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A MAXIMAL OPERATOR RELATED TO A CLASS OF SINGULAR INTEGRALS

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1. Introduction

We are interested in finding a class of kernels $\mathbf{M}$ such that we have the maximal operator

$$\sup_{K\in\mathbf{M}}\left|\text{p.v.}\int_{R^n}K(x-y)f(y)\right|$$

bounded on some L^p spaces. As a first approach, we consider the dimension of the space n to be bigger than 1 and let K have the form $h(|x|)\Omega(x')/|x|^n$ where Ω is a homogeneous function, continuous with mean 0 on S_{n-1}, and h is a radial function. These kernels could be gotten, for example, when we decompose a kernel K, satisfying the growth condition of Calderón-Zygmund kernels $|K(x)| \le C/|x|^n$, into its radial and spherical parts

$$\sum_k h_k(r)Y_k(x')\Big/|x|^n$$

where Y_k are the spherical harmonics. In this paper we consider the case when $\mathbf{M}$ is the set with the radial function h satisfying

$$\left(\int_0^\infty |h(r)|^S\frac{dr}{r}\right)^{1/S} \le 1.$$

We show that for $1 \le S \le 2$, the maximal operator is bounded on $L^p(R^n)$, $p > s_n/(s_{n-1})$. And this range of p is the best possible.

Here, we should remark that some ideas of the proof are from the paper of J. Duoandikoetxea and R.L. Rubio de Francia [2], and [3] of E.M. Stein.

2. Result and proof

THEOREM. *Let $n \ge 2$ and $\Omega \in C(S^{n-1})$ with $\int_{S^{n-1}}\Omega(\xi)\,d\sigma(\xi) = 0$ where $d\sigma$ is the surface measure of S^{n-1} and Ω is of homogeneous of degree zero. Let*

$$T(f)(x) = \sup_h\left|\int h(|y|)\frac{\Omega(y)}{|y|^n}f(x-y)\,dy\right|,$$

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where the supremum is over the set $\|h\|_{L^S(R^+, dr/r)} \le 1$. *Then*

$$\|T(f)\|_p \le A_p \|f\|_p$$

for $1 \le S \le 2$, $p > S_n/(S_{n-1})$. *This range of p is the best possible.*

Proof. We first show that the range of p is limited. We assume

$$f(x) = \begin{cases} \dfrac{1}{|x|^{n-\alpha}}, & 0 < |x| \le 10, \\ 0, & |x| > 10, \end{cases}$$

where $\alpha < 1$. Thus $T(f)$, by duality, is simply

$$\left[\int_0^\infty \left| \int_{S^{n-1}} \Omega(\xi) \frac{\chi(x - r\xi)}{|x - r\xi|^{n-\alpha}}\, d\sigma(\xi) \right|^{S'} \frac{dr}{r}\right]^{1/S'}$$

where χ is the characteristic function of the set $|x| \le 10$.
For each x, $|x| < 1$, let $x' = x/|x|$. Then

$$\begin{aligned}(Tf(x))^{S'} &\ge \int_{|x|}^{2|x|} \left| \int_{S^{n-1}} \Omega(\xi) \frac{\chi(|x - r\xi)}{|x - r\xi|^{n-\alpha}}\, d\sigma(\xi) \right|^{S'} \frac{dr}{r} \\ &\approx \frac{C}{|x|^{S'(n-\alpha)}} \int_1^2 \left| \int_{S^{n-1}} \Omega(\xi) \frac{1}{|x'/r - \xi|^{n-\alpha}}\, d\sigma(\xi) \right|^{S'} dr\end{aligned}$$

since $\chi(x - r\xi) \equiv 1$ when $|x| \le 1$. Denote by $I(x', r)$ the integral over S^{n-1}. Let B be a ball in R^n centered at x' with radius ε. We are going to pick $\varepsilon > 0$ small enough so that $\Omega(\xi)$ is basically constant on the set $B \cap S^{n-1}$. Now we wish to estimate the rate of growth of $I(x', r)$ as r approaches 1. We have

$$I(x', r) = \int_{B \cap S^{n-1}} + \int_{B^C \cap S^{n-1}} = I_1 + I_2.$$

It is clear that as r close to 1, I_2 is bounded by

$$\|\Omega\|_\infty \int_{S^{n-1}} \varepsilon^{\alpha - n}\, d\sigma(\xi) \le C_\varepsilon \|\Omega\|_\infty.$$

For I_1, we use a change of variable to the tangent plane of S^{n-1} at x'. Since

$r > 1$ we have

$$\left|x' - \frac{x'}{r}\right| < \left|\xi - \frac{x'}{r}\right| \quad \text{for } x' \neq \xi \in B \cap S^{n-1}.$$

We can pick a suitable point P on the tangent plane at x' such that

$$\left|x' - \frac{x'}{r}\right| + |P| = \left|\xi - \frac{x'}{r}\right|.$$

The change of variable is the mapping ϕ: $\xi \to P$. Letting $t = |P|$ and $u = 1 - 1/r$ we have

$$\begin{aligned} I_1(x', r) &\geq C\int_0^{\varepsilon} \frac{t^{n-2}}{\left(u^2 + t^2\right)^{(n-\alpha/2)}}\, dt \\ &= Cu^{-1+\alpha} \int_0^{\varepsilon/u} \frac{t^{n-2}}{\left(1 + t^2\right)^{(n-\alpha)/2}}\, dt. \end{aligned}$$

This means that $I(x', r)$ blows up at least on the order of $|r - 1|^{-1+\alpha}$ as r approaches 1. Thus $T(f)(x) = \infty$ when $\int_1^2 |I(x', r)|^{S'}\, dr = \infty$ or $\alpha \leq 1/S$. This implies that $T(f)$ is not in any L^q space when $\alpha \leq 1/S$, or correspondingly, when

$$f \in L^p(R^n) \quad \text{for } p < Sn/(Sn - 1).$$

To rule out the case $p = Sn/(Sn - 1)$, we simply let

$$f(x) = |x|^{-n+1/S}(\log 1/|x|)^{-1}\chi_{|x|<10}(x).$$

Now, let us consider the case $S' = 2$. By duality,

$$\begin{aligned} T(f)(x) &= \left(\int_0^{\infty} \left|\int_{S^{n-1}} \Omega(\xi) f(x - r\xi)\, d\sigma(\xi)\right|^2 \frac{dr}{r}\right)^{1/2} \\ &= \left(\sum_{k=-\infty}^{\infty} \int_1^2 \left|\int_{S^{n-1}} \Omega(\xi) f(x - 2^k r\xi)\, d\sigma(\xi)\right|^2 \frac{dr}{r}\right)^{1/2}. \end{aligned}$$

Let us take a smooth function $p(r)$ supported on $\{r \mid 1/2 < |r| < 2\}$ and $\Sigma_k p(2^k r) = 1$. We define the partial sum operators

$$\widehat{S_k f} = p(2^k|x|)\hat{f}(x).$$

Since $f = \Sigma_j(S_{k+j}f)$ for any k,

$$
\begin{aligned}
|T(f)(x)| &\le \left(\sum_k \int_1^2 \left|\sum_j \int_{S^{n-1}} \Omega(\xi)(S_{k+j}f)(x - 2^k r\xi)\, d\sigma(\xi)\right|^2 \frac{dr}{r}\right)^{1/2} \\
&\le \left(\sum_k \left(\sum_j \left(\int_1^2 \left|\int_{S^{n-1}} \Omega(\xi)(S_{k+j}f)(x - 2^k r\xi)\, d\sigma(\xi)\right|^2 \frac{dr}{r}\right)^{1/2}\right)^2\right)^{1/2} \\
&\le \sum_j \left(\sum_k \int_1^2 \left|\int_{S^{n-1}} \Omega(\xi)(S_{k+j}f)(x - 2^k r\xi)\, d\sigma(\xi)\right|^2 \frac{dr}{r}\right)^{1/2} \\
&\equiv \sum_j T_j(f)(x),
\end{aligned}
$$

where the last two inequalities are obtained by applying Minkowski's inequality. First, let us compute

$$\|T_j(f)\|_2^2 = \int_{R^n} \sum_k \int_1^2 \left|\int_{S^{n-1}} \Omega(\xi)(S_{k+j}f)(x - 2^k r\xi)\, d\sigma(\xi)\right|^2 \frac{dr}{r}\, dx.$$

By Plancherel's theorem, and Fubini's theorem, the last equality is dominated by

$$\sum_k \int_{2^{-(k+j)-1} \le |x| \le 2^{-(k+j)+1}} \left\{\int_1^2 \left|\int_{S^{n-1}} \Omega(\xi) e^{i2^k r\xi\cdot x}\, d\sigma(\xi)\right|^2 \frac{dr}{r}\right\} |\hat{f}(x)|^2\, dx.$$

We claim that the term in parentheses is bounded by

$$C \min\left\{2^k|x|, (2^k|x|)^{-\alpha}\right\},$$

for some positive number α. Applying the cancellation of Ω, it is easy to see the term in parentheses is bounded by $C2^k|x|$. On the other hand, by the second mean value theorem, the term in parentheses is bounded by

$$
\begin{aligned}
&\int_1^2 \left|\int_{S^{n-1}} \Omega(\xi) e^{i2^k r\xi\cdot x}\, d\sigma(\xi)\right|^2 dr \\
&\quad \le \|\Omega\|_\infty^2 \int_{S^{n-1}} \int_{S^{n-1}} \left|\int_1^2 e^{i2^k r|x|(\xi-\eta)\cdot x'}\, dr\right| d\sigma(\eta)\, d\sigma(\xi).
\end{aligned}
$$

The integral in the absolute value sign is bounded by 1 and $(2^k|x|(\xi - \eta) \cdot x')^{-1}$; hence it is less than $(2^k|x|(\xi - \eta) \cdot x')^{-\alpha}$ where $0 < \alpha < 1$. So

$$\|T_j(f)\|_2^2 \le C \min\{2^j, (2^j)^{-\alpha}\}\|f\|_2. \tag{1}$$

Next, we compute the L^p-norm of $T_j f$. For $p \ge 2$, there exists a function g in $L^{(p/2)'}$ such that

$$\|T_j(f)\|_p^2 \le C\|\Omega\|_\infty^2 \sum_k \int_{R^n} \int_1^2 \int_{S^{n-1}} |(S_{j+k}f)(x - 2^k r\xi)|^2 \, d\sigma(\xi) \frac{dr}{r} |g(x)| \, dx$$

By Fubini's theorem, the formula above becomes

$$\begin{aligned} C\sum_k \int_{R^n} |S_{j+k}f(x)|^2 \int_1^2 \int_{S^{n-1}} |g(x + 2^k r\xi)| \, d\sigma(\xi) \frac{dr}{r} \, dx \\ \le C\int_{R^n} \sum_k |S_{j+k}f(x)|^2 Mg(x) \, dx \\ \le C\left\| \sum_k |S_{k+j}f|^2 T \right\|_{p/2} \|Mg\|_{(p/2)'} \end{aligned}$$

where Mg denotes the classical Hardy-Littlewood Maximal function. By the Littlewood-Paley theorem and the fact that the maximal function, Mg, is bounded on $L^p(R^n)$ for $1 < p \le \infty$, we have

$$\|T_j(f)\|_p \le C\|f\|_p. \tag{2}$$

Interpolating between (1) and (2), and applying Minkowski's inequality, we have

$$\|T(f)\|_p \le C\|f\|_p,$$

if $2 \le p < \infty$.

Before we show the case $2n(2n - 1) < p < 2$, we need the following lemma.

LEMMA. *Let $g_k(x, r)$ be the arbitrary functions defined on $R^n \times R^+$. If $2n > p > 2$ then*

$$\left\| \left(\sum_k \int_1^2 \left| \int_{S^{n-1}} \Omega(\xi) g_k(x - 2^k r\xi, r) \, d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2} \right\|_p \le C \left\| \left(\sum_k \int_1^2 |g_k(\cdot, r)|^2 \frac{dr}{r} \right)^{1/2} \right\|_p .$$

Proof. As above, if $p > 2$, there exists a function h in $L^{(p/2)'}(R^n)$ such that the left hand side of above equation equals

$$\left(\sum_k \int_{R^n}\int_1^2 \left|\int_{S^{n-1}} \Omega(\xi) g_k(x - 2^k r\xi, r)\, d\sigma(\xi)\right|^2 \frac{dr}{r} h(x)\, dx\right)^{1/2}.$$

Following the same procedure as above, it is easy to see the above formula is dominated by

$$\left(\int_{R^n}\sum_k \int_1^2 |g_k(x,r)|^2 \int_{S^{n-1}} |h(x + 2^k r\xi)|\, d\sigma(\xi)\, \frac{dr}{r}\, dx\right)^{1/2}$$

$$\leq C\left(\left\|\sum_k \int_1^2 |g_k(\cdot, r)|^2 \frac{dr}{r}\right\|_{p/2} \|M_S(h)\|_{(p/2)'}\right)^{1/2},$$

where $M_S(h)$ denotes the spherical maximal function. The lemma follows by the fact that $M_S(h)$ is bounded on $L^r(R^n)$ if $r > n/(n-1)$ (see [1], [3]). The lemma is proved.

Now we prove the case $2n/(2n-1) < p < 2$. By a duality argument, there exist functions $g_k(x, r)$ defined on $R^n \times R^+$ with $\| \,\| \, \|g_k\|_{L^2(dr/r)}\|_{l^2}\|_{L^{p'}} \leq 1$ such that

$$\|T_j(f)\|_p = \int_{R^n}\sum_k \int_1^2 \int_{S^{n-1}} \Omega(\xi)(S_{k+j}f)(x - 2^k r\xi)\, d\sigma(\xi) g_k(x, r) \frac{dr}{r}\, dx.$$

After changing variables and applying Hölder's inequality and the lemma, the L^p-norm of $T_j(f)$ is dominated by $\|(\Sigma_k |S_{k+j}f|^2)^{1/2}|_p$. Again, by the Littlewood-Paley theorem, we have

$$\|T_j(f)\|_p \leq C\|f\|_p, \tag{3}$$

if $2 > p > 2n/(2n-1)$. The case, $S = 2$, is proved by interpolating between (1) and (3). On the other hand, it is clear that $T(f)$ is dominated by the Spherical maximal function if $S = 1$.

To show that Tf is bounded on $L^p(R^n)$, for $1 < S < 2$, it suffices to show that the operator

$$\int_0^\infty h(r, x) \int_{S^{n-1}} \Omega(\xi) f(x - r\xi)\, d\sigma(\xi)\, \frac{dr}{r}$$

is bounded, where $h(r, x)$ is an arbitrary measurable function and the L^S-norm of $h(\cdot, x)$ is not bigger than 1 for every x. Therefore we may define a family of operators,

$$T^{\alpha}f(x) = \int_0^{\infty} |h(r, x)|^{(1-\alpha/2)S} \operatorname{sign}\{h(r, x)\}$$
$$\times \int_{S^{n-1}} \Omega(\xi) f(x - r\xi)\, d\sigma(\xi)\, \frac{dr}{r},$$

where α are complex numbers. It is clear that $T^{\alpha}f = Tf$ if $\alpha = 2(1 - 1/S)$. Then we have our theorem by interpolating between $\operatorname{Re}(\alpha) = 0$ (the boundedness of the operator corresponds to $S = 1$) and $\operatorname{Re}(\alpha) = 1$ (the $S = 2$ case).

Remark. In [6], it was pointed out that when $S = \infty$, there exists a function $f \in L^{p}$, $1 < p < \infty$ so that the maximal operator acting on f yields an identically infinity function.

References

1. J. Bourgain, *Estimations de Certaines Functions Maximales*, C. R. Acad. Sci. Paris Sér. I Math., vol. 301 (1985), pp. 499–502.
2. J. Duoandikoetxea and R.L. Rubio de Francia, *Maximal and singular operators via Fourier transform estimates*", Invent. Math., vol. 84 (1986), pp. 541–561.
3. E.M. Stein, *Maximal functions: spherical means*, Proc. Nat. Acad. Sci. U.S.A., vol. 73 (1976), pp. 2174–2175.
4. ______, *Singular integrals and differentiability properties of functions*, Princeton University press, Princeton, N.J., 1970.
5. E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton, N. J., 1975.
6. J. Ash, P. Ash, C. Fefferman and R. Jones, *Singular integral operators with complex homogeneity*, Studia Math., vol. LXV (1979), pp. 31–50.

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