In this paper we study the extremal structure of the unit ball of a Lebesgue Bochner function space. Throughout, \( X \) will denote a Banach space, \( B_X \) the unit ball, \( S_X \) the unit sphere, \( X^* \) the dual space of \( X \), \( (\Omega, \Sigma, \mu) \) a positive measure space, and \( 1 < p, q < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

Let \( K \) be a subset of \( X \). A point \( x \) in \( K \) is a point of sequential continuity of \( K \) if for every sequence \( \{x_n\} \) in \( K \), \( \text{weak-lim}_n x_n = x \) implies \( \lim_n \|x_n - x\| = 0 \). The point of sequential continuity is a generalization of the point of continuity. A space \( X \) has the Kadec-Klee property if every point \( x \) in \( S_X \) is a point of sequential continuity of \( B_X \).

It is well-known that if \( (\Omega, \Sigma, \mu) \) is not purely atomic, then \( L^p(\mu, X) \) with the Kadec-Klee property must be strictly convex. This result, due to M. Smith and B. Turett [ST], is one of the most surprising results in the theory of Lebesgue-Bochner function spaces. Our first main result (Theorem 2.2) asserts that if \( (\Omega, \Sigma, \mu) \) is atom-free, then every point of sequential continuity of \( B_{L^p}(\mu, X) \) must be an extreme point of \( B_{L^p}(\mu, X) \). This gives a local version of the result of Smith and Turett.

Theorem 2.2 has several interesting consequences; for example, it implies that if \( (\Omega, \Sigma, \mu) \) is not purely atomic then:

(i) The Radon-Nikodym Property (RNP) and the Convex Point of Continuity Property (CPCP) are equivalent for \( L^p(\mu, X) \) and \( L^p(\mu, X)^* \).

(ii) The super-RNP and the super-CPCP are equivalent for \( L^p(\mu, X) \) and \( L^p(\mu, X)^* \).

Recall that the RNP implies the PCP (Point of Continuity Property) which, in turn, implies the CPCP, and that RNP, PCP, and CPCP are distinct [BR], [GMS1]. It follows that if \( X \) has the PCP but fails the RNP, and if \( (\Omega, \Sigma, \mu) \) is not purely atomic, then \( L^p(\mu, X) \) does not have the CPCP. Consequently, neither the PCP nor the CPCP can be "lifted" from \( X \) to \( L^p(\mu, X) \). We would like to mention (1) it is still an open problem whether the super-RNP and the super-CPCP are equivalent in general, (2) the RNP and the CPCP are equivalent for Banach spaces with the Krein-Milman Property [Sc], and
(3) the RNP and the PCP are equivalent for Banach lattices not containing isomorphic copies of $c_0$ [GM].

Let $f$ be a norm one element in $L^p(\mu, X)$. The condition that for almost all $t$ in the support of $f$ such that $f(t)/\|f(t)\|$ is an extreme point of $B_X$ is strictly stronger than the condition that $f$ is an extreme point of the unit ball of $L^p(\mu, X)$ [G]. We do not know whether the conclusion of Theorem 2.2 can be strengthened so that $f(t)/\|f(t)\|$ is an extreme point of $B_X$ for almost all $t$ in supp $f$. It is shown, however, that if $(\Omega, \Sigma, \mu)$ is atom-free and that $f$ is a $\sigma(L^p(\mu, X), L^q(\mu, X^*))$-point of sequential continuity of $B_{L^p(\mu, X)}$, then $f(t)/\|f(t)\|$ is a strongly extreme point of $B_X$ for almost all $t$ in supp $f$, thus $f$ is in fact a strongly extreme point of $B_{L^p(\mu, X)}$ in this case.

Another generalization of the point of continuity is the point of small combination of slices (SCS-points, for short). It is known [GGMS] that $X$ is strongly regular if and only if every non-empty bounded closed convex set $K$ in $X$ is contained in the norm-closure of SCS($K$). Schachermayer [Sc] proved that a Banach space has the RNP if and only if it is strongly regular and it has the Krein-Milman Property. We will show that the “point-version” of this result is also true; i.e., if $K$ is a closed convex set in $X$ and $x \in K$, then $x$ is a denting point of $K$ if and only if $x$ is both a SCS-point and an extreme point of $K$. An example is given to show that we can not replace the point of sequential continuity by the SCS-point in Theorem 2.2.

The main tool used in the proof of Theorem 2.2 is developed in Section I, where we study the weak-convergence of sequences of vector-valued Rademacher functions. The major part of Section II is devoted to the proof of Theorem 2.2 and its consequences.

Section I

The usual Rademacher functions are associated with the dyadic partitions of the unit interval. To define our “Rademacher functions” we use countable partitions of $\Omega$ and a special index set.

Let $T$ be the set consisting of all the finite sequences of positive integers with the natural partial order; i.e., $(i_1, \ldots, i_m) \leq (j_1, \ldots, j_n)$ if and only if $m \leq n$ and $i_k = j_k$, $k = 1, \ldots, m$, and with the empty set $\emptyset$ as the smallest element in $T$. For $\alpha \in T$, let $|\alpha|$ be the cardinality of $P_\alpha$ where $P_\alpha = \{\beta: \beta \in T, \beta < \alpha\}$ and let $T_n = \{\alpha: \alpha \in T, |\alpha| = n\}$, $n \geq 0$. If $\alpha = (i_1, \ldots, i_m)$ and $i$ is a natural number, then we also use $\alpha i$ to denote $(i_1, \ldots, i_m, i)$.

We call a “subset” $\{E_\alpha\}_{\alpha \in T}$ of $\Sigma$ a Rademacher tree of measurable sets if it satisfies the following conditions:

For all $k \geq 0$ and $\alpha \in T_k$, $\{E_{\alpha n}\}_{n \geq 1}$ is a partition of $E_\alpha$ and $\mu(E_{a2n-1}) = \mu(E_{a2n})$, and $\mu(E_\alpha) < \infty$.

We say that a sequence $\{f_\alpha\}$ of functions from $\Omega$ to $X$ is Rademacher if there are a Rademacher tree $\{E_\alpha\}_{\alpha \in T}$ in $\Sigma$ and $\{x_\alpha\}_{\alpha \in T}$ in $X$, $\alpha \in T$ such
that for \( k \geq 0, \)

\[
f_k = \sum_{\alpha \in T_k} x_\alpha \sum_{n \geq 1} (-1)^n \chi_{E_{\alpha n}}.
\]

Each \( f_k \) is called a Rademacher function, and \( \{E_{\alpha n}\}_{\alpha \in T} \) is called a Rademacher tree associated to \( \{f_k\} \), and \( \{f_k\} \) is said to be determined by \( \{E_{\alpha n}\}_{\alpha \in T} \). We use \( \Sigma(T) \) to denote the sub-\( \sigma \)-algebra of \( \Sigma \) generated by the tree \( \{E_{\alpha n}\}_{\alpha \in T} \). It is obvious that each \( f_k \) is \( \Sigma(T) \)-measurable.

**Proposition 1.1.** Every bounded Rademacher sequence in \( L^p(\mu, X) \) is null with respect to the \( \sigma(L^p(\mu, X), L^q(\mu, X^*)) \) topology. In particular, if \( X \) is an Asplund space, then every bounded Rademacher sequence in \( L^p(\mu, X) \) is weakly null.

**Proof.** Suppose \( \{f_k\} \) is a bounded Rademacher sequence in \( L^p(\mu, X) \). Let \( \{E_{\alpha n}\}_{\alpha \in T} \) be a Rademacher tree associated to \( \{f_k\} \). For \( x^* \) in \( X^* \), \( \tau \in T_m \), and \( k \geq m \), we have

\[
\int_{\Omega} \left( x^* \chi_{E_{\tau n}}, f_k(t) \right) d\mu(t) = 0.
\]

Since span\( \{x^* \chi_{E_{\tau n}} : x^* \in X \text{ and } \tau \in T\} \) is dense in \( L^q(\mu, \Sigma(T), X^*) \),

\[
\sigma(L^p(\mu, X), L^q(\mu, \Sigma(T), X^*)) - \lim_k f_k = 0.
\]

Let \( P \) be the conditional expectation projection from \( L^q(\mu, X^*) \) onto \( L^q(\mu, \Sigma(T), X^*) \) (see e.g. [Bi]), and suppose \( g \in L^q(\mu, X^*) \). Since \( f_k \) is \( \Sigma(T) \)-measurable,

\[
\int_{\Omega} \left( g(t), f_k(t) \right) d\mu(t) = \int_{\Omega} \left( Pg(t), f_k(t) \right) d\mu(t).
\]

Hence \( \{f_k\} \) is \( \sigma(L^p(\mu, X), L^q(\mu, X^*)) \)-null. Finally if \( X \) is an Asplund space, then \( L^q(\mu, X^*) \) is the dual of \( L^p(\mu, X) \) [DU], so \( \{f_k\} \) is weakly null.

**QED**

In general, it is not true that every bounded Rademacher sequence in \( L^p(\mu, X) \) is weakly null as shown by Example 1.2. In Theorem 1.3, we give a sufficient condition for a Rademacher sequence in \( L^p(\mu, X) \) to be weakly null.

**Example 1.2.** Let \( X \) be the space \( l^1 \) with the usual norm, and \( \mu \) the Lebesgue measure on \([0, 1)\). If \( \{r_k\} \) is the usual Rademacher sequence on
and \( \{e_k\} \) is the canonical basis for \( l^1 \). Define the \( X \)-valued sequence \( \{f_k\} \) by \( f_k(t) = r_k(t)e_{k+1} \) for \( t \) in \([0, 1)\) and \( k \geq 0 \). Then \( \{f_k\} \) is a bounded Rademacher sequence in \( L^2(\mu, X) \). It is easy to check that \( \text{co}\{f_k\} \) is a subset of the unit sphere. So the weak closure of \( \text{co}\{f_k\} \) still lies in the unit sphere. Therefore \( \{f_k\} \) is not weakly null.

**Theorem 1.3.** Suppose \( \{f_k\} \) is an \( X \)-valued Rademacher sequence determined by \( \{E_\alpha, x_\alpha\}_{\alpha \in T} \). If \( \{x_\alpha\} \) is bounded and there is \( \varepsilon_k > 0 \) such that

\[
\lim_{k \to \infty} \varepsilon_k = 0 \quad \text{and} \quad \|x_\beta - x_\alpha\| < \varepsilon_k \quad \text{for } k > 0, \alpha \in T_k, \text{ and } \beta \geq \alpha,
\]

then \( \{f_k\} \) is weakly null in \( L^p(\mu, X) \), \( 1 < p < \infty \).

**Proof.** Let \( Q_k \) be the natural projection from \( \bigcup_{i \geq 0} T_{k+i} \) to \( T_k \), i.e., for each \( \alpha \in T_{k+i} \), \( Q_k(\alpha) \) is the unique element in \( T_k \) such that \( Q_k(t) < \alpha \).

**Claim.** For all \( t \in \Omega \), \( k \geq 1 \) and \( i \geq 0 \),

\[
\left\| f_{k+i}(t) - \sum_{\alpha \in T_{k+i}} x_{Q_k(\alpha)} \sum_{n \geq 1} (-1)^n x_{E_n}(t) \right\| < \varepsilon_k x_{E_n}(t).
\]

We only need to prove this for \( t \in E_\delta \). Note that \( \{E_n; \alpha \in T_{k+i} \text{ and } n \geq 1\} \) is a partition of \( E_\delta \), so if \( t \in E_\delta \), then \( t \in E_{\gamma s} \) for some \( \gamma \in T_{k+i} \) and \( s \geq 1 \). Thus \( f_{k+i}(t) = (-1)^s x_\gamma \) and

\[
\sum_{\alpha \in T_{k+i}} x_{Q_k(\alpha)} \sum_{n \geq 1} (-1)^n x_{E_n}(t) = (-1)^s g(t_{Q_k(\gamma)}).
\]

So we have

\[
\left\| f_{k+i}(t) - \sum_{\alpha \in T_{k+i}} x_{Q_k(\alpha)} \sum_{n \geq 1} (-1)^n x_{E_n}(t) \right\| = \left\| (-1)^s x_\gamma - (-1)^s x_{Q_k(\gamma)} \right\| = \| x_\gamma - x_{Q_k(\gamma)} \| < \varepsilon_k
\]

Assume that \( \{f_k\} \) does not converge weakly to 0. Then there exists \( F \in L^p(\mu, X)^* \) with \( \|F\| = 1 \), a subsequence \( \{f_{n_k}\} \) of \( \{f_n\} \) and \( \delta > 0 \), such that \( F(f_{n_k}) > \delta \) for \( k \geq 1 \). It follows that for every \( h \in \text{co}\{f_{n_k}; k \geq 1\} \), \( \|h\| \geq F(h) > \delta \).

For \( k \geq 1 \), let \( h_k = \sum_{\alpha \in T_k} \sum_{n \geq 1} (-1)^n x_{E_n} \). Then \( \{h_k\} \) is a bounded Rademacher sequence in \( L^p(\mu) \). By Proposition 1.1, \( w\text{-lim}_k h_k = 0 \), so \( w\text{-lim}_k h_{n_k} = 0 \). Choose \( M \geq \mu(E)^{1/p} \) such that \( \|x_\alpha\| \leq M \) for all \( \alpha \in T \). Then choose \( k_0 > 1 \) with \( \varepsilon_{n_{k_0}} < \delta /3M \). Since \( \{h_{n_k}\} \) is weakly null, there exist
\( \lambda_i \geq 0, 1 \leq i \leq m, \) with \( \sum_{i=1}^{m} \lambda_i = 1 \) such that

\[
\left\| \sum_{i=1}^{m} \lambda_i h_{n_{k_0+i}} \right\| < \frac{\delta}{3M}.
\]

Then

\[
\left\| \sum_{i=1}^{m} \lambda_i f_{n_{k_0+i}} \right\| > \delta.
\]

On the other hand,

\[
\left\| \sum_{i=1}^{m} \lambda_i f_{n_{k_0+i}} \right\| \leq \left\| \sum_{i=1}^{m} \lambda_i \left( f_{n_{k_0+i}} - \sum_{\alpha \in T_{n_{k_0+i}}} x_{Q_{n_{k_0}}(\alpha)} \sum_{n \geq 1} (-1)^n \chi_{E_{an}} \right) \right\|
\]

\[
+ \left\| \sum_{i=1}^{m} \lambda_i \sum_{\alpha \in T_{n_{k_0+i}}} x_{Q_{n_{k_0}}(\alpha)} \sum_{n \geq 1} (-1)^n \chi_{E_{an}} \right\|
\]

\[
\leq \sum_{i=1}^{m} \lambda_i \| x_{n_{k_0}} \chi_{E_{an}} \| + \left\| \sum_{\alpha \in T_{n_{k_0}}} x_{\alpha} \chi_{E_{an}} \sum_{i=1}^{m} \lambda_i h_{n_{k_0+i}} \right\|
\]

\[
= \sum_{i=1}^{m} \lambda_i x_{n_{k_0}} \mu(E)^{1/p} + \left( \max_{\alpha \in T_{n_{k_0}}} \| x_{\alpha} \| \right) \left\| \sum_{i=1}^{m} \lambda_i h_{n_{k_0+i}} \right\|
\]

\[
\leq x_{n_{k_0}} \mu(E)^{1/p} + \left( \max_{\alpha \in T_{n_{k_0}}} \| x_{\alpha} \| \right) \left\| \sum_{i=1}^{m} \lambda_i h_{n_{k_0+i}} \right\|
\]

\[
< \frac{\delta}{3} + M \left( \frac{\delta}{3M} \right) < \delta,
\]

which is impossible. Therefore \( \{ f_k \} \) does converge weakly to 0. QED

Next we consider a special construction of Rademacher tree of measurable sets.

**Lemma 1.4** [D, p. 154]. Suppose \((\Omega, \Sigma, \mu)\) is atom-free. Then for any \( E \) in \( \Sigma \) with \( \mu(E) < \infty \), there exists a partition \( \{ E_1, E_2 \} \) of \( E \) such that \( \mu(E_1) = \mu(E_2) \).

Recall that an atom in \( \Sigma \) is a measurable set \( E \) in \( \Sigma \) such that for any measurable subset \( F \) of \( E \), either \( \mu(F) = 0 \) or \( \mu(F) = \mu(E) \). We say that \((\Omega, \Sigma, \mu)\) is atom-free if \( \Sigma \) does not contain any atoms of positive finite measure.
Lemma 1.5. Suppose that \((\Omega, \Sigma, \mu)\) is finite and that \(f_i\) is a separably valued measurable function from \(\Omega\) to Banach space \(X_i\) for \(1 \leq i \leq k\). Then for any \(\epsilon > 0\), there is a partition \(\{E_n^i\}\) of \(\Omega\) such that \(\text{diam} f_i(E_n^i) < \epsilon\), \(1 \leq i \leq k, n \geq 1\). If, in addition, \((\Omega, \Sigma, \mu)\) is atom-free, then we may also require that \(\mu(E_{2n-1}^i) = \mu(E_{2n}^i) > 0\).

Proof. The first conclusion is obvious. To prove the second one, first we choose a partition \(\{F_n^i\}\) of \(\Omega\) such that \(\mu(F_n^i) > 0\) and \(\text{diam} f_i(F_n^i) < \epsilon\), \(1 \leq i \leq k, n \geq 1\), then by Lemma 1.4, we choose for each \(n \geq 1\) a partition \(\{E_{2n-1}^i, E_{2n}^i\}\) of \(F_n^i\) such that \(\mu(E_{2n-1}^i) = \mu(E_{2n}^i)\). Then \(\{E_n^i\}\) is the partition of \(\Omega\) we wanted. QED

Using Lemma 1.5, it is easy to prove the following result.

Proposition 1.6. Suppose that \((\Omega, \Sigma, \mu)\) is atom-free and \(f_i\) is a separably valued measurable function from \(\Omega\) to Banach space \(X_i\) for \(1 \leq i \leq m\). Then for any \(\epsilon_k > 0\), \(k \geq 0\), and \(E \in \Sigma\) with \(0 < \mu(E) < \infty\), there is a Rademacher tree of measurable sets \(\{E_{\alpha}\}_{\alpha \in T}\) in \(\Omega\) such that

\[
E_\phi = E, \quad \mu(E_\alpha) > 0, \quad \text{diam} f_i(E_\alpha) < \epsilon_k
\]

for \(1 \leq i \leq m, k > 0\), and \(\alpha \in T_k\).

Section II

Recall that \(X\) is said to have the Schur property if every weakly convergent sequence in \(X\) is norm convergent. It is obvious that \(X\) has the Schur property if and only if 0 is a point of sequential continuity of \(B_X\). If \(K\) is a subset of \(X\), we use \(\text{psc} K\) (resp. \(\text{ext} K\)) to denote the set of points of sequential continuity (resp. extreme points) of \(K\).

Lemma 2.1. Suppose that \(K\) is a bounded closed convex set in \(X\) and that \(x \in \text{psc} K\). If \(x = \frac{1}{2}(y + z)\) for some \(y\) and \(z\) in \(K\), then both \(y\) and \(z\) are points of sequential continuity of \(K\). Thus if \(X\) fails the Schur property and \(x\) is a point of sequential continuity of \(B_X\), then \(\|x\| = 1\).

Proof. We only need to show that \(y \in \text{psc} K\). So let \((y_n)\) be a sequence in \(K\) which converges weakly to \(y\). Then \(\text{w-lim}_n \frac{1}{2}(y_n + z) = x\) and \(\frac{1}{2}(y_n + z) \in K\), thus \(\lim_n \frac{1}{2}(y_n + z) = x = \frac{1}{2}(y + z)\). It follows that \(\lim_n y_n = y\). Hence \(y \in \text{psc} K\). QED
Theorem 2.2. Suppose $(\Omega, \Sigma, \mu)$ is atom-free. Then every point of sequential continuity of $B_{L^p(\mu, X)}$ is an extreme point of $B_{L^p(\mu, X)}$.

Proof. Let $f \in \text{psc } B_{L^p(\mu, X)}$. Since $L^p(\mu, X)$ contains a copy of $L^p(\mu)$ which fails the Schur property, by Lemma 2.1, $\|f\| = 1$. Assume $f \notin \text{ext } B_{L^p(\mu, X)}$. There is $g \in L^p(\mu, X)$ with $\|g\| > 0$ and $\|f \pm g\| = 1$. Since $\|f\| = 1 = \|f \pm g\|$ and $f = \frac{1}{2}[(f + g) + (f - g)]$, and since $L^p(\mu)$ is strictly convex, we conclude that $\|f(t) \pm g(t)\| = \|f(t)\|$ for almost all $t \in \Omega$. Without loss of generality we may assume that $\|f(t) \pm g(t)\| = \|f(t)\|$ for all $t \in \Omega$ and that both $f(\Omega)$ and $g(\Omega)$ are separable.

Since $\|g\| > 0$, there is $M > 0$ and $E$ in $\Sigma$ such that $\mu(E) > 0$ and $\frac{1}{M} \leq \|g(t)\| \leq M$ for all $t$ in $E$. Then $\mu(E) < \infty$. By Proposition 1.6, there exists a Rademacher tree of measurable sets $\{E_\alpha\}_{\alpha \in T}$ in $\Omega$ such that for $k > 0$, and $\alpha \in T_k$, we have

$$E_\phi = E, \quad \mu(E_\alpha) > 0, \quad \text{diam } f(E_\alpha) < 2^{-k} \quad \text{and} \quad \text{diam } g(E_\alpha) < 2^{-k}.$$ 

For each $\alpha \in T$, pick an element $t_\alpha \in E_\alpha$ and define, for $k \geq 0$,

$$g_k = \sum_{\alpha \in T_k} g(t_\alpha) \sum_{n \geq 1} (-1)^n \chi_{E_{an}}.$$

By Theorem 1.3, $\{g_k\}$ converges weakly to 0.

Claim. $\lim_k \|f + g_k\| = 1$.

If $t \in \Omega \setminus E$, then $(f \pm g_k)(t) = f(t)$, so $\|(f \pm g_k)(t)\| = \|f(t)\|$. If $t \in E$, then for $k > 1$, there is $\alpha \in T_k$ and $n \geq 1$ such that $t \in E_{an}$. Thus $g_k(t) = (-1)^n g(t_\alpha)$. Since

$$\text{diam } f(E_\alpha) < 2^{-k}, \quad t \in E_\alpha, \quad t_\alpha \in E_\alpha, \quad \text{and} \quad \|f(t_\alpha) \pm g(t_\alpha)\| = \|f(t_\alpha)\|,$$

we have

$$\|(f \pm g_k)(t)\| = \|f(t) \pm (-1)^n g(t_\alpha)\|$$

$$\leq \|f(t) - f(t_\alpha)\| + \|f(t_\alpha) \pm (-1)^n g(t_\alpha)\|$$

$$= \|f(t) - f(t_\alpha)\| + \|f(t_\alpha)\|$$

$$\leq \|f(t)\| + 2\|f(t) - f(t_\alpha)\| < \|f(t)\| + 2^{-k+1}.$$

Therefore $\|f \pm g_k\| < \|f\| + 2^{-k+1} \mu(E)^{1/p}$. It follows that $\lim_k \|f \pm g_k\| = \|f\| = 1$. 

Since \( \lim_k \|f + g_k\| = 1 \) and weak-\( \lim_k (f + g_k) = f \), we have

\[
\lim_{k} (f + g_k) = f,
\]
i.e., \( \lim_k \|g_k\| = 0 \). On the other hand, since \( \|g(t)\| \geq 1/M \) for \( t \in E \), we have \( \|g_k\| \geq (1/M)\mu(E)^{1/p} > 0 \), which is impossible. Therefore \( f \in \text{ext } B_{L^p(\mu, X)} \). QED

We say that \((\Omega, \Sigma, \mu)\) is not purely atomic if there is \( E \) in \( \Sigma \) such that \( 0 < \mu(E) < \infty \), and \( E \) contains no atoms, that is, \((E, \Sigma_E, \mu_E)\) is atom-free, where \( \mu_E \) be the restriction of \( \mu \) to \( \Sigma_E = \{F: F \in \Sigma \text{ and } F \subset E\} \).

**Corollary 2.3 [ST].** Suppose that \((\Omega, \Sigma, \mu)\) is not purely atomic. If \( L^p(\mu, X) \) has the Kadec-Klee property, then \( X \) is strictly convex.

**Proof.** Since \((\Omega, \Sigma, \mu)\) is not purely atomic, there is \( E \) in \( \Sigma \) such that \( 0 < \mu(E) < \infty \) and \((E, \Sigma_E, \mu_E)\) is atom-free. Since \( L^p(\mu_E, X) \) is isometrically isomorphic to a subspace of \( L^p(\mu, X) \), the space \( L^p(\mu_E, X) \) has the Kadec-Klee property. By Theorem 2.2, every unit vector in \( L^p(\mu_E, X) \) is an extreme point of the unit ball, thus \( L^p(\mu_E, X) \) is strictly convex. Therefore \( X \) is also strictly convex. QED

If \( K \subset X \), the slice of \( K \) determined by the functional \( x^* \) in \( X^* \) and \( \delta > 0 \) is the subset of \( K \) given by

\[
S(x^*, K, \delta) = \{x \in K: x^*(x) > \sup K^* - \delta\}.
\]

Let \( x \in K \). Then \( x \) is called a denting point of \( K \) if the family of all slices of \( K \) containing \( x \) is a neighborhood base of \( x \) with respect to the relative norm topology on \( K \). And \( x \) is said to be a point of continuity of \( K \) if the relative weak and norm topologies on \( K \) coincide at \( x \). If \( K \subset X^*, K \neq \emptyset \), then weak* slices, weak* denting points, and weak* points of continuity of \( K \) are defined similarly. We use dent \( K \) (resp. pc \( K \), w*dent, w*-pc \( K \)) to denote the set of denting points (resp. points of continuity, weak* denting points, weak* points of continuity) of \( K \).

By definition, a denting point is a point of continuity, and a point of continuity is a point of sequential continuity. It is known that \( x \in \text{dent } K \) if and only if \( x \in \text{pc } K \) and \( x \in \text{ext } K \) [LLT]. Thus by Theorem 2.2, the following assertion follows.

**Corollary 2.4.** Suppose that \((\Omega, \Sigma, \mu)\) is atom-free and \( f \in L^p(\mu, X) \). Then \( f \) is a point of continuity of \( B_{L^p}(\mu, X) \) if and only if \( f \) is a denting point of \( B_{L^p}(\mu, X) \).
A Banach space $X$ has the RNP if every non-empty bounded closed set $K$ in $X$ has a denting point. $X$ has the CPCP (resp. PCP) if for every non-empty bounded closed convex (resp. bounded closed) set $K$ in $X$, $pc K \neq \phi$. It is obvious that the RNP implies the PCP, and the PCP implies the CPCP, but these three properties are distinct. The dual version of PCP, in which one considers weak* point of continuity, is the same as the corresponding dual version of RNP, which in turn is the same as RNP itself [St]. However, the dual version of CPCP, denoted by $C^*\text{PCP}$, is distinct from RNP [GMS2]. It is clear that $C^*\text{PCP}$ implies CPCP, though the converse is not true [DGHZ].

**Corollary 2.5.** Suppose $(\Omega, \Sigma, \mu)$ is not purely atomic. Then the RNP and the CPCP are equivalent in both $L^p(\mu, X)$ and $L^p(\mu, X)^*$. 

**Proof.** Suppose that $L^p(\mu, X)$ has the CPCP. Let $| |$ be an equivalent norm on $X$. Choose $E$ in $\Sigma$ such that $0 < \mu(E) < \infty$ and $(E, \Sigma_E, \mu_E)$ is atom-free. Since $L^p(\mu, X)$ has the CPCP, the space $L^p(\mu, (X, | |))$ which is isomorphic to a subspace of $L^p(\mu, X)$ also has the CPCP. Hence there exists $f$ in $pc B_{L^p(\mu, (X, | |))}$. Then $f$ must be a denting point of $B_{L^p(\mu, (X, | |))}$ following Corollary 2.4. By a result in [LL], it follows that $f(t)/|f(t)| \in \text{dent } B_{(X, | |)}$ for almost all $t \in \text{supp } f$. Thus $B_{(X, | |)}$ is not empty. Therefore $X$ has the RNP (see e.g. p. 30 [Bi]), and hence $L^p(\mu, X)$ has the RNP [DU]. The converse is obvious.

Now suppose that $L^p(\mu, X)^*$ has the CPCP. The space $L^q(\mu, X^*)$, being a subspace of $L^p(\mu, X)^*$, also has the CPCP. As a consequence of the previous paragraph, the space $L^q(\mu, X^*)$ has the RNP. Thus $X^*$ has the RNP, which implies that $L^p(\mu, X)^*$ has the RNP [DU]. The converse is also obvious. QED

Recall that a normed space $Y$ is said to be finitely representable in a normed space $E$, if for each $\epsilon > 0$ and finite dimensional subspace $F$ of $Y$, there is a 1-1 linear operator

$$T: F \to T(F) \subset E \text{ with } \|T\| \|T^{-1}\| \leq 1 + \epsilon.$$ 

If (P) is a property defined for Banach spaces, $X$ is said to have the property “Super (P)” if every Banach space finitely representable in $X$ has the property (P). It is known that $X$ is super-reflexive if and only if it is super-Radon-Nikodym. It is an open problem whether super-RNP and super-PCP are equivalent.

**Proposition 2.6.** Suppose $X \oplus_p X$ is finitely representable in $X$ for some $p$. 


Then $X$ has the super-RNP if and only if $X$ has the super-CPCP.

**Proof.** Suppose $X$ has the super-CPCP. Let $Y$ be a Banach space finitely representable in $X$. Then $l^p(Y_n)$, where $Y_n = Y$, $n \geq 1$, is finitely representable in $l^p(X_n)$, where $X_n = X$. Let $\mu$ be the Lebesgue measure on $[0, 1)$. Then $\mu$ is atom-free.

**Claim.** $L^p(\mu, Y)$ is finitely representable in $X$.

Let $E$ be the linear span of simple functions in $L^p(\mu, Y)$. Since $E$ is dense in $L^p(\mu, Y)$, the space $L^p(\mu, Y)$ is finitely representable in $E$. It is obvious that $E$ is finitely representable in $l^p(Y_n)$, in fact, every finite dimensional subspace $G$ of $E$ is isometric to a subspace of $l^p(Y_n)$. Thus $L^p(\mu, Y)$ is finitely representable in $l^p(X_n)$. Since $X \oplus_p X$ is finitely representable in $X$, it follows that $l^p(X_n)$ is also finitely representable in $X$. Thus $L^p(\mu, Y)$ is finitely representable in $X$.

Since $X$ has the super-CPCP, the space $L^p(\mu, Y)$ has the CPCP. By Corollary 2.5, $L'(\mu, Y)$ has the RNP. Thus $Y$ has the RNP. Therefore $X$ has the super-RNP. The converse is obvious. QED

**Corollary 2.7.** Suppose that $(\Omega, \Sigma, \mu)$ is a measure space which is not purely atomic or which contains infinitely many atoms of finite positive measure. Then in both $L^p(\mu, X)$ and $L^p(\mu, X)^*$, super-RNP and super-CPCP are equivalent.

**Proof.** In each case, $L^p(\mu, X) \oplus_p L^p(\mu, X)$ is finitely representable in $L^p(\mu, X)$. Thus $L^p(\mu, X)$ has the super-RNP if and only if it has the super-CPCP.

Now suppose that $L^p(\mu, X)^*$ has the super-CPCP, then $L^q(\mu, X^*)$, being a subspace of $L^p(\mu, X)^*$, also has the super-CPCP. Thus $L^q(\mu, X^*)$ has the super-RNP, and in particular $X^*$ has the RNP. Therefore $L^p(\mu, X)^* = L^q(\mu, X^*)$ [DU], and so $L^p(\mu, X)^*$ has the super-RNP. The converse is obvious. QED

Suppose $K$ is a subset of $X$ and $x \in K$. For a given $\varepsilon > 0$, we say that $x$ is an $\varepsilon$-strongly extreme point in $K$ if there is a $\delta > 0$ such that for any $y$ in $X$, the conditions $d(x + y, K) < \delta$ and $d(x - y, K) < \delta$ imply that $\|y\| < \varepsilon$, where $d(x, K)$ is the distance between $x$ and $K$. Then $x$ is called a strongly extreme point of $K$ if $x$ is an $\varepsilon$-strongly extreme point in $K$ for all $\varepsilon > 0$. We use str-ext $K$ to denote the set of the strongly extreme points of $K$. By definition, strongly extreme points are extreme points, but the converse is not true [M]. It is obvious that if $K$ is convex and $d(x \pm y, K) < \delta$ then for any $0 \leq \lambda \leq 1$, we have $d(x \pm \lambda y, K) < \delta$. Thus if $K$ is convex and $x$ is not
\( \varepsilon \)-strongly extreme in \( K \), then for any \( \delta > 0 \), there exists \( y \) in \( X \) such that 
\[ d(x \pm y, K) < \delta \] and \( \| y \| = \varepsilon \).

**Theorem 2.8.** Suppose that \((\Omega, \Sigma, \mu)\) is atom-free and \( f \) is a \( \sigma(L^p(\mu, X), L^q(\mu, X^*))\)-point of sequential continuity of \( B_{L^p}(\mu, X) \), i.e.,
\[ \lim_{k \to \infty} f_k = f \quad \text{if} \quad \sigma(L^p(\mu, X), L^q(\mu, X^*)) \quad \text{and} \quad \| f_k - f \| \to 0 \]
and \( \{ f_k \} \) is in \( B_{L^p}(\mu, X) \). Then \( \| f \| = 1 \) and \( f(t)/\| f(t) \| \) is \( 1/m \)-strongly extreme in \( B_X \) for almost all \( t \) in \( \text{supp } f \). Thus \( f \) is a strongly extreme point of \( B_{L^p}(\mu, X) \).

**Proof.** By Theorem 2.2, the norm \( \| f \| = 1 \). Without loss of generality, we may assume that \( f(\Omega) \) is separable. Define
\[ D = \left\{ t : t \in \text{supp } f \text{ and } f(t)/\| f(t) \| \text{ is not } 1/m\text{-strongly extreme in } B_X \right\} \]
and define, for each \( m \geq 1 \), the set
\[ D_m = \left\{ t : t \in D, \| f(t) \| > 1/m, \text{ and } f(t)/\| f(t) \| \text{ is not } 1/m\text{-strongly extreme in } B_X \right\}. \]

Then \( D \) is the union of \( D_m \). Assume that it is not true that \( f(t)/\| f(t) \| \in \text{ str-ext } B_X \) for almost all \( t \) in \( \text{supp } f \), that is, \( \mu^*(D) > 0 \), where \( \mu^* \) is the outer measure associated to \( \mu \). Then there is \( m \) such that \( \mu^*(D_m) > 0 \). Choose a measurable set \( E \subset \text{supp } f \) with \( \mu(E) = \mu^*(D_m) \) and \( D_m \subset E \). It is obvious that \( \mu(E) < \infty \). By Proposition 1.6, there is a Rademacher tree of measurable sets \( \{ E_\alpha \}_{\alpha \in T} \) in \( \Omega \) such that for \( 1 \leq i \leq m, k > 0, \) and \( \alpha \in T_k \), we have
\[ E_\phi = E, \quad \mu(E_\alpha) > 0, \quad \text{and} \quad \text{diam } f(E_\alpha) < 2^{-k}. \]

It is obvious that for each \( \alpha \in T \), \( \mu^*(A \cap E_\alpha) = \mu(E_\alpha) \). For each \( \alpha \in T \), pick an element \( t_\alpha \in A \cap E_\alpha \) and choose \( x_\alpha \in X \) such that
\[ \| x_\alpha \| = 1/m \quad \text{and} \quad \left\| \frac{f(t_\alpha)}{\| f(t_\alpha) \|} \pm x_\alpha \right\| \leq 1 + \frac{1}{2^{\| \alpha \|}}. \]
For each $k > 0$, define

$$g_k = \sum_{\alpha \in T_k} \|f(t_\alpha)\| x_\alpha \sum_{n \geq 1} (-1)^n \chi_{E_n}.$$ 

**Claim.** For $k \geq 1$, $\|g_k\| \leq 3\|f\| + 2^{-k+2}\mu(E)^{1/p}$ and $\lim_k \|f \pm g_k\| = \|f\| = 1.$

If $t \in \Omega \setminus E$, then $(f \pm g_k)(t) = f(t)$, so $\|(f \pm g_k)(t)\| = \|f(t)\|$. If $t \in E$, then for $k > 1$, there is $\alpha \in T_k$ and $n \geq 1$ such that $t \in E_n$. Thus $g_k(t) = (-1)^n\|f(t)\| x_\alpha$ and so we have

$$\|(f \pm g_k)(t)\| \leq \|f(t) - f(t_\alpha)\| + \|f(t_\alpha)\| \pm (-1)^n\|f(t)\| x_\alpha$$

$$= \|f(t) - f(t_\alpha)\| + \left(1 + \frac{1}{2|\alpha|}\right)\|f(t_\alpha)\|$$

$$\leq \|f(t) - f(t_\alpha)\| + \left(1 + \frac{1}{2|\alpha|}\right)\|f(t) - f(t_\alpha)\|$$

$$\leq \left(1 + \frac{1}{2|\alpha|}\right)\|f(t)\| + 2^{-k+2}.$$

Therefore $\|(f \pm g_k)(t)\| \leq (1 + 2^{-k})\|f\| + 2^{-k+2}\mu(E)^{1/p}$. It follows that

$$\|g_k\| \leq 3\|f\| + 2^{-k+2}\mu(E)^{1/p} \quad \text{and} \quad \lim_k \|f \pm g_k\| = \|f\| = 1.$$

Since $\{g_k\}$ is a bounded Rademacher sequence in $L^p(\mu, X)$, by Proposition 1.1, it is $\sigma(L^p(\mu, X), L^q(\mu, X^*))$-null. Thus

$$\sigma(L^p(\mu, X), L^q(\mu, X^*)) - \lim_k f + g_k = f \quad \text{and} \quad \lim_k \|f + g_k\| = \|f\| = 1.$$

Since $f$ is a $\sigma(L^p(\mu, X), L^q(\mu, X^*))$-point of sequential continuity of $B_{L^p}(\mu, X)$, we conclude that $\lim_k f + g_k = f$. Thus $\lim_k \|g_k\| = 0$. On the other hand, since $\|g_k(t)\| \geq 1/m^2$ for $t \in E$, the norm

$$\|g_k\| \geq (1/m^2)^{1/p} > 0,$$

which is a impossible. Therefore

$$f(t)/\|f(t)\| \in \text{str-ext } B_X$$

for almost all $t$ in supp $f$. Hence $f$ is a strongly extreme point of $B_{L^p}(\mu, X)$ [Sm2]. QED
In addition to its sequential generalization, the point of continuity has a "slice generalization", namely, the point of small combination of slices (SCS-point). Let \( K \) be a convex set of \( X \), the point \( x \in K \) is called a SCS-point of \( K \) [GGMS] if for each \( \epsilon > 0 \), there exist slices \( S_i \) of \( K \) and \( \lambda_i > 0 \), \( i = 1, \ldots, n \) with \( \sum_{i=1}^{n} \lambda_i = 1 \) such that \( \text{diam} \sum_{i=1}^{n} \lambda_i S_i < \epsilon \) and \( x \in \bigcup_{i=1}^{n} \lambda_i S_i \). Let SCS(\( K \)) denote the set of all SCS-points of \( K \). If \( K \) is in \( X^* \), a \( w^* \)-SCS-point of \( K \) is defined similarly except the slices \( S_i \) of \( K \) are weak* slices. It is clear that \( pc K \subset \text{SCS}(K) \) (resp. \( w^*-pc K \subset \text{w*-SCS}(K) \)) for all convex sets \( K \) in \( X \) (resp. \( X^* \)).

It is known [GGMS], [R1] that \( X \) (resp. dual space \( X^* \)) is strongly (resp. \( w*-strong \)) regular if and only if every non-empty bounded closed convex set \( K \) in \( X \) (resp. \( X^* \)) is contained in the norm-closure (resp. weak* closure) of SCS(\( K \)) (resp. \( w^*-SCS(K) \)). Schachermayer [Sc] proved that a Banach space has the RNP if and only if it is strongly regular and it has the Krein-Milman Property. The "point-version" of this result is also true and it extends the result in [LLT].

**Proposition 2.9.** Let \( K \) be a closed convex set in \( X^* \) and let \( \overline{K^*} \) be the weak* closure of \( K \). Then:

1. \( w^*-pc K = w^*-pc \overline{K^*} \).
2. \( w^*-SCS(K) = w^*-SCS(\overline{K^*}) \).
3. \( w^*-dent \overline{K^*} = w^*-dent K = (w^*-pc K) \cap \text{ext} K = w^*-SCS(K) \cap \text{ext} K \).

**Proof.** (1) Let \( x^* \in w^*-pc \overline{K^*} \). Since the weak* and norm topologies on \( \overline{K^*} \) coincide at \( x^* \), we have \( x^* \in K \subset K \). Thus \( x^* \in w^*-pc K \).

Conversely, if \( x^* \in w^*-pc K \), then for each \( \epsilon > 0 \), there are \( x_1, \ldots, x_n \) in \( X \) and \( \delta > 0 \) such that \( \text{diam} V < \epsilon \), where

\[
V = \{ y^* : y^* \in K, \ (y^*, x) > (x^*, x) - \delta, i = 1, \ldots, n \}.
\]

Let

\[
U = \{ y^* : y^* \in \overline{K^*}, \ (y^*, x) > (x^*, x) - \delta, i = 1, \ldots, n \}.
\]

Then \( U \) is a \( w^* \)-neighborhood of \( x^* \) in \( \overline{K^*} \) and \( V \) is weak* dense in \( U \). Thus \( \text{diam} U = \text{diam} V < \epsilon \). So \( x^* \in w^*-pc \overline{K^*} \).

(2) Let \( x^* \in w^*-SCS(\overline{K^*}) \). It is obvious that every weak* slice of \( \overline{K^*} \) contains a point of \( K \). Hence, by the definition of \( w^*-SCS \)-points, \( x^* \in \overline{K^*} = K \). Therefore \( x^* \in w^*-SCS(K) \).

Conversely, if \( x^* \in w^*-SCS(K) \), then for each \( \epsilon > 0 \), there exist \( w^* \)-slices \( S_i \) of \( K \) and \( \lambda_i > 0 \), \( i = 1, \ldots, n \) with \( \sum_{i=1}^{n} \lambda_i = 1 \) such that \( \text{diam} \sum_{i=1}^{n} \lambda_i S_i < \epsilon \). We assume \( S_i = S(x_i, K, \delta_i) \) for some \( x_i \) in \( X \) and \( \delta_i > 0 \). Since
\( \Sigma_{i=1}^{n} \lambda_i S(x_i, K^*, \delta_i) \) is a subset of the weak* closure of \( \Sigma_{i=1}^{n} \lambda_i S_i \), we have \( \text{diam} \Sigma_{i=1}^{n} \lambda_i S(x_i, K^*, \delta_i) < \epsilon \). Hence \( x^* \in w^*\text{-SCS } K^* \).

(3) It is obvious that

\[
w^*\text{-dent } K^* \subset w^*\text{-dent } K \subset (w^*\text{-pc } K) \cap \text{ext } K \subset w^*\text{-SCS}(K) \cap \text{ext } K.
\]

To complete the proof we only need to show

\[
(w^*\text{-SCS } K) \cap \text{ext } K \subset w^*\text{-dent } K^*.
\]

So let \( x^* \in w^*\text{-SCS}(K) \cap \text{ext } K \). For each \( \epsilon > 0 \), there exist weak* slices \( S_i \) of \( K \) and \( \lambda_i > 0 \), \( i = 1, \ldots, n \) with \( \Sigma_{i=1}^{n} \lambda_i = 1 \) such that \( \text{diam} \Sigma_{i=1}^{n} \lambda_i S_i \subset \epsilon \) and \( x^* \in \Sigma_{i=1}^{n} \lambda_i S_i \). Since \( x^* \in \text{ext } K \), \( x^* \) must belong to \( \bigcap_{i=1}^{n} S_i \). Thus \( \bigcap_{i=1}^{n} S_i \) is a weak* neighborhood of \( x^* \). Note that \( \text{diam} \bigcap_{i=1}^{n} S_i \leq \text{diam} \Sigma_{i=1}^{n} \lambda_i S_i < \epsilon \), so \( x^* \in w^*\text{-pc } K \).

Next we show that \( x^* \in \text{ext } K^* \). Assume \( x^* = (y^* + z^*)/2 \) for some \( y^*, z^* \in K^* \). Since \( x^* \in w^*\text{-pc } K = w^*\text{-pc } K^* \), it follows that \( y^*, z^* \in w^*\text{-pc } K^* \) (see the proof of Lemma 2.1). By (1), \( y^*, z^* \in K \). Thus \( x^* = y^* = z^* \) because \( x^* \in \text{ext } K \). So \( x^* \in \text{ext } K^* \). Since \( x^* \) is a weak* point of continuity and an extreme point of the weak* compact convex set \( K^* \cap B_{X^*}(x, 1) \), the weak* slices of \( K^* \cap B_{X^*}(x, 1) \) containing \( x^* \) is a norm neighborhood base at \( x^* \). Therefore \( x^* \in w^*\text{-dent } K^* \cap B_{X^*}(x, 1) \). Hence \( x^* \in w^*\text{-dent } K^* \) [B], QED

**Corollary 2.10.** Let \( K \) be a closed convex set in \( X \) and let \( K^* \) be the weak* closure of \( K \) in \( X^{**} \). Then:

(1) \( \text{pc } K = w^*\text{-pc } K^* \).

(2) \( w^*\text{-dent } K^* = \text{dent } K = \text{pc } K \cap \text{ext } K = \text{SCS}(K) \cap \text{ext } K \).

**Proof.** This follows immediately from Proposition 2.9 and the facts that \( w^*\text{-dent } K = \text{dent } K \), \( w^*\text{-pc } K = \text{pc } K \), and \( w^*\text{-SCS}(K) = \text{SCS}(K) \), QED

Note that for any \( f \in L^q(\mu, X^*) \) and \( g \in L^p(\mu, X) \), the action of \( f \) on \( g \) is defined by

\[
(f, g) = \int_{\Omega} (f(t), g(t)) \, d\mu(t) \quad [DU].
\]

It is obvious that the space \( L^q(\mu, X^*) \) is a subspace of \( L^p(\mu, X^*) \), and that \( L^q(\mu, X^*) \) norms \( L^p(\mu, X) \). So if \( K = B_{L^p(\mu, X^*)} \), then \( K^* = B_{L^q(\mu, X^*)} \). Hence the following result is a corollary of Proposition 2.9.
Corollary 2.11. The following assertions are true:

1. \( w^*-\text{pc } B_{L^q(\mu, X^*)} = w^*-\text{pc } B_{L^p(\mu, X^*)} \).
2. \( w^*-\text{SCS } B_{L^q(\mu, X^*)} = w^*-\text{SCS } B_{L^p(\mu, X^*)} \).
3. \( w^*-\text{dent } B_{L^p(\mu, X^*)} = w^*-\text{dent } B_{L^q(\mu, X^*)} \cap \text{ext } B_{L^q(\mu, X^*)} \).

If \( (\Omega, \Sigma, \mu) \) is atom-free, then every weak* point of continuity \( f \) of \( B_{L^q(\mu, X^*)} \) is an extreme point of \( B_{L^q(\mu, X^*)} \) (Corollary 2.4), by Corollary 2.11, it is a weak* denting point of \( B_{L^q(\mu, X^*)} \). Thus we have the following result.

Corollary 2.12. Suppose that \( (\Omega, \Sigma, \mu) \) is atom-free and \( f \) in \( L^p(\mu, X^*) \). Then \( f \) is a weak* point of continuity of \( B_{L^p(\mu, X^*)} \) if and only if \( f \) is a weak* denting point of \( B_{L^p(\mu, X^*)} \).

The next example shows that we can not replace the point of sequential continuity by SCS-point in Theorem 2.2.

Example 2.13. Let \( Y \) be a Banach space such that it contains no copies of \( l^1 \) but its dual \( Y^* \) does not have the RNP [GMS2]. Let \( X = Y^* \) and let \( K = B_{L^p(\mu, X^*)} \). By taking equivalent norms, we may assume that \( w^*-\text{dent } B_{Y^*} = \phi \) [Bi]. Let \( \mu \) be the Lebesgue measure on \([0, 1)\). Since \( Y \) contains no copy of \( l^1 \), the space \( L^q(\mu, X) \) also contains no copy of \( l^1 \) [P]. By a result of J. Bourgain [Ba], \( L^q(\mu, Y^*) \) is weak* strongly regular. Thus \( K \) is contained in the weak* closure of \( w^*-\text{SCS}(K) \). So the weak* closure of the \( w^*-\text{SCS}-\text{points} \) is \( B_{L^q(\mu, Y^*)} \). Were a \( w^*-\text{SCS} \) point \( f \) an extreme point, that point \( f \) would be a weak* denting point of \( B_{L^p(\mu, X^*)} \) by Corollary 2.11. But then by a result in [HL], for almost all \( t \) in the support of \( f \), \( f(t)/\|f(t)\| \) would be a weak* denting point of \( B_{Y^*} \), which contradicts the fact that \( w^*-\text{dent } B_{Y^*} = \phi \). Therefore none of these \( w^*-\text{SCS}-\text{points} \) is an extreme point of \( K \). By definition, \( w^*-\text{SCS}(K) \subset \text{SCS}(K) \), so in Theorem 2.2 we can not replace the point of sequential continuity by the SCS-point.

If \( (\Omega, \Sigma, \mu) \) is purely atomic and finite, then there exists an at most countable partition \( \pi \) of \( \Omega \) such that every element in \( \pi \) is an atom of positive measure. For each \( E \) in \( \pi \), let \( X_E \) be the space \( X \). Define mapping \( T \) from \( L^p(\mu, X) \) to \( l^p(X_E) \) by

\[
T(f)(E) = \mu(E)^{1/p} \int_E f(t) \, d\mu(t).
\]

Thus \( T(f)(E) = \mu(E)^{1/p} f(t) \) for almost all \( t \) in \( E \). It is obvious that \( T \) is an isometric embedding. Partly because of this, in the rest of this section we will consider the space \( l^p(X_i) \), instead of \( L^p(\mu, X) \) with \( (\Omega, \Sigma, \mu) \) being purely atomic.
Proposition 2.14. Let \( \{X_i\}_{i \in I} \) be a family of Banach spaces and let 
\( f = (f(i))_{i \in I} \) be a unit vector in \( l^p(X_i) \). Then \( f \) is psc \( B_{l^p(X_i)} \) (resp. pc \( B_{l^p(X_i)} \); 
\( \text{ext } B_{l^p(X_i)} \); or dent \( B_{l^p(X_i)} \)) if and only if \( f(i)/\|f(i)\| \) is psc \( B_{X_i} \) (resp. pc \( B_{X_i} \); 
\( \text{ext } B_{X_i} \); or dent \( B_{X_i} \)) for \( i \in \text{supp } f \).

Moreover, the weak* version of this statement is also true.

Proof. Suppose \( f \in \text{psc } B_{l^p(X_i)} \). Fix \( i \in I \) with \( f(i) \neq 0 \). We use \( B_X(x, r) \) to denote the ball in \( X \) with center \( x \) and radius \( r \). Let \( \{x_n\} \) be a sequence in \( B_{X_i}(0, \|f(i)\|) \) such that \( \omega-\lim_n x_n = f(i) \). For each \( n \) define

\[
f_n(j) = \begin{cases} 
  f(j) & \text{if } j \neq i \\
  x_n & \text{if } j = i
\end{cases}
\]

Then \( f_n \in B_{l^p(X_i)} \) and weak-\( \lim_n f_n = f \). Hence \( \lim_n \|f_n - f\| = 0 \) and so
\( \lim_n \|x_n - f(i)\| = 0 \). Therefore \( f(i) \in \text{psc } B_{X_i}(0, \|f(i)\|) \) which is equivalent to
\( f(i)/\|f(i)\| \in \text{psc } B_{X_i} \).

Conversely, suppose \( f_n \in B_{l^p(X_i)} \) with weak-\( \lim_n f_n = f \). Then weak-\( \lim_n f_n(i) = f(i) \), \( i \in I \) and w-\( \lim_n \frac{1}{2}(f_n + f) = f \). Since \( \|f\| = 1 \) we must have
\( \lim_n \frac{1}{2}(\|f_n(\cdot)\| + \|f(\cdot)\|) = 1 \) in \( l^p(I) \). By the uniform convexity of \( l^p(I) \),
\( \lim_n \|f_n(\cdot) - f(\cdot)\| = 0 \). So for each \( i \in I \), \( \lim_n \|f_n(i) - f(i)\| = 0 \). Using the fact that \( f(i) \in \text{psc } B_{X_i}(0, \|f(i)\|) \), we can conclude that \( \lim_n \|f_n(i) - f(i)\| = 0 \). Hence \( \lim_n \|f_n - f\| = 0 \), and so \( f \in \text{psc } B_{l^p(X_i)} \).

The proofs for pc, w*-psc and w*-pc points are similar while that for extreme points can be found in [Sm1]. The conclusion for denting (resp. w*-denting) points follows from Proposition 2.9. QED

As a corollary of Proposition 2.14, if \( (\Omega, \Sigma, \mu) \) is purely atomic and \( f \) is a
unit vector in \( L^p(\mu, X) \), then \( f \in \text{psc } B_{L^p(\mu, X)} \) (resp. pc \( B_{L^p(\mu, X)} \); dent \( B_{L^p(\mu, X)} \)) if and only if \( f(t)/\|f(t)\| \in \text{psc } B_X \) (resp. pc \( B_X \); dent \( B_X \)) for almost all \( t \in \text{supp } f \).

For the proof of our next result, we need the following facts: \( X \) has the
CPCP (resp. PCP) if and only if given \( \varepsilon > 0 \) and any non-empty bounded convex (resp. bounded) set \( K \) in \( X \), there is a relatively weakly open set \( V \) in \( K \) with diameter less than \( \varepsilon \); \( X^* \) has the C*PCP if and only if given \( \varepsilon > 0 \) and any non-empty bounded convex set \( K \) in \( X^* \), there is a relatively weak* open set \( V \) in \( K \) with diameter less than \( \varepsilon \) (see [R2]).

Theorem 2.15. Let \( \{X_i, i \in I\} \) be a family of Banach spaces. Then:

1. \( l^p(X_i) \) has the CPCP (resp. PCP) if and only if each \( X_i \) has the CPCP (resp. PCP).

2. \( l^p(X_i)^* \) which can be identified as \( l^q(X_i^*) \) has the C*PCP if and only if each \( X_i^* \) has the C*PCP.
Proof. Assume that each $X_i$ has the CPCP and $I = \{1, 2\}$. Since the CPCP is an isomorphic invariant, it suffices to show that the space

$$X = \{(x_1, x_2) : x_i \in X_i, i = 1, 2, \|x_1, x_2\| = \max(\|x_1\|, \|x_2\|)\}$$

has the CPCP.

Let $A$ be a non-empty bounded convex set in $X$ and let $P_i : X \to X_i$, $i = 1, 2$, be the natural projection. Let $A_1 = P_1(A)$. Since $X_1$ has the CPCP, there exist $x_j^*, a_j > 0$, $j = 1, \ldots, n$ such that $\text{diam} \bigcap_{j=1}^n S(x_j^*, A_1, a_j) < \epsilon$.

Let

$$A_2 = \bigcap \left( P_2^{-1}\left( \bigcap_{j=1}^n S(x_j^*, A_1, a_j) \right) \right).$$

Then $A_2$ is a non-empty bounded convex set in $X_2$. Since $X_2$ has the CPCP there are $y_k^*, b_k > 0$, $k = 1, \ldots, m$ such that $\text{diam} \bigcap_{k=1}^m S(y_k^*, A_2, b_k) < \epsilon$.

Put

$$V = \{(x_1, x_2) : (x_1, x_2) \in A, \sup_{j=1}^n x_j^*(A_1) - a_j, y_k^*(x_2) > \sup_{k=1}^m y_k^*(A_2) - b_k, j = 1, \ldots, n, k = 1, \ldots, m\}.$$

Then $V$ is a weakly open set in $A$ with diameter less than $\epsilon$. Therefore $X$ has the CPCP.

To prove the general case, let $E = l^p(X_i)$ and let $A$ be a non-empty bounded closed convex set in $E$. Without loss of generality, assume that $\text{sup}\{\|x\|, x \in A\} = 1$. Given $\epsilon > 0$, we can choose $0 < \epsilon_1 < 1 - [1 - (\epsilon/3)^p]^{1/p}$ and $x = (x_i)_{i \in I}$ in $A$ with $\|x\|^p > 1 - \epsilon_1$. Then there exists $i_k \in I$, $k = 1, \ldots, n$, such that $\Sigma_{k=1}^n \|x_{i_k}\|^p > 1 - \epsilon_1$. For each $k = 1, \ldots, n$, choose $x_k^*$ in $X_k^*$ such that $\|x_k^*\|^q = \|x_k\|^p$ and $(x_k^*, x_{i_k}) = \|x_k\|^p$. Let $x^* = (x_k^*)_{i \in I}$ where $x_{i_k} = 0$ for all $i \neq i_k$, $k = 1, \ldots, n$. Then $x^* \in l^q(X_k^*)$, $\|x^*\| \leq 1$ and $(x^*, x) = \Sigma_{k=1}^n \|x_{i_k}\|^p > 1 - \epsilon_1$.

Let $E_1 = l^p(X_i)$, $E_2 = l^p(X_i)$ and let $P : E \to E_1$ be the natural projection. Without loss of generality, we may regard $E_1$ and $E_2$ as subspaces of $E$. Let $\delta = \text{sup} x^*(A) - 1 + \epsilon_1$. Then for any $y = (y_i)_{i \in I}$ in $S(x^*, A, \delta)$ we have

$$\|Py\| \geq (x^*, Py) = (x^*, y) > 1 - \epsilon_1 > \left[1 - (\epsilon/3)^p\right]^{1/p}.$$ 

Hence

$$\|y - Py\| = (\|y\|^p - \|Py\|^p)^{1/p} < \epsilon/3.$$

By the first part of the proof, $E_1$ has the CPCP. So there is a weakly open set
Let $V = (V_1 \oplus E_2) \cap S(x^*, A, \delta)$. Then $V$ is non-empty and weakly open in $A$ and for any $y$ and $z$ in $V$, we have

$$\|y - z\| \leq \|y - Py\| + \|Py - Pz\| + \|Pz - z\| < \varepsilon.$$ 

Hence the diameter of $V$ is less than or equal to $\varepsilon$ and so $E$ has the CPCP.

The proofs of the remaining assertions are similar. QED

**Remark 2.16.** The PCP is a three-space property; i.e., if $Y$ is a subspace of $X$ such that both $Y$ and $X/Y$ have the PCP, then $X$ also has the PCP [R2], and this fact implies that $l^p(X_i)_{i \in I}$ has the PCP if $I$ is finite and $X_i$ has the PCP for every $i \in I$. However it is unknown whether CPCP or C*PCP is a three-space property.

**References**


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