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A FINITENESS THEOREM FOR THE SPECTRAL SEQUENCE OF A RIEMANNIAN FOLIATION

BY

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Introduction

Let M be a smooth closed manifold which carries a smooth foliation $\mathscr{F}$ of dimension p and codimension q. A differential form ω of degree r is said to be of filtration $\geq k$ if it vanishes whenever $r-k+1$ of the vectors are tangent to $\mathscr{F}$. In this way the deRham complex of the differential forms becomes a filtered differential algebra and we have the spectral sequence $(E_i(\mathscr{F}), d_i)$ which converges after a finite number of steps to the (finite dimensional) cohomology of M.

It is clear that $E_2^{0,0}(\mathscr{F})$, $E_2^{1,0}(\mathscr{F})$, $E_2^{q-1,p}(\mathscr{F})$ and $E_2^{q,p}(\mathscr{F})$ are of finite dimension but there are another vectorial spaces $E_2^{u,v}(\mathscr{F})$ that may be infinite-dimensional as shown in the examples of G.W. Schwarz [7].

In [6], K.S. Sarkaria proves that $E_2(\mathscr{F})$ is finite-dimensional when $\mathscr{F}$ is transitive. He uses techniques of functional analysis (constructing a 2-parametrix).

In [2], A. El Kacimi-Alaoui, V. Sergiescu and G. Hector prove that the basic cohomology, [which is equal to $E_2^{\cdot,0}(\mathscr{F})$) is finite-dimensional. They prove it step to step for Lie foliations, transversely parallelizable foliations and Riemannian foliations.

This paper establishes the following improvement of the two results above.

THEOREM. *If a smooth closed manifold M carries a Riemannian foliation $\mathscr{F}$ then $E_2(\mathscr{F})$ is finite-dimensional.*

To prove it we assume that $\mathscr{F}$ is transversely oriented and construct an operation of a Lie algebra in $E_1(\hat{\mathscr{F}})$, where $\hat{\mathscr{F}}$ is the horizontal lift of $\mathscr{F}$ to the principal fiberbundle of oriented orthonormal frames with the transverse

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Levi-Civita connection [4]. Then $E_2(\mathcal{F})$ and $E_2(\hat{\mathcal{F}})$ can be related by results of [1] and by the above result of [6], the theorem follows.

This result has also been obtained recently by Sergiescu [6] but using different techniques.

Finally, I want to express my deep gratitude to Xosé M. Masa Vázquez, who is guiding me through this subject.

1. The spectral sequence associated to a foliation

Let M be a smooth manifold which carries a foliation $\mathcal{F}$ of dimension p and codimension q. We may describe $\mathcal{F}$ by the exact sequence of vectorbundles

$$0 \to T\mathcal{F} \to TM \to Q \to 0, \tag{1.1}$$

where $T\mathcal{F} \subset TM$ denotes the integrable subbundle of vectors of M tangent to $\mathcal{F}$, and $Q = TM/T\mathcal{F}$ is the normal bundle.

The spectral sequence $(E_i(\mathcal{F}), d_i)$ associated to $\mathcal{F}$ arises from the following filtration of the deRham complex (A, d) of M:

$$F^k(A^r) = \{\alpha \in A^r / i_v(\alpha) = 0 \quad \text{for } v = X_1 \wedge \cdots \wedge X_{r-k+1},\ X_i \in \Gamma T\mathcal{F}\} \tag{1.2}$$

With this decreasing filtration, (A, d) is a graded filtered differential algebra. Since $F^{q+1}(A) = 0, (E_i(\mathcal{F}), d_i)$ collapses at the $(q+1)$-th term and is convergent to $H_{DR}(M)$.

The choice of a Riemannian metric on M defines a subbundle $\nu = T\mathcal{F}^{\perp} \subset TM$ and a splitting $\sigma\colon Q \to TM$ of (1.1) such that $\sigma(Q) = \nu$. Then (A, d) is a bigraded differential algebra if we define

$$A^{u,v} = \Gamma(\Lambda^v T^*\mathcal{F} \otimes \Lambda^u \nu^*) = \Gamma\Lambda^v T^*\mathcal{F} \otimes_{C^\infty(M)} \Gamma\Lambda^u \nu^* \tag{1.3}$$

for $0 \leq u \leq q$ and $0 \leq v \leq p$.

The exterior derivative d may be decomposed as the sum of the bihomogeneous operators $d_{\mathcal{F}}$, $d_{1,0}$ and $d_{2,-1}$ of bidegrees $(0,1)$, $(1,0)$ and $(2,-1)$ respectively, which satisfy

$$\begin{gathered} d_{\mathcal{F}}^2 = 0, \quad d_{2,-1}^2 = 0, \quad d_{\mathcal{F}} d_{1,0} + d_{1,0} d_{\mathcal{F}} = 0, \\ d_{1,0} d_{2,-1} + d_{2,-1} d_{1,0} = 0, \quad d_{1,0}^2 + d_{2,-1} d_{\mathcal{F}} + d_{\mathcal{F}} d_{2,-1} = 0. \end{gathered} \tag{1.4}$$

The filtration of A may be represented by

$$F^k(A) = \bigoplus_{u \geq k} A^{u,\cdot} .$$

Hence we have the following well known theorem.

(1.6) THEOREM [3]. *We have the following identities of bigraded differential algebras.*
(i) $(E_0(\mathcal{F}), d_0) = (A, d_{\mathcal{F}})$,
(ii) $(E_1(\mathcal{F}), d_1) = (H(A, d_{\mathcal{F}}), d_{1,0*})$.

It follows that $E_2(\mathcal{F}) = H(H(A, d_{\mathcal{F}}), d_{1,0*})$, $E_1^{\cdot,0}(\mathcal{F}) = A_b(\mathcal{F})$, and $E_2^{\cdot,0}(\mathcal{F}) = H_b(\mathcal{F})$, where $A_b(\mathcal{F})$ and $H_b(\mathcal{F})$ are respectively the algebra of basic forms and the basic cohomology of $\mathcal{F}$.

2. Riemannian foliations

Assume that in Section 1, $\mathcal{F}$ is Riemannian and transversely oriented. Let $\pi\colon \hat{M} \to M$ be the principal SO(q)-bundle of oriented orthonormal transverse frames. We have on $\hat{M}$ the transverse Levi-Civita connection ω with curvature Ω and the transversely parallelizable foliation $\hat{\mathcal{F}}$, where $T\hat{\mathcal{F}}$ is the horizontal lifting of $T\mathcal{F}$ [4], which satisfy

$$\dim(\hat{\mathcal{F}}) = p \quad \text{and} \quad \operatorname{codim}(\hat{\mathcal{F}}) = q + q_0,$$

where $q_0 = \dim(SO(q)) = \frac{1}{2}q(q-1)$. Let $\hat{\nu}$ denote the horizontal lifting of ν and V the vertical subbundle. $T\hat{\mathcal{F}}$, $\hat{\nu}$ and V are preserved by the action of SO(q) on TM.

Let $(\hat{A}, \hat{d})$ denote the deRham complex of M, which is a trigraded algebra if we set

$$\hat{A}^{s,t,v} = \Gamma(\Lambda^v T^*\hat{\mathcal{F}} \otimes \Lambda^t \hat{\nu}^* \otimes \Lambda^s V^*) \tag{2.1}$$

for $0 \leq s \leq q_0$, $0 \leq t \leq q$ and $0 \leq v \leq p$. Thus, if we define

$$\hat{A}^{u,v} = \bigoplus_{s+t=u} \hat{A}^{s,t,v} \tag{2.2}$$

for $0 \leq u \leq q_0 + q$ and $0 \leq v \leq p$, $(\hat{A}, \hat{d})$ is a bigraded differential algebra from which the spectral sequence $(E_i(\hat{\mathcal{F}}), \hat{d}_i)$ arises according to Section 1.

The exterior derivative $\hat{d}$ may be decomposed as the sum of the bihomogeneous operators $d_{\mathscr{F}}$, $\hat{d}_{1,0}$ and $\hat{d}_{2,-1}$ of bidegrees $(0,1)$, $(1,0)$ and $(2,-1)$ respectively, satisfying the analogue of (1.4). Then (1.6) shows that

$$\left(E_1(\hat{\mathscr{F}}), \hat{d}_1\right) = \left(H\left(\hat{A}, d_{\mathscr{F}}\right), \hat{d}_{1,0*}\right). \tag{2.3}$$

Let $(so(q), i, \theta, \hat{A}, \hat{d})$ be the operation of $so(q)$ associated with the principal $\mathrm{SO}(q)$-bundle $\pi\colon \hat{M} \to M$ and ω^* the algebraic connection, with curvature Ω^*, corresponding to ω, (Section 8.22 of vol. III of [1]). Let $(E_i(\hat{A}_{\theta=0}), \hat{d}_i)$ denote the spectral sequence corresponding to the bigraded differential algebra $(\hat{A}_{\theta=0}, \hat{d})$.

The homomorphism $\pi^*\colon A \to \hat{A}$ can be regarded as an isomorphism

$$\pi^*\colon A \xrightarrow{\cong} \hat{A}_{i=0,\theta=0}. \tag{2.4}$$

We will let $A = \hat{A}_{i=0,\theta=0}$.

For $Z \in so(q)$, i_Z and θ_Z are trihomogeneous of tridegrees $(-1,0,0)$ and $(0,0,0)$ respectively. Hence, comparing bidegrees in $i_Z\hat{d} + \hat{d}i_Z = \theta_Z$ and $\hat{d}\theta_Z = \theta_Z\hat{d}$ we obtain

$$\begin{gathered} i_Z d_{\mathscr{F}} + d_{\mathscr{F}} i_Z = 0, \quad i_Z\hat{d}_{1,0} + \hat{d}_{1,0} i_Z = \theta_Z, \\ \theta_Z d_{\mathscr{F}} = d_{\mathscr{F}}\theta_Z, \quad \theta_Z\hat{d}_{1,0} = \hat{d}_{1,0}\theta_Z, \end{gathered} \tag{2.5}$$

from which we can derive in cohomology the operation $(so(q), i_1, \theta_1, E_1(\hat{\mathscr{F}}), \hat{d}_1)$.

The algebraic connection $\omega^*\colon so(q)^* \to \hat{A}^{1,0,0} \subset \hat{A}^{1,0}$ satisfies

$$\operatorname{Im}(\omega^*) \subset \hat{A}^{1,0} \cap \operatorname{Ker}(d_{\mathscr{F}}) = E_1^{1,0}(\hat{\mathscr{F}}) \quad [2];$$

then

$$\omega_1^* = \omega^*\colon so(q)^* \to E_1^{1,0}(\hat{\mathscr{F}})$$

is an algebraic connection for $(so(q), i_1, \theta_1, E_1(\hat{\mathscr{F}}), \hat{d}_1)$.

We have the isomorphism of graded algebras (Section 8.4 of vol. III of [1]):

$$f\colon \hat{A}_{i=0} \otimes \Lambda so(q)^* \xrightarrow{\cong} \hat{A}, \quad \alpha \otimes \phi \mapsto \alpha \cdot \omega_\Lambda^*(\phi). \tag{2.6}$$

According to the identification given by f we obtain (Section 8.7 of vol. III of [1]):

$$i_Z = w \otimes i_{so(q)Z}, \tag{2.7}$$

$$\theta_Z = \theta_Z \otimes 1 + 1 \otimes \theta_{so(q)Z}, \tag{2.8}$$

$$\hat{d} = w \otimes d_{so(q)} + \hat{d}_\theta + h_\Omega + \nabla_{i=0} \otimes 1, \tag{2.9}$$

where $Z \in so(q)$, w is the degree involution, ∇ is the covariant derivative in $\hat{A}$ associated with ω^*, and $d_{so(q)}$, d_θ and d_Ω are defined by

$$d_{so(q)} = \frac{1}{2}\sum_l \mu(e^{*l})\theta_{so(q)e_l}, \tag{2.10}$$

$$d_\theta = \sum_l w\theta_{e_l} \otimes \mu(e^{*l}), \tag{2.11}$$

$$h_\Omega = \sum_l w\mu(\Omega^*(e^{*l})) \otimes i_{so(q)e_l}, \tag{2.12}$$

being e^{*l}, e_l a pair of dual bases for $so(q)^*$ and $so(q)$, and $\mu(e^{*l})$ is the multiplication by e^{*l}.

Over $\hat{A}_{\theta=0}$ we have

$$\hat{d} = -w \otimes d_{so(q)} + h_\Omega + \nabla_{i=0} \otimes 1. \tag{2.13}$$

For all $X, Y \in \Gamma T\hat{M}$, $\omega(X) = 0$ and $Y \in \Gamma T\hat{F}$ implies that $\omega([X, Y]) = 0$ [4]. Hence we can regard Ω^* as

$$\Omega^*\colon so(q)^* \to \Gamma\Lambda^2\hat{\nu}^* = \hat{A}^{0,2,0} = \hat{A}^{2,0}_{i=0} \otimes 1, \tag{2.14}$$

and so h_Ω is trihomogeneous of tridegree $(-1, 2, 0)$.

According to the bigradation of $\hat{A}_{i=0}$, $\nabla_{i=0}$ may be decomposed as the sum of the bihomogeneous operators $\nabla_{i=0;0,1}$, $\nabla_{i=0;1,0}$ and $\nabla_{i=0;2,-1}$ of bidegrees $(0,1)$, $(1,0)$ and $(2,-1)$ respectively. Then, by comparing bidegrees in (2.9) and (2.13) we obtain that over $\hat{A}$,

$$d_{\hat{\mathcal{F}}} = \nabla_{i=0;0,1} \otimes 1 = d_{\hat{\mathcal{F}}} \otimes 1, \tag{2.15}$$

$$\hat{d}_{1,0} = w \otimes d_{so(q)} + \hat{d}_\theta + \nabla_{i=0;1,0} \otimes 1 + h_\Omega, \tag{2.16}$$

and over $\hat{A}_{\theta=0}$,

$$\hat{d}_{1,0} = -w \otimes d_{so(q)} + \nabla_{i=0;1,0} \otimes 1 + h_{\Omega}. \tag{2.17}$$

We have analogous results for $(so(q), i_1, \theta_1, E_1(\hat{\mathscr{F}}), \hat{d}_1)$ with ω_1^*.

(2.18) PROPOSITION. *$H(E_1(\hat{\mathscr{F}})_{\theta_1=0})$ is finite-dimensional if and only if $H(E_1(\hat{\mathscr{F}})_{i_1=0,\theta_1=0})$ is finite-dimensional.*

Proof. Since $so(q)$ is reductive it follows that $H(E_1(\hat{\mathscr{F}})_{\theta_1=0})$ has finite type if and only if $H(E_1(\hat{\mathscr{F}})_{i_1=0,\theta_1=0})$ has finite type (Corollary VI of Section 9.5 of vol. III of [1]). The proof is completed because we have that $E_1^{u,v}(\hat{\mathscr{F}}) = 0$ if $u > q + q_0$ or $v > p$. □

3. Invariant cohomology

Let M and N be smooth manifolds. N is assumed to be connected, oriented and of dimension n. Let π_M and π_N denote the canonical projections of $M \times N$ over M and N respectively. By $\int_N$: $A_{cv}(M \times N) \to A(M)$ we mean the integration along the fiber of the trivial oriented fiberbundle π_M: $M \times N \to M$.

For $r \geq 0$ and any $\phi \in A_c^r(N)$ we may define the linear homogeneous operator of degree $r - n$

$$I_\phi \colon A(M \times N) \to A(M), \quad \alpha \mapsto \int_N \alpha \Lambda \pi_N^*(\phi). \tag{3.1}$$

Now let ϕ denote a fixed element of $A_c^n(N)$ such that $\int_N \phi = 1$. Then $I_\phi d = dI_\phi$ and $I_\phi \pi_M^* = 1$. Fix $b \in N$ and let j_b: $M \to M \times N$ denote the inclusion opposite b.

(3.2) THEOREM (Section 4.4 of vol. II of [1]). *There exists a linear homogeneous operator l: $A(M \times N) \to A(M)$ of degree -1 such that $I_\phi - j_b^* = dl + ld$.*

Proof. Let U be a contractible open neighbourhood of b. Given $\psi \in A_c^n(U)$ such that $\int_U \psi = 1$ there exists $X \in A_c^{n-1}(N)$ such that $\phi - \psi = dX$.

Let λ: $M \times U \to M \times N$ denote the inclusion. ψ determines an operator

$$\tilde{I}_\psi \colon A(M \times U) \to A(M)$$

such that $\tilde{I}_\psi \lambda^* = I_\psi$.

Let H: $U \times I \to U$ be any homotopy connecting 1_U with cte_b: $U \to b$ ($I = [0,1]$). Thus we have the homogeneous linear operator of degree -1,

$$\tilde{h}: A(M \times U) \to A(M \times U), \quad \alpha \mapsto \int_I i_{\partial/\partial t}(1_M \times h)^* \alpha \cdot dt,$$

satisfying $(1_M \times cte_b)^* - 1 = d\tilde{h} + \tilde{h}d$. If we define $l = I_X w - \tilde{I}_\psi \tilde{h} \lambda^*$, the theorem follows. □

Let G be a compact Lie group of dimension n and T: $M \times G \to M$ an action. For each $a \in G$ we define T_a to be the diffeomorphism of M given by the restriction of T to $M \times \{a\}$, and let R_a and L_a be the right and left translations of G. Assume that G has a left-invariant orientation and let Δ denote the unique left-invariant n-form such that $\int_G \Delta = 1$. We obtain the homogeneous linear operator

$$\rho = I_\Delta T^*: A(M) \to A(M), \quad \phi \mapsto \int_G T^* \phi \Lambda \pi_G^* \Delta. \tag{3.3}$$

By $A_I(M)$ and $H_I(M)$ we mean the differential subalgebra of T-invariant differential forms and the T-invariant cohomology of M respectively. Let j: $A_I(M) \to A(M)$ be the inclusion.

(3.4) Proposition (Section 4.3 of vol. II of [1]). $\rho j = 1$.

(3.5) Theorem (Section 4.3 of vol. II of [1]). *If G is compact and connected then*

$$j^*: H_I(M) \xrightarrow{\cong} H(M).$$

Proof. From (3.4) we obtain $\rho_* j_* = 1$. Let e denote the identity element of G. According to (3.2) we can define a linear homogeneous operator

$$l: A(M \times G) \to A(M)$$

of degree -1 such that $I_\Delta - j_e^* = dl + ld$. Then for $h = lT^*$ we obtain $j\rho - 1 = dh + hd$. □

Let $J\colon G \times G \to G$ be the smooth map defined by $(a, g) \mapsto a^{-1}ga$. For any $a \in G$ the restriction of J to $\{a\} \times G$ determines the interior automorphism $J_a = R_a L_{(a^{-1})}$. Then we define the homogeneous linear operator of degree 0,

$$\eta = I_\Delta J^*\colon A(G) \to A(G), \quad \phi \mapsto \int_G J^*\phi \Lambda \pi_G^* \Delta. \tag{3.6}$$

We have the differential subalgebra of $A(G)$ given by

$$A_{J^*=1}(G) = \bigcap_{a \in G} \operatorname{Ker}(J_a^* - 1). \tag{3.7}$$

Let $H_{J^*=1}(G)$ denote its cohomology and let $i\colon A_{J^*=i}(G) \to A(G)$ be the inclusion. With the same arguments as in (3.4) and (3.5) we get the following two results.

(3.8) PROPOSITION. $\eta i = 1$.

(3.9) THEOREM. *If G is compact and connected then* $i_*\colon H_{J^*=1}(G) \xrightarrow{\cong} H(G)$.

(3.10) LEMMA. *If G is compact and connected for defining l in the proof of* (3.5) (*following the proof of* (3.2)) *we can choose* ψ *and* X *belonging to* $A_{J^*=1}(G)$ *and* $H\colon U \times I \to U$ *satisfying* $R_a H_g = L_a H_{J_a(g)}$ *for any* $a \in G$ *and any* $g \in U$ *where* $H_g\colon I \to U$ *is the restriction of* H *to* $\{g\} \times I$.

Proof. Because G is compact we can take the canonical biinvariant Riemannian metric on G. For $\varepsilon > 0$ such that $\exp\colon B(0, \varepsilon) \xrightarrow{\cong} B(e, \varepsilon)$, $U = B(e, \varepsilon)$ is contractible and we can take the homotopy

$$H\colon U \times I \to U, \quad (g, t) \mapsto \exp((1 - t) \cdot \log(g))$$

connecting 1_U with $cte_e\colon U \to e$.

For $g \in U$, H_g is the unique geodesic in U joining g with e and defined in I. Hence $R_a H_g = L_a H_{J_a(g)}\colon I \to B(a, \varepsilon)$ because both ones are the unique geodesic in $B(a, \varepsilon)$ joining ga with a and defined in I.

Let Θ be the biinvariant volume form corresponding to the above Riemannian metric on G. Then $\Delta = (1/\int_G \Theta) \cdot \Theta$ is biinvariant and thus $\eta\Delta = \Delta$.

Let us take $\psi \in A_c(U)$ and $X \in A^{n-1}(G)$ such that $\int_G \psi = 1$ and $dX = \Delta - \psi$. For any $a \in G$, since J_a is an isometry with e as fixed point we have $\eta\psi \in A_c^n(U)$. From (3.8) and (3.9) we obtain $\int_G \eta\psi = \int_G \psi = 1$, and on the other hand $d\eta X = \eta dX = \Delta - \eta\psi$. So we can define l using $\eta\psi$, ηX and this H. □

(3.11) PROPOSITION. *In the proof of* (3.5) *l can be taken such that* $T_a h = h T_a$ *for any* $a \in G$.

Proof. Assume that l is defined as in (3.10). Fix $\phi \in A^r(M)$ and $x \in M$. Then

$$(T^*\phi)_x \in \sum_{s+t=r} A^s\big(G, \Lambda^t T_x^* M\big),$$

and its component of degree 1 may be represented by

$$(T^*\phi)_x^1 = \sum_i \alpha_i \otimes \gamma_i \in \Lambda^{r-1} T_x^* M \otimes A^1(G).$$

For any $a \in G$ we have $T_a T = T(1 \times R_a)$ and $T = T(T_{(a^{-1})} \times L_a)$, so we obtain

$$T^* T_a^* = (1 \times R_a)^* T^* \quad \text{and} \quad (T_{(a^{-1})} \times L_a)^* T^*.$$

On the other hand, since G is connected the right and left translations in G are orientation-reserving. Hence we have

$$\begin{aligned}
(I_X T^* T_a^* \phi)(x) &= \int_G (T^* T_a^* \phi)_x^1 \cdot X \\
&= \int_G (1 \times R_a)^* (T^*\phi)_x^1 \cdot X \\
&= \sum_i \alpha_i \cdot \int_G R_a \gamma_i^* \cdot X \\
&= \sum_i \alpha_i \cdot \int_G \gamma_i \cdot R_{(a^{-1})}^* X \\
&= \sum_i \alpha_i \cdot \int_G \gamma_i \cdot L_{(a^{-1})}^* X \\
&= T_a^* \sum_i T_{(a^{-1})}^* \alpha_i \cdot \int_G (L_a^* \gamma_i) \cdot X \\
&= T_a^* \int_G (T_{(a^{-1})} \times L_a)^* (T^*\phi)_x^1 \cdot X \\
&= T_a^* \int_G (T^*\phi)_{xa}^1 \cdot X \\
&= T_a^* (I_X T^*\phi)(xa) \\
&= (T_a^* I_X T^* \phi)(x).
\end{aligned}$$

Let $\xi = \tilde{I}_\psi \tilde{h}\lambda^* T^* T_a^* \phi$ and $\zeta = T_a^* \tilde{I}_\psi \tilde{h}\lambda^* T^* \phi$. We have

$$\begin{aligned}
\xi(x) &= \int_G \left(\int_I i_{\partial/\partial t}(1 \times H)^* T^* T_a^* \phi \cdot dt \right)_x \cdot \psi \\
&= \int_G \left(\int_I i_{\partial/\partial t}(1 \times H)^*(1 \times R_a)^*(T^*\phi)^1_x \cdot dt \right) \cdot \psi \\
&= \sum_i w(\alpha_i) \cdot \int_G \left(\int_I i_{\partial/\partial t} H^* R_a^* \gamma_i \cdot dt \right) \cdot \psi, \\
\zeta(x) &= T_a^* \left(\tilde{I}_\psi \tilde{h}\lambda^* T^* \phi \right)(xa) \\
&= T_a^* \int_G \left(\int_I i_{\partial/\partial t}(1 \times H)^*(T^*\phi)^1_{xa} \cdot dt \right) \cdot \psi \\
&= T_a^* \int_G \left(\int_I i_{\partial/\partial t}(1 \times H)^*(T_{(a^{-1})} \times L_a)^*(T^*\phi)^1_x \cdot dt \right) \cdot \psi \\
&= \sum_i w(\alpha_i) \cdot \int_G \left(\int_I i_{\partial/\partial t} H^* L_a^* \gamma_i \cdot dt \right) \cdot \psi.
\end{aligned}$$

For any $g \in U$ and for any i it is easy to prove that

$$\begin{aligned}
\left(i_{\partial/\partial t} H^* R_a^* \gamma_i \cdot dt \right)_g &= H_g^* R_a^* \gamma_i, \\
\left(i_{\partial/\partial t} H^* L_a^* \gamma_i \cdot dt \right)_g &= H_g^* L_a^* \gamma_i
\end{aligned}$$

in $A^1(I)$. Then

$$\begin{aligned}
\left(\int_I i_{\partial/\partial t} H^* R_a^* \gamma_i \cdot dt \right)(g) = \int_I H_g^* R_a^* \gamma_i &= \int_I H_{J(g)}^* L_a^* \gamma_i \\
&= \left(\int_I i_{\partial/\partial t} H^* L_a^* \gamma_i \cdot dt \right)(J(g))
\end{aligned}$$

and so we obtain

$$\begin{aligned}
\xi(x) &= \sum_i w(\alpha_i) \cdot \int_G \left(\left(\int_I i_{\partial/\partial t} H^* L_a^* \gamma_i \cdot dt \right) J_a \right) \cdot \psi \\
&= \sum_i w(\alpha_i) \cdot \int_G J_a^* \left(\left(\int_I i_{\partial/\partial t} H^* L_a^* \gamma_i \cdot dt \right) \cdot \psi \right) \\
&= \sum_i w(\alpha_i) \cdot \int_G \left(\int_I i_{\partial/\partial t} H^* L_a^* \gamma_i \cdot dt \right) \cdot \psi = \zeta(x).
\end{aligned}$$

Therefore, recalling the definition of l and h, the theorem follows. $\square$

4. $E_2(\hat{A}_{\theta=0}) = E_2(\hat{\mathscr{F}})$

We consider a Riemannian and transversely oriented foliation $\mathscr{F}$ of a manifold M. In this section the notation established in Section 2 remains in force. Then, let T: $\hat{M} \times SO(q) \to \hat{M}$ be the action of $SO(q)$ on $\hat{M}$. It follows that the algebra $\hat{A}_{\theta=0}$ is equal to the algebra of T-invariant differential forms on $\hat{M}$, and let j: $\hat{A}_{\theta=0} \to \hat{A}$ be the inclusion. Since $SO(q)$ is compact and connected, according to Section 3 we can construct the linear homogeneous operators ρ: $\hat{A} \to \hat{A}_{\theta=0}$ and h: $\hat{A} \to \hat{A}$ of degrees 0 and -1 respectively, such that $\rho\hat{d} = \hat{d}\rho$, $\rho j = 1$ and $j\rho - 1 = \hat{d}h + h\hat{d}$.

The deRham complex of $\hat{M} \times SO(q)$ may be decomposed as the direct sum of the following spaces

$$A^{s,t,u,v}(\hat{M} \times SO(q)) = \Gamma\big(\Lambda^v(T^*\hat{\mathscr{F}} \times SO(q)) \otimes \Lambda^u(\hat{\nu}^* \times SO(q)) \otimes \Lambda^t(V^* \times SO(q)) \otimes \Lambda^s(\hat{M} \times T^*SO(q))\big) \quad (4.1)$$

for $s, t, u, v \geq 0$, where $SO(q)$ and $\hat{M}$ are identified with the trivial vector-bundles over themselves. Then, recalling the definitions of ρ and h we have an analogous decomposition for the deRham complex of $\hat{M} \times U$ and the following three lemmas have easy but tedious proofs.

(4.2) LEMMA. $T^*(\hat{A}^{t,u,v}) \subset \sum_{0 \leq s \leq t} A^{s,t-s,u,v}(\hat{M} \times SO(q))$.

(4.3) LEMMA. *If* $\phi \in A(SO(q))$ *then*

$$I_\phi\big(A^{s,t,u,v}(\hat{M} \times SO(q))\big) \subset \hat{A}^{t,u,v},$$

and it is 0 *if* $s \neq q_0 - \deg(\phi)$.

(4.4) LEMMA. $\tilde{h}(A^{s,t,u,v}(\hat{M} \times U)) \subset A^{s-1,t,u,v}(\hat{M} \times U)$.

Applying (4.2), (4.3) and (4.4) we get:

(4.5) PROPOSITION. *ρ and h are trihomogeneous of tridegrees* $(0,0,0)$ *and* $(-1,0,0)$ *respectively.*

Therefore for all $i \geq 0$ we have

$$E_i(\hat{A}_{\theta=0}) \underset{\rho_i}{\overset{j_i}{\rightleftarrows}} E_i(\hat{\mathscr{F}}) \quad (4.6)$$

where $\rho_i j_i = 1$. And comparing bidegrees we have

$$d_{\hat{\mathscr{F}}}h + hd_{\hat{\mathscr{F}}} = 0 \quad \text{and} \quad \hat{d}_{1,0}h + h\hat{d}_{1,0} = j\rho - 1. \quad (4.7)$$

Hence for $u, v \geq 0$ we obtain

$$h_1 \colon E_1^{u,v}(\hat{\mathscr{F}}) \to E_1^{u-1,v}(\hat{\mathscr{F}}) \tag{4.8}$$

where $h_1\hat{d}_1 + \hat{d}_1 h_1 = j_1\rho_1 - 1$. Thus $j_2\rho_2 = 1$ and we have the following result.

(4.9) THEOREM. $j_2 \colon E_2(\hat{A}_{\theta=0}) \xrightarrow{\cong} E_2(\hat{\mathscr{F}})$.

5. $E_2(\mathscr{F})$ is finite-dimensional for $\mathscr{F}$ Riemannian and M closed

A smooth foliation is called transitive if evaluating all its infinitesimal transformations at each point we get all the tangent vectors [5].

(5.1) THEOREM [5]. *If a smooth closed manifold carries a transitive foliation then the second term of its spectral sequence is finite-dimensional.*

Clearly every transversely parallelizable foliation is transitive, (this is false for Riemannian foliations). Then, going back to our cases in Sections 2 and 4, we see that $\hat{\mathscr{F}}$ is transitive, and if M is closed so is $\hat{M}$. Thus we have the following consequence.

(5.2) COROLLARY. *If M is closed then $E_2(\hat{\mathscr{F}})$ is finite-dimensional.*

By (3.11), $h \colon \hat{A} \to \hat{A}$ can be taken such that $h\theta_Z = \theta_Z h$ for each Z in $so(q)$, then $h_1\theta_{1Z} = \theta_{1Z}h_1$ and

$$h_1\left(E_1(\hat{\mathscr{F}})_{\theta_1=0}\right) \subset E_1(\hat{\mathscr{F}})_{\theta_1=0}. \tag{5.3}$$

(5.4) PROPOSITION. $j_2 \colon E_2(\hat{A}_{\theta=0}) \xrightarrow{\cong} H(E_1(\hat{\mathscr{F}})_{\theta_1=0})$.

Proof. It follows because we have the restrictions

$$E_1\left(\hat{A}_{\theta=0}\right) \underset{\rho_1}{\overset{j_1}{\rightleftarrows}} E_1(\hat{\mathscr{F}})_{\theta_1=0} \quad \text{and} \quad h_1 \colon E_1(\hat{\mathscr{F}})_{\theta_1=0} \to E_1(\hat{\mathscr{F}})_{\theta_1=0}$$

where $\rho_1 j_1 = 1$ and $j_1\rho_1 - 1 = h_1\hat{d}_1 + \hat{d}_1 h_1$. $\square$

By (2.6), (2.8) and (2.15) we have $\hat{A} = \hat{A}_{i=0} \otimes \Lambda so(q)^*$, $1_Z = 1 \otimes i_{so(q)Z}$ for $Z \in so(q)$, and $d_{\hat{\mathscr{F}}} = d_{\hat{\mathscr{F}}} \otimes 1$. Then $E_1(\hat{\mathscr{F}}) = H(\hat{A}_{i=0}, d_{\hat{\mathscr{F}}}) \otimes \Lambda so(q)^*$ and $i_{1Z} = 1 \otimes i_{so(q)Z}$, so

$$E_1(\hat{\mathscr{F}})_{i_1=0} = H\left(\hat{A}_{i=0}, d_{\hat{\mathscr{F}}}\right). \tag{5.5}$$

Since $\hat{A}_{i=0} = \hat{A}^{0,\cdot,\cdot}$ and ρ preserves the trigraduation of $\hat{A}$ we have the restrictions

$$A = \hat{A}_{i=0,\theta=0} \underset{\rho}{\overset{j}{\rightleftarrows}} \hat{A}_{i=0}, \tag{5.6}$$

which are compatible with $d_{\mathscr{F}}$ and $d_{\hat{\mathscr{F}}}$, and such that $\rho j = 1$. Hence we obtain the homomorphisms of bigraded differential algebras,

$$E_1(\mathscr{F}) \underset{\rho_1}{\overset{j_1}{\rightleftarrows}} H(\hat{A}_{i=0}, d_{\hat{\mathscr{F}}})_{\theta_1=0} = E_1(\hat{\mathscr{F}})_{i_1=0,\theta_1=0}, \tag{5.7}$$

where $\rho_1 j_1 = 1$. Also, since h is of tridegree $(-1,0,0)$ we have $h(\hat{A}_{i=0}) = 0$, and by (5.5),

$$h_1(E_1(\hat{\mathscr{F}})_{i_1=0,\theta_1=0}) = 0. \tag{5.8}$$

Thus, as in (5.4), we obtain

$$E_1(\mathscr{F}) = E_1(\hat{\mathscr{F}})_{i_1=0,\theta_1=0} \quad \text{and} \quad E_2(\mathscr{F}) = H(E_1(\hat{\mathscr{F}})_{i_1=0,\theta_1=0}). \tag{5.9}$$

(5.10) THEOREM. *If a smooth closed manifold M carries a Riemannian foliation $\mathscr{F}$ then $E_2(\mathscr{F})$ is finite-dimensional.*

Proof. Let $\tilde{\mathscr{F}}$ be the lift of $\mathscr{F}$ to the 2-sheeted covering $\tilde{M}$ of transverse orientation of $\mathscr{F}$. Since M is closed so is $\tilde{M}$, thus from (2.18), (4.9), (5.2), (5.4) and (5.9), $E_2(\tilde{\mathscr{F}})$ is finite-dimensional. Then so is $E_2(\mathscr{F})$ by standard arguments. □

6. The spaces $E_2^{0,\cdot}(\mathscr{F})$ and $E_2^{1,\cdot}(\mathscr{F})$

In the preceding section we have $\hat{A}_{\theta=0} = A \oplus (\hat{A}_{i=0} \otimes \Lambda^+ so(q)^*)_{\theta=0}$ where $d_{\hat{\mathscr{F}}} = d_{\mathscr{F}}$ over A and $d_{\hat{\mathscr{F}}} = d_{\hat{\mathscr{F}}} \otimes 1$ over $(\hat{A}_{i=0} \otimes \Lambda^+ so(q)^*)_{\theta=0}$. This implies that for $0 \le v \le p$ and $0 \le u \le q$,

$$E_1^{u,v}(\hat{A}_{\theta=0}) = E_1^{u,v}(\mathscr{F}) \oplus H^{1,u-1,v} \oplus \cdots \oplus H^{u,0,v} \tag{6.1}$$

where $H^{s,t,v} = H^v((\hat{A}^{t,\cdot}_{i=0} \otimes \Lambda^s so(q)^*)_{\theta=0}, d_{\hat{\mathscr{F}}})$ for $s + t = u$. Then, from (2.17), for $0 \leq v \leq p$, we obtain

$$\begin{tikzcd}
E_1^{0,v}(\hat{A}_{\theta=0}) \arrow[d, "\hat{d}_1"'] & = & E_1^{0,v}(\mathscr{F}) \arrow[d, "d_1"'] & & & & \\
E_1^{1,v}(\hat{A}_{\theta=0}) \arrow[d, "\hat{d}_1"'] & = & E_1^{1,v}(\mathscr{F}) \arrow[d, "d_1"'] & \oplus & H^{1,0,v} \arrow[dll, "h_\Omega{}^*"] \arrow[d, "-(w\otimes d_{so(q)})_*"'] \arrow[drr, "(\nabla_{i=0,1,0}\otimes i)_*"] & & \\
E_1^{2,v}(\hat{A}_{\theta=0}) & = & E_1^{2,v}(\mathscr{F}) & \oplus & H^{1,1,v} & \oplus & H^{2,0,v}
\end{tikzcd} \tag{6.2}$$

where the derivative $\hat{d}_1$ is decomposed as the sum of the operators on the right side. Hence we have the following result.

(6.3) PROPOSITION. (i). $E_2^{0,\cdot}(\hat{\mathscr{F}}) = E_2^{0,\cdot}(\mathscr{F})$.
(ii) $E_2^{1,\cdot}(\hat{\mathscr{F}}) = E_2^{1,\cdot}(\mathscr{F}) \oplus \operatorname{Ker}(\hat{d}_1 \colon H^{1,0,v} \to E_1^{2,v}(\hat{A}_{\theta=0}))$.

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