$|\langle J^s_{\bar{\phi}} f, g\rangle| \le C\|\rho Df\|_{T^p_2}\|\rho^s[M_\phi, C_s]g\|_{T^{p'}_2} \le C\|f\|_{H^p}\|g\|_{L^{p'}}.$$

It therefore follows from Lemma 3 that $J^s_{\bar{\phi}}$ is bounded from $H^p$ to $L^p$. If $[M_\phi, C_s]$ is bounded from $L^\infty$ to $BMOA_{-s}$ then the same argument shows that $J^s_{\bar{\phi}}$ is bounded from $H^1$ to $L^1$. This proves the first assertion of Lemma 5.

For $1 < p' < \infty$, the second assertion follows from Lemma 4. For the remaining case where $p' = \infty$, if $[M_\phi, C_s]$ is a bounded operator from $L^\infty$ to $BMOA_{-s}$ then the constant function $1 \in L^\infty$, and therefore $\phi - C_s\phi = [M_\phi, C_s]1 \in BMOA_{-s}$. Since $\phi \in BMOA_{-s}$, it follows that $C_s\phi \in BMOA_{-s}$. We show that this implies that $\phi \in BMOA$. From Lemma 2.1 in [AC] we have the formula

$$\frac{1}{1 - \zeta\bar{\eta}} = s\int_0^1 \frac{(1-t)^{s-1}}{(1 - t\zeta\bar{\eta})^{s+1}}\,dt.$$

Therefore if $|\zeta| < 1$ then

$$\phi(\zeta) = s\int_0^1 (1-t)^{s-1} C_s\phi(t\zeta)\,dt.$$

Since $C_s\phi \in BMOA_{-s}$, it follows that $I^s C_s\phi \in BMOA$, where

$$I^s C_s\phi(\zeta) = \frac{1}{\Gamma(s)}\int_0^1 \left(\log\frac{1}{t}\right)^{s-1} C_s\phi(t\zeta)\,dt.$$

The estimate

$$\left|(1-t)^{s-1} - \left(\log\frac{1}{t}\right)^{s-1}\right| \le C(1-t)^s$$

for $1/2 < t < 1$ shows that

$$\left|\phi(\zeta) - \frac{\Gamma(s)}{s} I^s C_s\phi(\zeta)\right| \le C\int_0^1 (1-t)^s |C_s\phi(t\zeta)|\,dt + C\|\phi\|_{H^1},$$

where $C$ is a constant independent of $\zeta$. Since $C_s\phi \in BMOA_{-s}$, it follows that $|C_s\phi(t\zeta)| \le (1 - t|\zeta|)^{-s}$. Therefore $\phi - \frac{\Gamma(s)}{s} I^s C_s\phi$ is a bounded function and it follows that $\phi \in BMOA$. This proves the second assertion if $p' = \infty$. $\square$

LEMMA 6. *Let $\phi \in H^1$, $s > 0$, and $1 \le p < \infty$. Suppose $H^s_{\bar\phi}$ is bounded from $H^p$ to $L^p$. Then:*

1. *$[M_\phi, C_s]$ is bounded from $L^{p'}$ to $H^{p'}_{-s}$.*
2. *$\phi \in BMOA$.*

*Proof.* First, since $H^s_{\bar\phi}1 = \bar\phi - \bar\phi(0)$, it follows from the hypothesis that $\phi \in H^p$. Let $g$ be smooth on $T$. It can be verified that

$$|[M_\phi, C_s]g(0)| \le C\|\phi\|_{H^p}\|g\|_{L^{p'}}. \tag{9}$$

Let $h$ be the holomorphic function on $D$ given by

$$h(\zeta) = \zeta^{-1}\left([M_\phi, C_s]g(\zeta) - [M_\phi, C_s]g(0)\right).$$

Then the tent space characterization of $H^{p'}_{-s}$ and (9) show that

$$\|[M_\phi, C_s]g\|_{H^{p'}_{-s}}\| \doteq \|h\|_{H^{p'}_{-s}} + \|\phi\|_{H^p}\|g\|_{L^{p'}}.$$

Suppose $1 < p < \infty$. It is enough to show that

$$\|h\|_{H^{p'}_{-s}} \le C\|\phi\|_{H^p}\|g\|_{L^{p'}}$$

for a constant $C$ independent of $g$. The characterization of $H^{p'}_{-s}$ in terms of tent space and the duality between $T^p_2$ and $T^{p'}_2$ shows that there is a bounded function $F$ with compact support in $D$ and $\|F\|_{T^p_2} \le C$ such that

$$\|h\|_{H^{p'}_{-s}} = \int_D F(\zeta)\overline{h(\zeta)}(1-|\zeta|^2)^{s-1}\,d\bar\zeta \wedge d\zeta.$$

We claim that $h \in L^1(dm_{s+1})$. To see this let $\zeta = r\eta$ where $\eta \in T$. Since $g$ is smooth on $T$, we may write

$$|h(\zeta)| \le |C_s\phi(g - g(\eta))(\zeta)| + |g(\eta)C_s\phi(\zeta)|.$$

It is easy to see that

$$|C_s\phi(g - g(\eta))(\zeta) \le C(1-r)^{-s}\|\phi\|_{H^1}.$$

Also, the arguments used in Lemma 2.1 and 2.2 of [AC] show that

$$|C_s\phi(\zeta)| \le C(1-r)^{-s}N\phi(\eta).$$

It follows that $h \in L^1(dm_{s+1})$ and we may use the weighted Bergman projection $P_{s+1}$ to write

$$h(\zeta) = \frac{s+1}{2\pi i}\int_D h(\eta)\frac{(1-|\eta|^2)^s}{(1-\zeta\bar{\eta})^{s+2}}\,d\bar{\eta}\wedge d\eta.$$

We may use this formula to express

$$\int_D F(\zeta)\overline{h(\zeta)}(1-|\zeta|^2)^{s-1}\,d\bar{\zeta}\wedge d\zeta$$

as an iterated integral and since $F$ has compact support we may interchange to order of integration to get

$$\|h\|_{H^{p'}_{-s}} = \int_D G(\eta)\overline{h(\eta)}(1-|\eta|^2)^s\,d\bar{\eta}\wedge d\eta$$

where

$$G(\eta) = \frac{s+1}{2\pi i}\int_D F(\zeta)\frac{(1-|\zeta|^2)^{s-1}}{(1-\eta\bar{\zeta})^{s+2}}\,d\bar{\zeta}\wedge d\zeta.$$

It follows from Theorem D that the $T_2^p$ norm of the function $\rho G$ is less than $C\|F\|_{T_2^p}$. Therefore by (3), $G(\eta) = Df(\eta)$ where $f$ is holomorphic on the closed disk and $\|f\|_{H^p} \le C$. Thus

$$\begin{aligned}\|h\|_{H^{p'}_{-s}} &= \left|\int_D Df(\eta)\overline{h(\eta)}(1-|\eta|^2)^s\,d\bar{\eta}\wedge d\eta\right| \\ &= \left|\langle J^s_{\bar{\phi}}f, g\rangle - \overline{[M_\phi, C_s]g}(0)\int_D Df(\eta)(1-|\eta|^2)^s\,d\bar{\eta}\wedge d\eta\right|.\end{aligned}$$

The estimate $|Df(re^{i\theta})(1-r)| \le CNf(e^{i\theta})$ shows that the second term in the sum on the right hand side above is less than a constant times $\|\phi\|_{H^p}\|g\|_{H^{p'}}\|f\|_{H^p}$. Since $J^s_{\bar{\phi}}$ is bounded from $H^p$ to $L^p$, the same estimate holds for the first term and it follows that $[M_\phi, C_s]$ is a bounded operator from $L^{p'}$ to $H^{p'}_{-s}$. The same argument shows that if $H^s_{\bar{\phi}}$ is bounded from $H^1$ to $L^1$ then $[M_\phi, C_s]$ is bounded from $L^\infty$ to $BMOA_{-s}$. This proves the first assertion. The second assertion follows from the first assertion and the second assertion of Lemma 5.

With Lemmas 5 and 6 established, to finish the proof of Theorems 1 and 2 it is enough to show that if $\phi \in BMOA$ then $H^s_{\bar\phi}$ is a bounded operator. By Lemma 1, it is enough to prove that there is a constant $C$ such that if $f$ and $\phi$ are holomorphic in a neighborhood of the closed disk and $g$ is a smooth function on $T$ then the apriori estimate

$$|\langle H^s_{\bar\phi} f, g\rangle| \leq C\|f\|_{H^p}\|g\|_{L^{p'}}\|\phi\|_{BMOA}$$

is verified. Here, $C$ must be independent of $f$, $\phi$, and $g$. We may also assume $f(0) = 0$ to simplify matters when we apply Stokes' theorem.

Lemma 7 below is based on the fact that $\frac{1}{\zeta - z}$ is a fundamental solution to the equation $\bar{D}u = 2\pi i \delta_z$.

LEMMA 7. *Suppose $s > 0$, and $f$ and $\phi$ are holomorphic on a neighborhood of the closed disk. Then for $|z| \leq 1$,*

$$H^s_{\bar\phi} f(z) = -\frac{1}{2\pi i}\int_D f(\zeta)\overline{\phi'(\zeta)}\left(\frac{1-|\zeta|^2}{1-\bar\zeta z}\right)^s \frac{d\bar\zeta \wedge d\zeta}{\zeta - z}.$$

*Proof.* Suppose $\Psi$ is smooth on the closed disk. Then for $|z| < 1$,

$$\begin{aligned}
-2\pi i \Psi(z) &= \int_D \bar{D}\left(\Psi(\zeta)\left(\frac{1-|\zeta|^2}{1-\bar\zeta z}\right)^s\right)\frac{d\bar\zeta \wedge d\zeta}{\zeta - z} \\
&= \int_D \bar{D}\Psi(\zeta)\left(\frac{1-|\zeta|^2}{1-\bar\zeta z}\right)^s \frac{d\bar\zeta \wedge d\zeta}{\zeta - z} + \int_D \Psi(\zeta)\bar{D}\left(\frac{1-|\zeta|^2}{1-\bar\zeta z}\right)^s \frac{d\bar\zeta \wedge d\zeta}{\zeta - z} \\
&= \int_D \bar{D}\Psi(\zeta)\left(\frac{1-|\zeta|^2}{1-\bar\zeta z}\right)^s \frac{d\bar\zeta \wedge d\zeta}{\zeta - z} - 2\pi i P_s\Psi(z).
\end{aligned}$$

Applying this formula with $\Psi = f\bar\phi$ yields the result for $|z| < 1$. The full statement follows from the dominated convergence theorem and the fact that, by Lemma 1, $H^s_{\bar\phi} f$ is smooth on the closed disk.

We now complete the proof of Theorem 1.

Suppose $f$ and $\phi$ are holomorphic on a neighborhood of the closed disk, $f(0) = 0$, and $g$ is smooth on $T$. Let $h(z) = zg(z)$. Apply Lemma 7 to get

$$\begin{aligned}
-\langle H^s_{\bar\phi} f, g\rangle &= C\int_T\int_D f(\zeta)\overline{\phi'(\zeta)}\frac{(1-|\zeta|^2)^s \bar z}{(1-\bar\zeta z)^s(1-\zeta\bar z)}\, d\bar\zeta \wedge d\zeta\ \overline{g(z)}|dz| \\
&= C\int_D f(\zeta)\overline{\phi'}(\zeta)\overline{P_{s-1,0}h}(\zeta)\, d\bar\zeta \wedge d\zeta.
\end{aligned}$$

Apply Stokes' theorem to get

$$\langle H^s_{\bar{\phi}} f, g\rangle = C\int_D (Df(\zeta)\overline{\phi'}(\zeta)\overline{P_{s-1,0}h}(\zeta) + f(\zeta)\overline{\phi'}(\zeta)D\overline{P_{s-1,0}h}(\zeta))(1-|\zeta|^2)\,\frac{d\bar{\zeta}\wedge d\zeta}{\bar{\zeta}}.$$

It follows that

$$|\langle H^s_{\bar{\phi}} f, g\rangle| \le C(\|\rho Df\|_{T^p_2}\|\rho\phi' P_{s-1,0}h\|_{T^{p'}_2} + \|\rho\phi' f\|_{T^p_2}\|\rho\bar{D}P_{s-1,0}h\|_{T^{p'}_2}).$$

The desired apriori estimates follow now from Theorems B and C. □

*Proof of Theorem* 3. Let $0 < p < \infty$. If $\Phi \in BMOA_{-s}$ then $\rho^s\Phi \in T^\infty_2$ and it follows from Theorem B that

$$\|\Phi g\|_{H^p_{-s}} \doteq \|\rho^s\Phi g\|_{T^p_2} \le C\|\rho^s\Phi\|_{T^\infty_2}\|g\|_{H^p}.$$

Conversely, if $\|\Phi g\|_{H^p_{-s}} \le C\|g\|_{H^p}$ for all $g$ in $H^p$ then it follows that

$$\int_T\left(\frac{1}{2\pi i}\int_{\Gamma(\eta)}|\rho^s\Phi g|^2\,\frac{d\bar{z}\wedge dz}{(1-|z|)^2}\right)^{\frac{p}{2}}|d\eta| \le C\int_T |g(\eta)|^p\,|d\eta|$$

for all $g \in H^p$. Let $\frac{p}{2} = q$. Since every $G \in H^q$ is of the form $G = Ig^2$ where $I$ is an inner function and $g \in H^p$, it follows that

$$\int_T\left(\frac{1}{2\pi i}\int_{\Gamma(\eta)}\rho^{2s-1}|\Phi|^2|G|\,\frac{d\bar{z}\wedge dz}{1-|z|}\right)^q \le C\int_T|G(\eta)|^q\,|d\eta|,$$

for all $G \in H^q$. It follows from [C3] that $-i\rho^{2s-1}|\Phi|^2 d\bar{z}\wedge dz$ is a Carleson measure and therefore $\Phi \in BMOA_{-s}$. This completes the proof. □

Lemma 7 leads to a useful formula for the difference $T^{s+1}_{\bar{\phi}} f - T^s_{\bar{\phi}} f$. In what follows we will let $T^0_u$ be the usual Toeplitz operator $T_u$ and $H^0_u$ be the usual Hankel operator $H_u$.

LEMMA 8. *Let $s \ge 0$. Suppose $f$ is holomorphic in a neighborhood of the closed disk. Assume that $\phi$ is holomorphic on $D$ and that $\phi \in H^1$ if $s = 0$, and that $\phi' \in L^1(dm_s)$ if $s > 0$. Then*

$$T^{s+1}_{\bar{\phi}} f(z) - T^s_{\bar{\phi}} f(z) = \frac{1}{(s+1)2\pi i}\int_D f'(\zeta)\bar{\phi}'(\zeta)\left(\frac{1-|\zeta|^2}{1-\bar{\zeta}z}\right)^{s+1} d\bar{\zeta}\wedge d\zeta.$$

*Proof.* First assume that $\phi$ is holomorphic on a neighborhood of the closed disk. Using Lemma 7 and the fact that $T_u^{s+1} - T_u^s = H_u^s - H_u^{s+1}$ it follows that

$$T_{\bar{\phi}}^{s+1} f(z) - T_{\bar{\phi}}^s f(z) = \frac{1}{2\pi i} \int_D f(\zeta)\bar{\phi}'(\zeta)\bar{\zeta} \frac{(1-|\zeta|^2)^s}{(1-\bar{\zeta}z)^{s+1}} d\bar{\zeta} \wedge d\zeta.$$

The formula of the lemma now follows from Stokes' theorem, since $D((1-|\zeta|^2)^{s+1}) = -(s+1)\bar{\zeta}(1-|\zeta|^2)^s$. The result for general $\phi$ follows by applying the result to $\phi_r$ where $\phi_r(z) = \phi(rz)$ and taking the limit as $r \to 1$. (The case $s = 0$ is included since $\phi \in H^1$ implies $\phi' \in L^1(dm_2)$; see the proof of Lemma 6.) □

*Proof of Theorem* 4. Suppose $\phi \in BMOA$. Since $T_u^1 - T_u^0 = H_u^0 - H_u^1$, it follows from Theorem 1 that $T_{\bar{\phi}}^1 - T_{\bar{\phi}}^0$ is bounded from $H^p$ to $H^p$. Conversely, if $T_{\bar{\phi}}^1 - T_{\bar{\phi}}^0$ is bounded from $H^p$ to $H^p$, then it follows from Lemma 8 and Fubini's theorem that

$$\begin{aligned} \left| \int_D f'(\zeta)\bar{\phi}'(\zeta)\bar{g}(\zeta)(1-|\zeta|^2)\, d\bar{\zeta} \wedge d\zeta \right| &= C|\langle T_{\bar{\phi}}^1 f - T_{\bar{\phi}}^0 f, g\rangle| \\ &\le C\|f\|_{H^p}\|g\|_{H^{p'}} \end{aligned}$$

for all functions $f$ and $g$ which are holomorphic on the closed disk. Let $G$ be the holomorphic function vanishing at 0 such that $G'(\zeta) = \phi'(\zeta)g(\zeta)\zeta$. Then applying Stokes' theorem twice as in the proof of Lemma 8 yields

$$\lim_{r\to 1} \int_T f(\zeta)\bar{G}(r\zeta) d\zeta = \int_D f'(\zeta)\bar{\phi}'(\zeta)\bar{g}(\zeta)(1-|\zeta|^2)\, d\bar{\zeta} \wedge d\zeta$$

and it follows that

$$\lim_{r\to 1} \left| \int_T f(\zeta)\bar{G}(r\zeta)\, d\zeta \right| \le C\|f\|_{H^p}\|g\|_{H^{p'}},$$

for all $f$ holomorphic on the closed disk. Thus $G \in H^{p'}$ and $\|G\|_{H^{p'}} \le C\|g\|_{H^{p'}}$. We have shown therefore that there is a constant $C$, independent of $g$ such that

$$\|\phi' g\|_{H^{p'}_{-1}} \le C\|g\|_{H^{p'}}$$

for all $g$ holomorphic in a neighborhood of the closed disk. Theorem 3 implies that $\phi' \in BMOA_{-1}$ and this completes the proof. □

*Proof of Theorem* 5. First suppose that $T_{\bar{\phi}}^{s+1} - T_{\bar{\phi}}^s$ is a bounded operator from $H^p$ to $H^p$. Then there is a constant $C$ independent of $z$ and $f$ such that

$$|T_{\bar{\phi}}^{s+1} f(z) - T_{\bar{\phi}}^s f(z)| \le C(1-|z|)^{-1}\|f\|_{H^p}^p.$$

If $f$ is holomorphic on the closed disk, then the proof of Lemma 8 shows that

$$\left| \int_D f(\zeta)\bar{\phi}'(\zeta)\bar{\zeta} \frac{(1-|\zeta|^2)^s}{(1-\bar{\zeta}z)^{s+1}} d\bar{\zeta} \wedge d\zeta \right|^p \le C(1-|z|)^{-1}\|f\|_{H^p}^p. \qquad (10)$$

If we let $f(\zeta) = (1 - \bar{z}\zeta)^{-s-2}$ in equation (10) then we get

$$\left|\frac{\phi'(z)z}{(1-|z|^2)^{s+1}}\right|^p \leq C(1-|z|)^{-1}(1-|z|)^{-sp-2p+1}$$

which implies that $|\phi'(z)| = O((1-|z|)^{-1})$.

Next, suppose that $|\phi'(\zeta)| = O((1-|\zeta|)^{-1})$. By Lemma 8, if $f$ is holomorphic on the closed disk, then

$$T_{\bar{\phi}}^{s+1} f(z) - T_{\bar{\phi}}^{s} f(z) = \frac{1}{(s+1)2\pi i}\int_D f'(\zeta)\bar{\phi}'(\zeta)\left(\frac{1-|\zeta|^2}{1-\bar{\zeta}z}\right)^{s+1} d\bar{\zeta}\wedge d\zeta.$$

Thus

$$T_{\bar{\phi}}^{s+1} f(z) - T_{\bar{\phi}}^{s} f(z) = \int_D v(\zeta)\frac{(1-|\zeta|)^{s-1}}{(1-\bar{\zeta}z)^{s+1}} d\bar{\zeta}\wedge d\zeta$$

where $v(\zeta) = \frac{1}{(s+1)2\pi i} f'(\zeta)\bar{\phi}'(\zeta)(1-|\zeta|^2)^2$. Since $|\phi'(\zeta)| = O((1-|\zeta|)^{-1})$, it follows that there is a constant $C$ independent of $f$ such that

$$\|v\|_{T_2^p} \leq C\|f\|_{H^p}.$$

The desired conclusion follows now from Theorem D and the tent space characterization of $H^p$. □

*Proof of Theorem* 6. First suppose that $(1-|z|)d\mu(z)$ is a Carleson measure and $h = g + \bar{\phi}$ where $g$ is holomorphic and $\phi \in BMOA$. By Littlewood's theorem, (see [Ts] Theorem IV.33), $\lim_{r\to 1}\|G\mu_r\|_{L^1} = 0$. It follows that

$$H_u^s = H_{\bar{\phi}}^s - T_{G\mu}^s,$$

which is bounded by Theorems A and 1.

Conversely, suppose $u = h + G\mu$ and $H_u^s$ is bounded from $H^p$ to $L^p$. We argue very much as in the proof of the sufficiency statements of Theorems 1 and 2 in [C2]. Use Littlewood's theorem again to deduce that if $f$ is holomorphic on the closed disk then

$$H_u^s f = H_h^s f - T_{G\mu}^s f.$$

Since $H_u^s$ is a bounded operator, we have the pointwise bounds

$$|H_u^s f(z)|^p \leq C\frac{1}{1-|z|}\|f\|_{H^p}^p \tag{11}$$

for all functions $f$ that are holomorphic in a neighborhood of the closed disk. Let

$$f_z(\zeta) = \left(\frac{1-|z|^2}{1-\bar{z}\zeta}\right)^{s+1}.$$

By Lemma 7 of [C2],

$$H_u^s f_z(z) = -T_{G\mu}^s f_z(z).$$

This combined with (11) yields the estimate

$$|T_{G\mu}^s f_z(z)| \leq C$$

for a constant $C$ independent of $z$. The argument on page 18 in [C2], beginning with equation (10) of that paper, shows that $(1-|z|)d\mu(z)$ is therefore a Carleson measure. By Theorem A, $T_{G\mu}^s$ is bounded and therefore $H_h^s$ is bounded. Let $h_r(z) = h(rz)$. Then $H_h^s 1 = \lim_{r\to 1} H_{h_r}^s 1 = \bar{\phi} - \bar{\phi}(0)$ and it follows that $\phi \in H^1$. Therefore $h = g + \bar{\phi}$ with both $g$ and $\phi$ in $L^1(dm_s)$ and it is easy to see that $H_u^s = H_{\bar{\phi}}^s$. It follows from Theorem 1 that $\phi \in BMOA$ and this completes the proof. □

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