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ON THE SEQUENCES THAT ARE GOOD IN THE MEAN FOR POSITIVE L_p-CONTRACTIONS, $1 \leq p < \infty$

DOĞAN ÇÖMEZ AND SEBASTIAN FERRANDO

1. Introduction

It is a well-known fact that, if a weight **n** is good a.e. for all operators induced by measure preserving transformations (MPTs), then it is also good a.e. for any Dunford-Schwartz operator (i.e., $L_1 - L_\infty$-contraction) [BO]. Similar results have been obtained in various other settings [JO], [JOW], [ÇLO]. When T is an L_1-contraction induced by an MPT, various types of sequences, such as $\mathbb{Z}$, block sequences [BL], sequences satisfying the cone condition [BL], [RW], sequence of squares and sequence of primes [RW] are good in the mean for T. Recently, it was proved in [F] that, sequences satisfying the cone condition are good a.e. and in the mean for the class of bounded superadditive processes relative to MPTs.

In this article, our aim is to show that sequences which are good in the mean for invertible MPTs are also good in the mean for T-(super)additive processes relative to positive L_p-contractions (when $1 < p < \infty$), or positive Dunford-Schwartz operators on L_1.

Let (X, Σ, μ) be a finite measure space, and let T: $L_p(X) \longrightarrow L_p(X)$ be a positive linear contraction where $1 \leq p < \infty$ is fixed. In order to avoid certain difficulties we will assume that (X, Σ, μ) is a Lebesgue space. A strictly increasing sequence $\mathbf{n} = \{n_k\}$ of integers is called *good in the p-mean for T* if, for every $f \in L_p$, $\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} T^{n_i} f$ exists in the L_p-norm . If τ is a measurable transformation on X, we say that **n** is *good in the p-mean for* τ when it is good in the p-mean for the operator T induced by τ. As usual, **n** is called *good in the p-mean* if it is good in the p-mean for all MPTs.

A family $F = \{F_n\}_{n\geq 0}$ of functions in L_p is called a *T-superadditive process* if $F_{n+m} \geq F_n + T^n F_m$ a.e. for all $n, m \geq 0$ $(F_0 = 0)$, where T is a positive linear operator on L_p. If the reverse inequality holds, it is called *T-subadditive*, and if the equality holds, i.e., $F_n = \sum_{i=0}^{n-1} T^i F_1$, it is called *T-additive*. A nonnegative T-superadditive process F is called *bounded* if $\gamma_F = \sup_{n\geq 1} \frac{1}{n} \int F_n d\mu < \infty$. It is well known that, if $F \subset L_1^+$ is bounded, then $\lim_{n\to\infty} \frac{1}{n} \int F_n d\mu = \gamma_F$.

In order to define the "averages" of a T-superadditive process $F = \{F_n\}$ along a general sequence **n**, it will be convenient to view F as a collection of functions $\{f_k\}$

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in L_p with partial sums $F_n = f_0 + f_1 + \cdots + f_{n-1}$ satisfying the condition

$$T^m F_n \leq F_{m+n} - F_m, \quad m, n \geq 0.$$

Following [JO], in the sequel we will use the generalized notation for sequences. Namely, a sequence $\mathbf{n}$ will be a family of integers $\mathbf{n} = \{n(k,l)\}$ such that

$$n(k,l_1) < n(k,l_2) \text{ if } l_1 < l_2, \quad \text{and} \quad n(k_1,l) \leq n(k_2,l) \text{ if } k_1 \leq k_2 .$$

Given a sequence $\mathbf{n} = \{n(k,l)\}$, we define the average of a superadditive process F along $\mathbf{n}$ as, for $K \geq 1$ and $L \geq 1$,

$$\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)}(x). \tag{$*$}$$

(When $\mathbf{n} = \{n_k\}$, the averages of F along $\mathbf{n}$ will be $\frac{1}{N}\sum_{k=0}^{N-1} f_{n_k}$.) A sequence $\mathbf{n} = \{n(k,l)\}$ is called *good in the p-mean for T-superadditive processes* if, for every T-superadditive process F, $\lim_{K,L\to\infty} \frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)}$ exists as a double limit in L_p-norm.

2. The apparatus and the main theorem

The problem of determining when a sequence $\mathbf{n}$ is good in the mean for MPTs was settled by Rosenblatt [R, Theorem 1]:

(i) *For* $1 \leq p < \infty$, *a sequence* $\mathbf{n}$ *is good in the p-mean for invertible MPTs if and only if* $\lim_{K,N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi i n(K,j)\beta}$ for all $\beta \in [0,1)$.

(ii) *When* τ *is a MPT,* $\frac{1}{L}\sum_{j=0}^{L-1} f(\tau^{n(K,j)}x)$ *converges in* L_p*-norm* $\forall f \in L_p$, $1 \leq p < \infty$, *if and only if it converges in* L_2*-norm for all* $f \in L_2$.

Naturally, one asks if the existence of the limit in (i) would imply that the sequence is good in the p-mean for some operators on L_p. In this section, we will show that if $\mathbf{n}$ is good in the p-mean for (super)additive processes relative to MPTs, then it is good in the p-mean for (super)additive processes relative to positive L_p-contractions, when $1 < p < \infty$, or relative to positive Dunford-Schwartz operators on L_1. The main tool in obtaining this result is the *apparatus* below. Parts of the construction of the apparatus are standard, therefore we will only give an outline of it here (adapted to the superadditive setting). The reader is referred to [JO], [J] for the details. In [JO], due to the intended purpose there, only the case $1 < p < \infty$ was studied. Clearly, the complications with $p = 1$ in [JO] are due to Chacon's counterexample for a.e. convergence for positive linear isometries in L_1. We are mainly interested in norm convergence, hence it is natural to consider also the case $p = 1$.

The apparatus. Let T and S be linear operators induced by the nonsingular invertible point transformations τ and σ on X and Y, respectively, with w and z as the associated weight functions. (If σ is measure preserving, $z = 1$.) Then

$$T^n(f)(x) = f(\tau^n x)w_n(x) \quad \text{and} \quad S^n(f)(x) = f(\sigma^n x)z_n(x), \quad n \geq 1.$$

where $w_n = w(w \circ \tau)\dots(w \circ \tau^{n-1})$ and $z_n = z(z \circ \tau)\dots(z \circ \tau^{n-1})$. Also assume that τ is aperiodic. Fix $K \geq 1$ and $L \geq 1$, and choose $\delta > 0$ such that

$$\int_E \left| \frac{1}{L} \sum_{j=0}^{L-1} f_{n(K,j)} \right|^p d\mu < \epsilon \quad \text{if} \quad \mu(E) < \delta.$$

Let $m \in \mathbb{Z}^+$ such that $\frac{1}{m} < \frac{\delta}{2}$ and $n(K, L) < m$. Construct a Rohlin-Kakutani tower $\{A_k\}_{k=1}^{m^2}$ for τ with error less than $\frac{\delta}{2}$, and $\mu(\cup_{k=m^2-m+1}^{m^2} A_k) < \frac{\delta}{2}$. Similarly, construct a Rohlin-Kakutani tower $\{B_k\}_{k=1}^{m^2}$ for σ with error less than $\frac{\delta}{2}$. Pick a constant β such that $\mu(A_1) = \beta\nu(B_1)$. Let ϕ: $A_1 \to B_1$ be an invertible MPT and define H: $L_p(A_1) \to L_p(B_1)$ by $H(f)(y) = f(\phi^{-1}y)\beta^{1/p}$, $1 \leq p < \infty$, for supp$(f) \subset A_1$. Then $\int_{B_1} |Hf(y)|^p d\nu = \int_{A_1} |f(x)|^p d\mu$. Let $A = \cup_{k=1}^{m^2} A_k$, and $B = \cup_{k=1}^{m^2} B_k$, and then extend H: $L_p(A) \to L_p(B)$ as

$$H(f)(y) = f(\Phi^{-1}y)\frac{w_k(\tau^{-k}\Phi^{-1}y)}{z_k(\sigma^{-k}y)}\beta^{1/p},$$

where Φ: $A \to B$ is the extension of ϕ defined by $\Phi(x) = (\sigma^k \phi \tau^{-k})(x)$, $1 \leq k \leq m^2$, for $x \in A_k$. Consequently, if supp$(f) \subset A$, then supp$(Hf) \subset B$, and

$$\int_A |f|^p d\mu = \int_B |Hf|^p d\nu. \tag{1}$$

From the construction, it follows that, for $f \in L_1 \cap L_\infty$, $\|Hf\|_{L_\infty(B)} = \|f\|_{L_\infty(A)}$.

LEMMA 2.1. *Let the operators T, S, and H be as above. If $\{F_n\} \subset L_p(X)$ is a T-superadditive process, then $\{HF_n\} \subset L_p(Y)$ is an S-superadditive process.*

Proof. It is enough to show that $(HF_n)(\sigma^m y)z_m(y) = H[(F_n \circ \tau^m)w_m](y)$. Now,

$$\begin{aligned} H[(F_n \circ \tau^m)w_m](y) &= [(F_n \circ \tau^m)(\Phi^{-1}y)]w_m(\Phi^{-1}y)\frac{w_k(\tau^{-k}\Phi^{-1}y)}{z_k(\sigma^{-k}y)}\beta^{1/p} \\ &= [(F_n(\Phi^{-1}\sigma^m y)]w_m(\tau^{-m}\Phi^{-1}\sigma^m y)\frac{w_k(\tau^{-k-m}\Phi^{-1}\sigma^m y)}{z_k(\sigma^{-k}y)}\beta^{1/p}. \end{aligned}$$

Since $w_{k+m}(\tau^{-k-m}\Phi^{-1}\sigma^m y) = w_m(\tau^{-m}\Phi^{-1}\sigma^m y)w_k(\tau^{-k-m}\Phi^{-1}\sigma^m y)$, and $z_{k+m}(\sigma^{-k-m}\sigma^m y) = z_m(y)z_k(\sigma^{-k}y)$, we have

$$\begin{aligned} H[(F_n \circ \tau^m)w_m](y) &= F_n(\Phi^{-1}\sigma^m y)\frac{w_{k+m}(\tau^{-k-m}\Phi^{-1}\sigma^m y)}{z_{k+m}(\sigma^{-k-m}\sigma^m y)}z_m(y)\beta^{1/p} \\ &= HF_n(\sigma^m y)z_m(y) \end{aligned}$$

proving the desired equality. □

Now we are ready to obtain the main result.

THEOREM 2.2. *Let T be a positive L_p-contraction, $1 < p < \infty$, or a positive Dunford-Schwartz operator on L_1. If $\mathbf{n} = \{n(k,l)\}$ is a sequence of positive integers which is good in the p-mean for a class of superadditive processes relative to MPTs, then it is good in the p-mean for T-superadditive processes of the same class.*

Proof. First we will prove the theorem when T is a positive invertible linear isometry of $L_p(X)$, $1 < p < \infty$. Let T be induced by an aperiodic invertible nonsingular transformation τ, and let $(Y, \Sigma', \nu, \sigma)$ be an invertible measure preserving system. Fix K, L, and consider the apparatus above (where S is the isometry induced by σ). Let $x \in A_k \cap \operatorname{supp}(\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)})$. If $y = \Phi x$, then

$$\frac{1}{L}\sum_{j=0}^{L-1} Hf_{n(K,j)}(y) = H\left[\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)}(x)\right]. \tag{2}$$

Now, for any two pairs K, L and K', L', let δ_1 and δ_2 be chosen such that

$$\int_E \left|\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)}\right|^p d\mu < \frac{\epsilon}{2} \quad \text{and} \quad \int_E \left|\frac{1}{L'}\sum_{j=0}^{L'-1} f_{n(K',j)}\right|^p d\mu < \frac{\epsilon}{2}$$

if $\mu(E) < \delta_1$ and $\mu(E) < \delta_2$, respectively, as in the apparatus. Let $\delta = \min\{\delta_1, \delta_2\}$. Pick m so that $\frac{1}{m} < \frac{\delta}{2}$ and $n(K,L), n(K',L') < m$, and construct the mapping H as in the apparatus. Consequently, (2) holds for both $\sum_{j=0}^{L-1} f_{n(K,j)}$ and $\sum_{j=0}^{L'-1} f_{n(K',j)}$. Then

$$\begin{aligned}
&\left\|\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)} - \frac{1}{L'}\sum_{j=0}^{L'-1} f_{n(K',j)}\right\|_{L_p(X)}^p \\
&= \int_A \left|\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)} - \frac{1}{L'}\sum_{j=0}^{L'-1} f_{n(K',j)}\right|^p d\mu \\
&\quad + \int_{A^c} \left|\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)} - \frac{1}{L'}\sum_{j=0}^{L'-1} f_{n(K',j)}\right|^p d\mu \\
&\le \int_A \left|\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)} - \frac{1}{L'}\sum_{j=0}^{L'-1} f_{n(K',j)}\right|^p d\mu + \epsilon \quad \text{by the choice of } \delta, \\
&= \int_B \left|H\left[\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)} - \frac{1}{L'}\sum_{j=0}^{L'-1} f_{n(K',j)})\right]\right|^p d\nu + \epsilon \qquad \text{by (1)}, \\
&\le \left\|\frac{1}{L}\sum_{j=0}^{L-1} Hf_{n(K,j)} - \frac{1}{L'}\sum_{j=0}^{L'-1} Hf_{n(K',j)}\right\|_{L_p(Y)}^p + \epsilon \qquad \text{by (2)}.
\end{aligned}$$

Letting $\epsilon \to 0$, and then using the fact that $\frac{1}{L}\sum_{j=0}^{L-1} Hf_{n(K,j)}$ is Cauchy in the norm by hypothesis (since S is induced by measure preserving transformation, and Hf_n is S-superadditive by Lemma 2.1), we have that the averages of the original T-superadditive process is Cauchy in the norm. This proves the assertion when τ is aperiodic. If τ is periodic with period d, the same argument applies with minor modifications after replacing the sets A_i by disjoint sets $A_1, A_2, \dots A_d$, where if $x \in A_1$, then $\tau^k x \in A_k$, $k = 1, 2, \dots d$, and $\tau^{d+1}x = x$.

Next, let T be a positive L_p-contraction. By Akcoglu-Sucheston dilation theorem [AS] there exists another (larger) L_p-space, say L, and a positive invertible isometry Q: $L \to L$ so that $DT^n = EQ^nD$ for $n \geq 0$, where D: $L_p(X) \to L$ is a positive isometric imbedding of $L_p(X)$ into L and E: $L \to L$ is a positive projection. Here, the process $\{Df_k\}$ is Q-superadditive in $D(L_p)$, and, for any sequence $\mathbf{n}$,

$$D\left[\frac{1}{L}\sum_{j=0}^{L-1} f_{n(K,j)}\right] = E\left[\frac{1}{L}\sum_{j=0}^{L-1} Df_{n(K,j)}\right]. \tag{3}$$

By the first part, $\mathbf{n}$ is good in the p-mean for superadditive processes relative to positive invertible isometries, thus it is good in the p-mean for Q, and by (3), it is good in the p-mean for T.

When $p = 1$, the same argument in the first part implies that if $\mathbf{n}$ is good in the 1-mean for superadditive processes relative to MPTs, then it is good in the 1-mean for superadditive processes relative to positive invertible isometries which are also L_∞-contractions. Again, we use Akcoglu-Sucheston dilation theorem (for $p = 1$) to obtain (3). Since E and D preserve L_∞-norm for $f \in L_1 \cap L_\infty$ (see also [A]), the assertion follows from the same argument as in the case $1 < p < \infty$. $\square$

Not every sequence which is good in the p-mean is good in the p-mean for superadditive processes (see the example below). However, the method of proof of Theorem 2.2, adapted to the additive processes, also gives:

THEOREM 2.3. *Let T be a positive L_p-contraction, $1 < p < \infty$, or a positive Dunford-Schwartz operator on L_1. If $\mathbf{n} = \{n(k, l)\}$ is a sequence of positive integers which is good in the p-mean, then it is good in the p-mean for T.*

Remark. If $\mathbf{n}$ is good in the mean for invertible isometries, then an argument similar to that of Theorem 2.2 above shows that it is also good in the p-mean for T-additive processes when T is a positively dominated operator on L_p, $1 < p < \infty$, or is a (not neccesarily positive) Dunford-Schwartz operator, or is a power bounded Lamperti operator.

Combining Theorem 2.3 and the theorem of Rosenblatt [R], if we define the *Fourier coeffícent function* $C(\beta)$ of $\mathbf{n}$, by $C(\beta) = \lim_{K,N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi i n(K,j)\beta}$, whenever the limit exists for all $\beta \in [0, 1)$, we obtain:

THEOREM 2.4. *If* **n** *is a sequence whose Fourier coefficent function* $C(\beta)$ *exists for all* $\beta \in [0, 1)$, *then it is good in the p-mean for all positively dominated* L_p-*contractions,* $1 < p < \infty$, *or Dunford-Schwartz operators on* L_1.

Recall that the result (i) of Rosenblatt provides a criterion for determining the sequences that are good in the 2-mean for additive processes (relative to MPTs). There is, yet, no such criteria known for superadditive processes. We will show in the following section that for a certain family of sequences (B-sequences) if the process F has some additional properties, or if the averages of F along subsequences are defined differently, one can say more about the convergence in the mean. Indeed, the following shows that the question of which sequences **n** are good in the p-mean for superadditive processes is very delicate (which will be discussed in a separate article).

Example. Let $f_n = (-1)^n$, $n = 0, 1, 2, \ldots$ Clearly $F = \{F_n\}$ is a bounded subadditive process (on a one point space). Now, we will define, inductively, a sequence $\{n_k\}$ such that $\lim_N \frac{1}{N}\sum_{k=0}^{N-1} f_{n_k}$ fails to exist. For, let $n_0 = 0$, $n_1 = 1$, $n_2 = 2$, $n_3 = 4$, $n_4 = 5$, $n_5 = 7$, and

$$\begin{aligned} &n_{3^i2+j},\ 0 \le j < 3^i2, \text{ are the next } 3^i2 \text{ even numbers after } n_{3^i2-1} \\ &n_{3^i4+j},\ 0 \le j < 3^i2, \text{ are the next } 3^i2 \text{ odd numbers after } n_{3^i4-1}. \end{aligned}$$

Then $\frac{1}{N}\sum_{k=0}^{N-1} f_{n_k} = 0$ if $N = 3^i2$, and is $\frac{1}{2}$ if $N = 3^i4$. Hence, $\liminf_N \frac{1}{N}\sum_{k=0}^{N-1} f_{n_k} = 0$, whereas $\limsup_N \frac{1}{N}\sum_{k=0}^{N-1} f_{n_k} = \frac{1}{2}$.

Remarks. **1.** The Fourier coefficent function of this sequence exists for all $\beta \in [0, 1)$, hence, by Theorem 2.4, it is good in the p-mean (for additive processes).

2. For the process F in the example above, the sequence of even (or odd) integers, sequence of squares, sequence of primes, and block sequences (i.e., sequences of the form $\mathbf{n} = \cup B_k$, where $B_k = \{n_k + i\}_{i=0}^{l_k}$, and $n_k \uparrow$, $l_k \uparrow$ with $n_k + l_k < n_{k+1}$) are good in the p-mean.

3. The superadditive case revised

In [F] it was shown that if **n** is a B-sequence (see definition below), then the averages of bounded superadditive processes relative to MPTs along **n** (called *moving averages*) are good in the 1-mean as well as a.e. However, the definition of moving averages used there is different than $(*)$. In this section we will show in Proposition 3.1 that if the definition $(*)$ is used, then B-sequences and block sequences need not be good in the p-mean (nor a.e.) for bounded superadditive processes. We emphasize that in these two cases the additive averages do converge. For these subsequences we study two different solutions: the first one is to consider a more restrictive class of superadditive processes (the Chacon processes) and use the definition $(*)$, and the second solution is to redefine the superadditive averages for these subsequences using the definition in [F]. In both cases, we will show that B-sequences (and block sequences), which are good in the 1-mean for additive processes, are good in the 1-mean for bounded superadditive processes. In the sequel, we only concentrate on B-sequences given that the case of block sequences is similar. We will also concentrate only on the case $p = 1$.

A sequence $\mathbf{n} = \{(v_n, r_n)\}_{n=0}^{\infty}$ in $\mathbb{Z} \times \mathbb{Z}$ such that $r_n > 0$ for all n is called a *B-sequence* if there is a constant satisfying

$$|\{k : \exists n, \quad k + [v_n, v_n + r_n) \subset I\}| \leq B|I|$$

for every interval $I \subset \mathbb{Z}$, where $|S|$ denotes the cardinality of a set $S \subset \mathbb{Z}$. Notice that, if $l = l(j) = r_j$, $k = k(j) = v_j$, and $n(k, i) = k(j) + i$, then B-sequences can be viewed in terms of the general definition of subsequences. We observe that the above definition of superadditive averages along a subsequence can be written as follows:

$$\frac{1}{r_n}(F_{v_n + r_n} - F_{v_n}), \tag{4}$$

PROPOSITION 3.1. *Let $\{(v_k, r_k)\}_{k=1}^{\infty}$ be a sequence of integers satisfying*

$$\sum_{k=1}^{\infty} \frac{r_k}{v_k + r_k} < \frac{1}{4},$$

where $\{v_n\}$ and $\{r_n\}$ are strictly increasing sequence of positive integers. Then there exists a nonnegative, bounded superadditive process $\{F_n\}_{n=0}^{\infty}$ (on a one-point measure space) such that (4) *diverges.*

Proof. Let $\Omega = \{w_0\}$, $\mu(w_0) = 1$. We will define a new sequence

$$\{(v'_k, r'_k)\}_{k=1}^{\infty} \subset \{(v_k, r_k)\}_{k=1}^{\infty}$$

as follows: let $(v'_1, r'_1) = (v_1, r_1)$, and

$$v'_2 = v_{n_2}, \quad v_{n_2} \text{ such that } v_{n_2} > v'_1 + r'_1 \quad \text{and} \quad r'_2 = r_{n_2}, \text{ and}$$

$$r'_3 = r_{n_3}, \quad r_{n_3} \text{ such that } \tfrac{v'_2 + r'_2}{r_{n_3}} \leq \tfrac{1}{3} \quad \text{and} \quad v'_3 = v_{n_3}.$$

In general

$$v'_{2k} = v_{n_{2k}} \text{ where } v_{n_{2k}} > v'_{2k-1} + r'_{2k-1} \text{ and } r'_{2k} = r_{n_{2k}} \text{ for } k = 1, 2, \ldots$$

$$r'_{2k+1} = r_{n_{2k+1}} \text{ where } \tfrac{v'_{2k} + r'_{2k}}{r_{n_{2k+1}}} \leq \tfrac{1}{3} \text{ and } v'_{2k+1} = v_{n_{2k+1}} \quad \text{for } k = 1, 2, \ldots.$$

Define $X^{2k} = (X_j^{2k})_{j=1}^{\infty}$ where

$$X_j^{2k} = \begin{cases} 1 & \text{if } j \in [nv'_{2k} + (n-1)r'_{2k}, nv'_{2k} + nr'_{2k}) \text{ for some } n \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

for $k = 1, 2, \ldots$, and $X^{2k+1} = (0)_{j=1,\ldots}$ for $k = 0, 1, 2, \ldots$ Moreover we also define

$$Y_p = \sum_{i=1}^{\infty} X_p^i, \quad Y_0 = 0 \qquad \text{and} \qquad F_n = \sum_{p=1}^{n-1} Y_p, \quad F_0 = 0.$$

A picture for the above definitions is as follows: X^i is the infinitely long i^{th} row, and Y_p is the sum of all the elements in the p^{th} column.

Because $v_n \to \infty$ we see that $F_n < \infty$ for all $n \geq 1$. We first check that $\sup_{n\geq 1} \frac{1}{n} F_n < \frac{1}{3}$. To prove this, notice that if we fix $n \geq 1$ then there exists v'_{k_n} satisfying $v'_{k_n} \leq n$ and $v'_{k_n+1} > n$. For a given i, $1 \leq i \leq k_n$, we write

$$n = \delta_i(v'_i + r'_i) + \gamma_i \tag{5}$$

where δ_i is an integer and $0 \leq \gamma_i < v'_i + r'_i$ we notice that the condition $\sum_{i=1}^{\infty} \frac{r'_i}{v'_i + r'_i} < \frac{1}{4}$ implies:

$$\frac{4}{3}\left(\frac{v'_i}{v'_i + r'_i}\right) > 1. \tag{6}$$

Hence using (5), (6) and the fact that S_n is the sum of the ones in the first n columns we estimate as follows: It is enough to consider the case $\gamma_i \geq v'_i$ for each i:

$$\begin{aligned} \frac{1}{n} F_n &\leq \frac{1}{n} \sum_{i=1}^{k_n} (r'_i \delta_i + r'_i) \leq \frac{1}{n} \sum_{i=1}^{k_n} \frac{4}{3} r'_i \left(\delta_i + \frac{v'_i}{v'_i + r'_i}\right) \\ &\leq \frac{4}{3n} \sum_{i=1}^{k_n} r'_i \left(\delta_i + \frac{\gamma_i}{v'_i + r'_i}\right) = \frac{4n}{3n} \sum_{i=1}^{k_n} \frac{r'_i}{(v'_i + r'_i)} < \frac{1}{3}. \end{aligned}$$

The next property we verify is $F_n + F_m \leq F_{m+n}$, $\forall m, n \geq 0$ (superadditivity). This follows from the inequality

$$\sum_{p=0}^{n-1} Y_p \leq \sum_{p=m}^{m+n-1} Y_p \qquad m, n \geq 0,$$

which in turn follows from $\sum_{j=1}^{n-1} X_j^i \leq \sum_{j=m}^{m+n-2} X_j^i$, $\forall i \geq 1$. This last inequality follows from the definition given for X_j^{2k}. So superadditivity is proved.

From the definitions it follows easily that

$$\frac{(F_{v'_{2k} + r'_{2k}} - F_{v'_{2k}})}{r'_{2k}} \geq 1, \qquad \forall k = 1, 2, \ldots \tag{7}$$

We finally claim that

$$\frac{(F_{v'_{2k+1}+r'_{2k+1}} - F_{v'_{2k+1}})}{r'_{2k+1}} \leq \frac{1}{3}, \qquad k = 0, 1, 2, \ldots \tag{8}$$

To check (8) take $k \geq 0$. By the construction, only the first $2k$ rows contribute with ones to the difference $(F_{v'_{2k+1}+r'_{2k+1}} - F_{v'_{2k+1}})$. Write

$$r'_{2k+1} = q_{2j}(v'_{2j} + r'_{2j}) + \gamma_{2j} \tag{9}$$

for $j = 1, \ldots, p$; q_{2j} an integer and $0 \leq \gamma_{2j} < (v'_{2j} + r'_{2j})$. We notice that by the definitions,

$$\frac{1}{3} \geq \frac{v'_{2k} + r'_{2k}}{r'_{2k+1}} \geq \frac{v'_{2j} + r'_{2j}}{r'_{2k+1}} \qquad 1 \leq j \leq k,$$

hence,

$$\frac{1}{3}\left(\frac{r'_{2j}}{v'_{2j} + r'_{2j}}\right) \geq \frac{r'_{2j}}{r'_{2k+1}}, \quad j = 1, \ldots, k. \tag{10}$$

Therefore, using (9) and (10) it follows from the definitions that

$$\begin{aligned} \frac{(F_{v'_{2k+1}+r'_{2k+1}} - F_{v'_{2k+1}})}{r'_{2k+1}} &\leq \sum_{j=1}^{k} \frac{(q_{2j}r'_{2j} + r'_{2j})}{r'_{2k+1}} \leq \sum_{j=1}^{k}\left(\frac{q_{2j}r'_{2j}}{r'_{2k+1}} + \frac{1}{3}\frac{r'_{2j}}{v'_{2j} + r'_{2j}}\right) \\ &\leq \sum_{j=1}^{k}\left(\frac{r'_{2k+1}r'_{2j}}{r'_{2k+1}(v'_{2j} + r'_{2j})} + \frac{1}{3}\frac{r'_{2j}}{(v'_{2j} + r'_{2j})}\right) \leq \frac{1}{3}. \end{aligned}$$

Combining (7) and (8) we conclude that $\lim_{n\to\infty} \frac{1}{r'_n}(F_{v'_n+r'_n} - F_{v'_n})$ does not exist. □

Remarks. **1.** We notice that a subsequence of a B-sequence is also a B-sequence. For instance, consider the example $v_k = (k+1)!$, $r_k = k!$. Hence, $r_k/v_k \to 0$ and then taking an appropriate subsequence the condition $\sum_{k=1}^{\infty} \frac{r_k}{v_k+r_k} < \frac{1}{4}$ used in the proposition above can be satisfied for some B-sequence.

2. The example above may be adapted to give a counterexample for the case of averages of bounded superadditive processes along Block sequences.

The counterexample constructed in Proposition 3.1 raises the question: Is there a class of one-parameter superadditive processes for which $\frac{1}{r_k}(F_{v_k+r_k}(w) - F_{v_k}(w))$ converges in the mean (and possibly pointwise) as $r_k \to \infty$, when F is in that class? Proposition 3.2 below proves that the so called Chacon's admissible processes give an affirmative answer to this question.

Definition. A collection $\{f_0, f_1, \ldots\}$ of functions in $L_1(X, \Sigma, \mu)$ is said to be a *bounded Chacon admissible process* with respect to a positive linear contraction S if

$$Sf_i \leq f_{i+1}, \qquad i \geq 0,$$

$$\sup_{n\geq 1} \frac{1}{n}\int_X F_n < \infty \qquad \text{where } F_n = \sum_{i=0}^{n-1} f_i, \;\; n \geq 1.$$

It follows that $F_k + S^k F_n \le F_{k+n}$, $k, n \ge 0$ (take $F_0 \equiv 0$). Therefore, a bounded Chacon admissible process is a bounded superadditive process.

We notice that the superadditive process constructed in Proposition 3.1 is not a Chacon process (neither is the process in the example following Theorem 2.4).

For the purpose of the next proposition we will assume that S is an operator on L_1, induced by an isomorphism on the base Lebesgue space. It is known that in this case S admits a weak type (1,1) maximal inequality along B-sequences for additive processes (in fact this is true for an arbitrary MPT [F]).

PROPOSITION 3.2. *Let S be an operator as above. Assume that $\{f_0, f_1, \ldots\}$, a collection of $L_1(X, \Sigma, \mu)$, is a bounded Chacon admissible process with respect to S. If $F_n \equiv \sum_{i=0}^{n-1} f_i$, $n \ge 1$, and $\{(v_n, r_n)\}_{n=1}^{\infty}$ is a B-sequence with $r_n \to \infty$, then*

$$\frac{(F_{v_n+r_n}(w) - F_{v_n}(w))}{r_n} \qquad \textit{converges a.e. and in the mean as} \quad n \to \infty.$$

Proof. Noticing that $\sum_{j=0}^{n-1} S^j f_0 \le F_n$, we can assume without loss of generality that $f_i \ge 0$, $i \ge 0$. For convenience, define $P_i = f_i - Sf_{i-1}$, $i \ge 1$; also we set $P_0 = f_0$. Using the fact that the process is bounded it can be proved that

$$\lim_{k\to\infty} \int f_k < \infty. \tag{11}$$

To obtain this inequality we compute as follows ($P_i \ge 0$):

$$\begin{aligned}
\int f_k &= \frac{1}{(r+1)} \sum_{j=0}^{r} \int S^j f_k \le \frac{1}{(r+1)} \sum_{j=0}^{r} \int f_{k+j} \\
&= \frac{1}{(r+1)} \left(\int F_{k+r+1} - \int F_k \right) \le \frac{(k+r+1)}{(r+1)} \left(\frac{1}{(k+r+1)} \int F_{k+r+1} \right) \\
&\le \frac{(k+r+1)}{(r+1)} \sup_{n\ge 1} \frac{1}{n} \int F_n.
\end{aligned}$$

Taking $r \to \infty$, this implies that $\lim_{k\to\infty} \int f_k \le \sup_{n\ge1} \frac{1}{n} \int F_n$.

For a given $k \ge 1$ define

$$g_n^k(w) = \begin{cases} S^{n-k} f_k(w) & \text{for } n > k \\ f_n(w) & \text{for } 0 \le n \le k. \end{cases}$$

Hence it follows that

$$f_n(w) - g_n^k(w) = \begin{cases} 0 \text{ if } \ 0 \le n \le k \\ \sum\limits_{i=1}^{m} S^{m-i} P_{k+i}(w) & \text{for} \quad n > k, \quad \text{where} \quad m = n - k. \end{cases} \tag{12}$$

Define $M_i(f - g^k) = \sum_{n=v_i}^{v_i+r_i-1} f_n - g_n^k$. Using (12) we estimate that

$$M_i(f - g^k)(w) \leq \sum_{j=v_i}^{v_i+r_i-1} \sum_{r=k+1}^{v_i} S^{j-r} P_r(w).$$

Also define $b_{k,q}(w) = \sum_{r=k+1}^{q} S_r P_r(w)$ and $b_k(w) = \lim_{q\to\infty} b_{k,q}(w)$. By an application of the Lebesgue monotone convergence theorem and equation (11) we obtain

$$\int_X b_k(w) d\mu(w) = \lim_{q\to\infty} \int_X b_{k,q}(w) d\mu(w) \leq \sum_{r=k+1}^{\infty} \int P_r \leq \lim_{k\to\infty} \int f_k < \infty. \tag{13}$$

Because $b_{k,q} \uparrow b$ and $b_k \in L_1$ we conclude $S^j b_{k,q} \uparrow S^j b_k$. Therefore (13) implies

$$M_i(f - g^k) \leq \sum_{j=v_i}^{v_i+r_i-1} S^j b_k. \tag{14}$$

As usual, define

$$f^1(w) = \limsup_{n\to\infty} \frac{1}{r_n} \sum_{j=v_n}^{v_n+r_n-1} f_j(w) \quad f^2(w) = \liminf_{n\to\infty} \frac{1}{r_n} \sum_{j=v_n}^{v_n+r_n-1} f_j(w).$$

For an arbitrary $\alpha > 0$ define $E = \{w \mid f^1(w) - f^2(w) > \alpha\}$. Then to finish the proof we need to show that $\mu(E) = 0$. For $k \geq 1$ set

$$G_k(w) \equiv \lim_{n\to\infty} \frac{1}{r_n} \sum_{j=v_n}^{v_n+r_n-1} (S^{j-k} f_k)(w) = \lim_{n\to\infty} \frac{1}{r_n} \sum_{j=v_n}^{v_n+r_n-1} g_j^k(w),$$

where the last equality is obvious for the cases in which $v_n \to \infty$ or $v_n \leq M, \ \forall n \geq 1$, and it also follows in the other case by an application of Theorem 3.3 in [F] (see the remark following Theorem 3.3 in [F]). Therefore

$$\begin{aligned} E &= \left\{w \mid \left[(f^1(w) - G_k(w)) - (f^2(w) - G_k(w))\right] > \alpha\right\} \\ &\subset \left\{w \mid \left(\sup_{n\geq 1} \frac{1}{r_n} M_n(f - g^k)\right) > \frac{\alpha}{2}\right\} \\ &\subset \left\{w \mid \left(\sup_{n\geq 1} \frac{1}{r_n} \sum_{j=v_n}^{v_n+r_n-1} S^j b_k(w)\right) > \frac{\alpha}{2}\right\}, \end{aligned}$$

where we used (14) to obtain the last inclusion. By hypothesis, S admits a maximal inequality along the B-sequence for the additive process $\{S^j b\}_{j=0}^{\infty}$. Hence we obtain

$$\mu(E) \leq \frac{2C}{\alpha} \int_\Omega b(w) d\mu(w) = \frac{2C}{\alpha} \sum_{r=k+1}^{\infty} \int |P_r| < \infty,$$

where we used (12). Then taking $k \to \infty$ gives $\mu(E) = 0$. Now we show how this a.e result implies convergence in L_1. Given the hypothesis on the operator S and the fact that the sequence $\{f_k\}$ is a bounded Chacon process it follows that $G_k \le G_{k+1}$ and $\int G_k = \int f_k$, where the last equality follows from the identification of the limit result given in [F]. Therefore Lebesgue monotone convergence theorem guarantees the existence of the L_1-limit $G_\infty = L_1 - \lim_{k\to\infty} G_k$.

The proof will be finished by showing that $\lim_{n\to\infty} \|\frac{1}{r_n}(F_{v_n+r_n} - F_{v_n}) - G_\infty\|_1 = 0$. Given $\epsilon \ge 0$, use equation (11) to find K such that $\int f_n - \int f_m \le \frac{\epsilon}{2}$ for all $K \le m \le n$ and $\|G_\infty - G_k\|_1 \le \frac{\epsilon}{2}$ for all $k \ge K$. It has been shown in [F] that L_1 convergence holds for additive processes along B-sequences; therefore $\lim_{n\to\infty} \|G_k - \frac{1}{r_n}\sum_{j=v_n}^{v_n+r_n-1}(S^{j-k} f_k)\|_1 = 0$. Hence

$$\begin{aligned}\left\|\frac{1}{r_n}(F_{v_n+r_n} - F_{v_n}) - G_\infty\right\|_1 &\le \left\|\frac{1}{r_n}\left[(F_{v_n+r_n} - F_{v_n}) - \sum_{j=v_n}^{v_n+r_n-1}(S^{j-k} f_k)\right]\right\|_1 \\ &\quad + \left\|G_k - \frac{1}{r_n}\sum_{j=v_n}^{v_n+r_n-1}(S^{j-k} f_k)\right\|_1 + \|G_k - G_\infty\|_1 \\ &\le \int f_{v_n+r_n} - f_k + \frac{\epsilon}{2} \le \epsilon. \qquad \square\end{aligned}$$

To extend the mean result of Proposition 3.2 to the operator case we need a version of Theorem 2.2 for Chacon processes. First notice that Proposition 3.2 implies that the averages $\frac{1}{r_n}(F_{v_n+r_n} - F_{v_n})$ converge in the mean, where F is any Chacon process with respect to some operator S (S as described above). To obtain this result for an arbitrary positive L_1 contraction T we can use the proof of Theorem 2.2 as HF is *S-Chacon* admissible, where the transformation H is as in Theorem 2.2. But this is easily checked along the lines of Lemma 2.1. With these remarks we obtain:

COROLLARY 3.3. *Let $\{F_n\}_{n=0}^{\infty}$ be a bounded Chacon's admissible process with respect to T, where T is a positive Dunford-Schwartz operator on L_1. If $\{(v_k, r_k)\}_{k=1}^{\infty}$ is a B-sequence with $r_k \to \infty$. Then*

$$\frac{(F_{v_k+r_k}(w) - F_{v_k}(w))}{r_k} \quad \textit{converges in the mean as } k \to \infty.$$

The second solution to the problem of defining averages of superadditive processes is actually to define them as follows. The "averages" of a T-superadditive process $F = \{F_n\}$ along a B-sequence $\mathbf{n} = \{(v_n, r_n)\}$ can also be defined by

$$\frac{1}{r_n} T^{v_n} F_{r_n}. \tag{**}$$

The ordinary (nonmoving) averages correspond to the case where $v_n = 0$ for all n. Observe that the averages of F along $\mathbf{n} = \{(v_n, r_n)\}$ using the definition (*) corresponds to $\frac{1}{r_n}[F_{v_n+r_n} - F_{v_n}]$. Both definitions are equivalent in the additive case. The same apparatus used in the proof of Theorem 2.2 leads to the same conclusion when the averages are defined by (**). For, since $\mathbf{n} = \{(v_n, r_n)\}$ is good in the 1-mean for bounded superadditive processes relative to MPTs in this case [F], it is enough to prove that if T and S are as in Theorem 2.2 and $\mathbf{n}$ is good in the 1-mean for bounded superadditive processes relative to MPTs, then it is also good for bounded T-superadditive processes. That is why we only state this result (without proof):

THEOREM 3.4. *Let $\{F_n\}_{n=0}^{\infty}$ be a bounded T-superadditive process, where T is a positive Dunford-Schwartz operator on L_1. If $\mathbf{n} = \{(v_k, r_k)\}_{k=1}^{\infty}$ is a B-sequence with $r_k \to \infty$. Then*

$$\frac{1}{r_k} T^{v_k} F_{r_k} \text{ converges in the mean as } k \to \infty.$$

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Doğan Çömez, Department of Mathematics, North Dakota State University, Fargo, ND 58105
comez@plains.nodak.edu

Sebastian Ferrando, Department of Mathematics, Physics and Computer Science, Ryerson Polytechnic University, Toronto, Ontario, M5B 2K3 Canada
ferrando@acs.ryerson.ca