

STRONG SWEEPING OUT FOR BLOCK SEQUENCES AND RELATED ERGODIC AVERAGES

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ABSTRACT. In this paper we consider the block sequences introduced by Bellow and Losert, and show that in several cases, not only does divergence occur, but that we also have strong sweeping out. Related averages are also considered.

1. Introduction and notation

Let (X, Σ, m) denote a probability space, and τ an ergodic measure preserving point transformation from X onto itself. Let $\{(n_k, \ell_k)\}$ be a sequence of pairs of positive integers. Define the sequence of averages,

$$A_k f(x) = \frac{1}{\ell_k} \sum_{j=0}^{\ell_k-1} f(\tau^{n_k+j} x).$$

In [4] necessary and sufficient conditions on $\{(n_k, \ell_k)\}$ were given so that the averages $A_k f$ converge a.e. Later in [6] it was shown that one can find sequences $\{(n_k, \ell_k)\}$, with $\ell_k \rightarrow \infty$, such that not only do the averages diverge, but the Cesaro averages of the block averages also diverge a.e. The argument in [6] shows the existence of a sequence, $\{(n_k, \ell_k)\}$, with this property, but does not actually give an example. Using different techniques from those in [4], in this paper we obtain many of the same results, but the techniques also establish much more. We give a family of examples that show the Cesaro averages of these moving averages can fail to converge, and in fact have the strong sweeping out property. We also consider the “block sequences” considered by Bellow and Losert in [5]. That is, we consider the subsequence of integers determined by the set $\cup_{k=1}^{\infty} [n_k, n_k + \ell_k)$. We give several examples of cases where such subsequences are “bad universal”, and in fact have the strong sweeping out property.

Definition 1.1. A sequence of $L^1 - L^\infty$ contractions, $\{T_k\}$, is said to be *strong sweeping out* if given $\epsilon > 0$ there is a set E such that $m(E) < \epsilon$ but such that

$$\limsup_k T_k \chi_E(x) = 1 \text{ a.e. and } \liminf_k T_k \chi_E(x) = 0 \text{ a.e..}$$

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The strong sweeping out property was introduced by Bellow, and subsequently studied by several authors. Showing that a sequence of operators is strong sweeping out implies that the operators diverge in the worst possible way. In [1] it was shown that many subsequence averages, such as $\frac{1}{n} \sum_{j=1}^n f(\tau^{2^j} x)$, have this property. In fact it is shown that the sequence $\{2^j\}$ can be replaced by any lacunary sequence, and we still have strong sweeping out. To do this the $C(\alpha)$ condition was introduced.

Definition 1.2. Let $0 < \alpha < \frac{1}{2}$. A sequence of real numbers, $\{w_k\}$, is said to satisfy the $C(\alpha)$ condition if given any finite sequence of real numbers, x_1, x_2, \dots, x_L , there is a real number θ so that $\theta w_k \in x_k + (\alpha, 1 - \alpha) + \mathbb{Z}$ for $k = 1, 2, \dots, L$.

In [1] it is shown that any lacunary sequence, after possibly neglecting the first few terms, satisfies the $C(\alpha)$ condition. A modification of what is given there shows that finite unions of lacunary sequences will also satisfy the $C(\alpha)$ condition. See [8] where the details of this modification are given. Further, in [1] it is shown that the averages associated with any sequence that satisfies the $C(\alpha)$ condition will have the strong sweeping out property. While the sequences considered in this paper do not satisfy the $C(\alpha)$ condition, we show that similar ideas can be used to establish strong sweeping out.

Throughout the paper, if $\{\nu_n\}$ denotes a sequence of measures on \mathbb{Z} , and τ is a measurable, measure preserving point transformation of a probability space onto itself, we will also use $\{\nu_n\}$ to denote the associated sequence of operators given by $\nu_n f(x) = \sum_k \nu_n(k) f(\tau^k x)$.

2. The main results

We begin this section with a definition of a condition that is similar to the $C(\alpha)$ condition, but is associated with a sequence of measures, rather than a sequence of real numbers.

Definition 2.1. Let $\{\nu_n\}$ be a sequence of probability measures. Let $0 < \alpha < \frac{1}{4}$. The sequence of measures is said to satisfy the $B(\alpha)$ condition if given ϵ , $0 < \epsilon < 1$, and a positive integer L , we can find k_1, k_2, \dots, k_L and sets J_1, J_2, \dots, J_L such that $\nu_{k_j}(J_j) \geq 1 - \epsilon$ for $j = 1, 2, \dots, L$, and such that for any sequence $\{x_j\}_{j=1}^L$, of real numbers, we can find a real number θ such that $\theta w \in x_j + (\alpha, 1 - \alpha) + \mathbb{Z}$ for all $w \in J_j$, $j = 1, 2, \dots, L$.

We can now state the following theorem which shows the importance of the $B(\alpha)$ condition.

THEOREM 2.2. Let $\{\nu_n\}$ be a dissipative sequence of measures with the $B(\alpha)$ condition for some α , $0 < \alpha < \frac{1}{4}$. Then the associated sequence of operators $\{\nu_n\}$ has the strong sweeping out property.

Proof. It will be enough to show that given $\epsilon > 0$ and N_0 , there is a Lebesgue measurable set G such that the density of G , $d(G) < \epsilon$, and

$$\left\{x: \sup_{n \geq N_0} v_n \chi_G(x) > 1 - \epsilon\right\} = \mathbb{R}.$$

Once we have this fact, the result will follow by standard arguments. The argument below is contained in [1], but is included here for completeness.

Let $E = (\alpha/2, 1 - \alpha/2) + \mathbb{Z}$ and $F = (\alpha, 1 - \alpha) + \mathbb{Z}$. Then there are finitely many translates F_1, F_2, \dots, F_S of F with the property that for each $x \in \mathbb{R}$ there is an integer s , $1 \leq s \leq S$, such that $x + F_s \subset E$.

Select Q such that $d(E)^Q < \epsilon$. Denote by $\Sigma = \{1, 2, \dots, S\}^{\{1, 2, \dots, Q\}}$. Let $L = S^Q$. We note that there are only L vectors in Σ . Using the fact that the measures satisfy the $B(\alpha)$ condition, we can select k_1, k_2, \dots, k_L such that the ν_{k_j} have the properties required by the definition. (We can assume that $k_1 > N_0$ since if not, we could have simply have chosen $L = S^Q + N_0$ and discarded the first N_0 terms.) For each vector $\sigma \in \Sigma$, let ν_σ denote one of the ν_{k_j} 's. (Make this assignment in a one-to-one way. This is possible since there are exactly L vectors and L measures.) Because of the assumptions, we know that for each $\sigma \in \Sigma$ we can find a real number r_q such that $r_q \cdot J_\sigma \subset F_{\sigma_q}$ where σ_q denotes the q th coordinate of σ . We know by Theorem 2.1 of [1] that for almost every choice of the real numbers r_1, r_2, \dots, r_Q we have

$$d\left(\bigcap_{q=1}^Q \frac{E}{r_q}\right) = d(E)^Q.$$

Thus we can select r_1, r_2, \dots, r_Q so that $r_q \cdot J_\sigma \subset F_{\sigma_q}$ and $d\left(\bigcap_{q=1}^Q \frac{E}{r_q}\right) < \epsilon$.

Let $G = \bigcap_{q=1}^Q \frac{E}{r_q}$. We have $d(G) < \epsilon$. We now need to show that for each $x \in \mathbb{R}$ there is a $\sigma \in \Sigma$ with $\nu_\sigma \chi_G(x) > 1 - \epsilon$. This will follow if we can show $x + r \in G$ for each $r \in J_\sigma$. This means that for each q , $1 \leq q \leq Q$, we need $x + r \in \frac{E}{r_q}$, that is $r_q x + r_q r \in E$. We know that there is s_q so that $r_q x + F_{s_q} \subset E$. We just take $\sigma = (s_1, s_2, \dots, s_Q)$. \square

The first application is an immediate corollary.

COROLLARY 2.3. *If $\{\nu_n\}$ is a dissipative sequence of measures with support on a sequence that satisfies the $C(\alpha)$ condition, then the operators associated with $\{\nu_n\}$ have the strong sweeping out property.*

Since it has been shown that lacunary sequences have the $C(\alpha)$ condition (see Proposition 2.5 below) this implies

COROLLARY 2.4. *If $\{\nu_n\}$ is a dissipative sequence of measures with support on a lacunary sequence, then the operators associated with $\{\nu_n\}$ have the strong sweeping out property.*

However, many sequences of measures that do not have support on a sequence that satisfies the $C(\alpha)$ condition can be shown to have the $B(\alpha)$ condition. To see this we will need the following proposition. The proof is a small adaptation of the proof given by Losert in [8] to show that a finite union of lacunary sequences satisfies the $C(\alpha)$ condition. (Also see [9] and [1].)

PROPOSITION 2.5. *Let $\{n_k\}$ be a lacunary sequence with $\frac{n_{k+1}}{n_k} \geq \lambda > 1$. Then there exists an α , $0 < \alpha < \frac{1}{4}$, such that given a sequence of real numbers, x_1, x_2, \dots , and a positive integer k_0 , we can find a real number $\theta \in [-1/n_{k_0}, 1/n_{k_0}]$ such that*

$$n_k \theta \in x_k + (2\alpha, 1 - 2\alpha) + \mathbb{Z}$$

for $k \geq k_0$.

Proof. First find an integer $h > 0$ such that $\lambda^h \geq 2h + 2$. Define $\alpha = \frac{1}{8} \frac{1}{1+h\lambda^{2h}}$. By adding some terms, if necessary, we can assume $\frac{v_{k+1}}{v_k} < \lambda^2$ for all k . Let $I_0 = [-1/n_{k_0}, 1/n_{k_0}]$. Since $n_{k_0}|I_0| \geq 1$, by reducing I_0 , we can in fact assume

$$n_{k_0} \lambda^{2h} |I_0| = 1 - 4\alpha.$$

We will show below that there is a closed subinterval $I_1 \subset I_0$, with $n_{k_0+h} \lambda^{2h} |I_1| = 1 - 4\alpha$, and such that for all $\theta \in I_1$, we have

$$n_k \theta \in x_k + (2\alpha, 1 - 2\alpha) + \mathbb{Z}$$

for $k = k_0 + 1, k_0 + 2, \dots, k_0 + h$. Then we can iterate the construction, replacing I_0 by I_1 , and starting at n_{k_0+h+1} , that is, replacing k_0 by $k_0 + h$. We get a nested sequence of closed intervals, I_0, I_1, I_2, \dots , and any θ in the intersection has the desired property. The new interval, I_1 , is found as follows. For $k_0 < k \leq k_0 + h$, put

$$G_k = \{x \in I_0 | n_k x \in x_k + (-2\alpha, 2\alpha) + \mathbb{Z}\}.$$

Since by assumption, $n_{k_0+h} \leq n_{k_0} \lambda^{2h}$, we have $|n_k I_0| \leq |n_{k_0+h} I_0| \leq n_{k_0} \lambda^{2h} |I_0| = 1 - 4\alpha$. From this we see that $x_k + n_k G_k$ is an interval of length at most 4α . If $G = [-\frac{1}{2}, \frac{1}{2}] \cap (\cup_{k=k_0+1}^{k_0+h} G_k)$ then

$$|G| \leq \frac{4\alpha h}{n_{k_0}}.$$

Since $I_0 \setminus G$ is a union of at most $h + 1$ intervals, at least one of these intervals has length at least

$$\frac{|I_0 \setminus G|}{h + 1} \geq \frac{\frac{1-4\alpha}{n_{k_0} \lambda^{2h}} - \frac{4\alpha h}{n_{k_0}}}{h + 1}.$$

Select one such interval and call it I_1 . We need to check that $|I_1|$ satisfies $n_{k_0+h}\lambda^{2h}|I_1| \geq 1 - 4\alpha$. From the above, we see that

$$n_{k_0+h}\lambda^{2h} \frac{|I_0 \setminus G|}{h+1} \geq \frac{n_{k_0+h}}{n_{k_0}} \frac{1 - 4\alpha - 4\alpha h\lambda^{2h}}{h+1}.$$

If we could show this is at least $1 - 4\alpha$ we would be done.

Using the definition of α given above, and the fact that $n_{k_0+h} \geq n_{k_0}\lambda^h$, we have

$$\frac{n_{k_0+h}}{n_{k_0}} \frac{1 - 4\alpha - 4\alpha h\lambda^{2h}}{h+1} \geq \lambda^h \frac{1}{2(h+1)} \geq 1 > 1 - 4\alpha$$

as required. \square

The first application of Proposition 2.5, combined with Theorem 2.2 is to give a different proof of some of the results from [4]. This is the simplest application of our technique. Modifications of the same idea will be used to obtain the later results.

THEOREM 2.6. *Let $\{n_k\}$ be a lacunary sequence of positive integers, and $\{\ell_k\}$ a non-decreasing sequence of positive integers such that given L , we have $\liminf_{k \rightarrow \infty} \frac{\ell_{k+L}}{n_k} = 0$. Then the moving averages*

$$A_k f(x) = \frac{1}{\ell_k} \sum_{j=0}^{\ell_k-1} f(\tau^{n_k+j} x)$$

have the strong sweeping out property.

Proof. We will show that the sequence of measures associated with the averages A_k satisfy the $B(\alpha)$ condition. Let α be the α given by Proposition 2.5, using the lacunary sequence $\{n_k\}$, that is the sequence of starting points of the blocks. Let L be given. In this case we take $\epsilon = 0$. The sets J_j will just be the intervals $[n_{k_0+j}, n_{k_0+j} + \ell_{k_0+j})$, where k_0 is chosen to have the property that $\frac{\ell_{k_0+L}}{n_{k_0}} < \alpha$. Let x_1, x_2, \dots, x_L be given. By Proposition 2.5 above, we can select θ so that $\theta n_{k_0+j} + x_j \in (2\alpha, 1 - 2\alpha) + \mathbb{Z}$. Since $|\theta| < \frac{1}{n_{k_0}}$, we have $|\theta w| < \frac{\ell_{k_0+j}}{n_{k_0}} < \alpha$ for all $w \in [0, \ell_{k_0+j})$. Hence for each j , $j = 1, 2, \dots, L$, we have $w\theta \in x_j + (\alpha, 1 - \alpha) + \mathbb{Z}$ for all $w \in [n_{k_0+j}, n_{k_0+j} + \ell_{k_0+j})$. \square

We have the following corollaries, which are also contained in [4].

COROLLARY 2.7. *Let $n_k = a^k$ and let $\ell_k = b^k$ where $1 < b < a < \infty$. Then the moving averages associated with the sequence (n_k, ℓ_k) have the strong sweeping out property.*

Proof. We just need to show that $\frac{\ell_{k+L}}{n_k} \rightarrow 0$, but in this case it is the same as showing

$$\frac{b^{k+L}}{a^k} = \left(\frac{b}{a}\right)^k b^L \rightarrow 0.$$

Since L , and $b < a$ are fixed, this certainly holds. \square

We can also obtain the following by the same argument.

COROLLARY 2.8. *Let $n_k = [k^a]$ and let $\ell_k = o(n_k)$ where $1 < a < \infty$. Then the moving averages associated with the sequence (n_k, ℓ_k) have the strong sweeping out property.*

Proof. Since it will be enough to show that we have strong sweeping out for a subsequence, we can consider only the lacunary subsequence obtained by restricting k to be of the form 2^n . Hence we are looking at n_k of the form $[(2^k)^a] = [2^{ak}]$, and $\ell_k = o(2^{ak})$. We need to know that

$$\frac{\ell_{k_0+L}}{n_{k_0}} \rightarrow 0.$$

However, since $\ell_{k_0+L} = o(n_{k_0+L})$, this means we need to know

$$\frac{o(2^{(k_0+L)a})}{2^{k_0a}} \rightarrow 0.$$

If we define ϵ_n by $\ell_n = \epsilon_n 2^{na}$, then we know that ϵ_{k_0+L} goes to zero as $k_0 \rightarrow \infty$. What we need to show is equivalent to showing

$$\epsilon_{k_0+L} \left(\frac{2^{(k_0+L)a}}{2^{k_0a}} \right) = \epsilon_{k_0+L} 2^{La} \rightarrow 0,$$

and for any fixed L this clearly holds since 2^{La} is fixed, and $\epsilon_{k_0+L} \rightarrow 0$. \square

In fact, more is true. By extracting a lacunary subsequence from a given sequence, as above, we can give a new proof of the result in [4] on moving averages.

Let Ω be an infinite collection of lattice points with positive second coordinate. Define

$$\Omega_\alpha = \{(z, s) : |z - y| < \alpha(s - r) \text{ for some } (y, r) \in \Omega, \\ (z, s) \text{ a lattice point}\}.$$

The cross section of Ω_α at height $s > 0$ is denoted by $\Omega_\alpha(s)$ and is defined by $\Omega_\alpha(s) = \{k : (k, s) \in \Omega_\alpha\}$. We will say that Ω satisfies the cone condition if there is a constant c (which can depend on α) such that for every $\lambda > 0$ we have $|\Omega_\alpha(\lambda)| \leq c\lambda$. In [4] it was shown that the moving averages $A_k f(x) = \frac{1}{\ell_k} \sum_{j=n_k}^{n_k+\ell_k} f(\tau^j x)$ converge a.e. if and only if the set $\Omega = \{(n_k, \ell_k)\}$ satisfies the cone condition. The following theorem is contained in [4]. However, the proof given below is new.

THEOREM 2.9. *Assume that the set $\Omega = \{(n_k, \ell_k)\}$ fails to satisfy the cone condition. Then the moving averages $A_k f(x) = \frac{1}{\ell_k} \sum_{j=n_k}^{n_k+\ell_k} f(\tau^j x)$ are strong sweeping out.*

Proof. We first show that we can extract a subsequence $\{(n'_k, \ell'_k)\}$ of $\{(n_k, \ell_k)\}$ which has the property that $\{n_k\}$ is lacunary, and $\{(n'_k, \ell'_k)\}$ fails to satisfy the cone condition. The result will then follow from Theorem 2.6 if we can show that given $L > 0$, we have $\liminf_{k \rightarrow \infty} \ell'_{k+L}/n'_k = 0$. To see that this holds, assume not. Then there would be a constant b such that $\ell'_{k+L} \geq bn'_k$. From this it would follow that the cone condition holds, and we know this is not the case.

To see that we can extract the desired subsequence, we argue by induction. Fix $\alpha = 1$, and let $n'_1 = n_1$. If n'_1, n'_2, \dots, n'_k have been selected, let n'_{k+1} denote the first $n_j > n'_k$ such that $|\Omega_1(\lambda)| > 2^k \lambda$ for some $\lambda > 0$. It is possible to find such a n'_{k+1} , since if not, the remaining (n_j, ℓ_j) (that is the points with $n_j > n'_k$) will all be contained in the cone with vertex at (n'_k, ℓ'_k) and $\alpha = 2^k$. This will allow us to conclude that the cone condition is satisfied, which we know is false. \square

We now show that the averages along certain block sequences have the strong sweeping out property. This generalizes some results of Bellow and Losert [5], who show that the averages diverge for some $f \in L^1$.

THEOREM 2.10. *Let $1 < \rho < \lambda$. Assume (n_k) form a lacunary sequence, with $\frac{n_{k+1}}{n_k} \geq \lambda$, and let (ℓ_k) be a sequence such that $\ell_k = [\rho^k]$ for each k . Let (w_j) be the integers in the set $\cup_{k=1}^{\infty} [n_k, n_k + \ell_k)$. Then the Cesaro averages along the sequence (w_j) have the strong sweeping out property.*

Proof. Define $S_k = \cup_{j=1}^k [n_j, n_j + \ell_j]$. Then we see that

$$\#(S_k) = \sum_{j=1}^k [\rho^j] \approx \rho \frac{\rho^k - 1}{\rho - 1}.$$

(Note that we only get an approximation because we are using the sum up to $[\rho^j]$ and ρ is not necessarily an integer.) Given $\epsilon > 0$, we will select $L + 1$ integers, $N_0, N_1, N_2, \dots, N_L$, such that

$$\frac{\#(S_{N_{k-1}})}{\#(S_{N_k})} < \epsilon$$

for $k = 1, 2, \dots, L$. We have

$$\#(S_{N_0}) = \sum_{j=1}^{N_0} [\rho^j] \approx \frac{\rho}{\rho - 1} \rho^{N_0}.$$

Hence we need to select N_1 so that $\#(S_{N_1}) \approx \#(S_{N_0})^{\frac{1}{\epsilon}}$, and in general select N_k so that $\#(S_{N_k}) \approx \#(S_{N_0})^{\left(\frac{1}{\epsilon}\right)^k}$. From this we see that we have $\#(S_{N_L}) \approx \#(S_{N_0})^{\left(\frac{1}{\epsilon}\right)^L}$. Since

$$\#(S_{N_L}) \approx \frac{\rho}{\rho - 1} \rho^{N_L}$$

we now have

$$\frac{\rho}{\rho - 1} \rho^{N_L} \approx \#(S_{N_0}) \left(\frac{1}{\epsilon}\right)^L \approx \frac{\rho}{\rho - 1} \rho^{N_0} \left(\frac{1}{\epsilon}\right)^L.$$

Consequently, $\rho^{N_L} \approx \rho^{N_0} \left(\frac{1}{\epsilon}\right)^L$. Define the sets J_j , $j = 1, 2, \dots, L$, by $J_j = S_j \setminus S_{j-1}$. By construction these sets have the property that $\nu_{N_j}(J_j) > 1 - \epsilon$. We just need to check that the other property of the $B(\alpha)$ condition is satisfied. As before, if we can be sure θ is so small that $\theta r < \alpha$ for $|r|$ less than the length of the last block, then we will be done. Since $\theta \in [-1/n_{N_0}, 1/n_{N_0}]$, we need to be able to take N_0 so large that

$$\frac{1}{n_{N_0}} \rho^{N_0} \left(\frac{1}{\epsilon}\right)^L < \alpha.$$

Since $n_{N_0} \geq \lambda^{N_0}$, it is enough to have

$$\frac{1}{\lambda^{N_0}} \rho^{N_0} \left(\frac{1}{\epsilon}\right)^L < \alpha.$$

Since $\rho < \lambda$, and both L and ϵ are fixed, we see that by taking N_0 large enough, we can make the left hand side as small as desired, and in particular, less than α . \square

With almost the same argument, we can also prove the following result.

THEOREM 2.11. *Let $1 < \lambda$, and let $d > 0$. Assume (v_i) form a lacunary sequence, with $\frac{v_{i+1}}{v_i} \geq \lambda$, and let (ℓ_i) be a sequence such that $\ell_i = [i^d]$ for each i . Let (w_k) denote the sequence formed by the integers in the union of these blocks. Then the Cesaro averages along the sequence (w_k) have the strong sweeping out property.*

Remark 2.12. To make the argument work, we need to know how many terms are in each block. That is why we need to give the length ℓ_k as we do above. If some ℓ_k are allowed to be much shorter, then we do not have control of the number of terms needed, consequently, we will not be able to control N_L . If we take an extreme case, by allowing $\ell_k = 0$ for many k , that is, we have mostly empty blocks, we can obtain the “good universal” block sequences considered by Bellow and Losert [5], that is with $n_k = 2^{2^k}$ and $\ell_k = \sqrt{n_k}$. Since these are known to be good, strong sweeping out cannot occur.

Remark 2.13. In Theorems 2.11 and 2.10 it was important that we considered Cesaro averages, or at least very specific averages. In [7] it is shown that for any sequence of moving averages, where the lengths of the blocks increase to infinity, the Cesaro averages of the moving averages have asymptotically trivial transforms. Consequently, applying a result in [3] we have convergence of some subsequence of the Cesaro averages of the moving averages for all f in L^p , $p > 1$. (Indeed, by taking the right choice of a dissipative sequence of measures, we can obtain a subsequence of the moving averages which will even converge for $f \in L^1$.) In particular we cannot have strong sweeping out for all dissipative sequences of probability measures on the sequence (w_j) .

In [6] (Corollary 2.5) it is shown that if we take the Cesaro averages of moving averages (with the lengths of the moving averages going to infinity) it is possible to construct a sequence of moving averages so that the Cesaro averages of the moving averages have the strong sweeping out property. With an argument similar to that used above, we can prove the following generalization of the result in that paper.

THEOREM 2.14. *Let $1 < \lambda$, and let $d > 0$. Assume (n_i) form a lacunary sequence, with $\frac{n_{i+1}}{n_i} \geq \lambda$, and let (ℓ_i) be a sequence such that $\ell_i \leq i^d$ for each i . Then the Cesaro averages of the moving averages associated with the blocks $[n_k, n_k + \ell_k)$ have the strong sweeping out property.*

Proof. The proof is very similar to the proof of Theorem 2.10, above. If $A_k f(x)$ denotes the moving average $\frac{1}{\ell_k} \sum_{j=0}^{\ell_k-1} f(\tau^{n_k+j}x)$ then we want to show that the averages $\frac{1}{N} \sum_{k=1}^N A_k f(x)$ have the strong sweeping out property. If N_0 is selected, then as before, we need to select $N_k \approx N_0(\frac{1}{\epsilon})^k$, $k = 1, 2, \dots, L$. Since $N_L \approx N_0(\frac{1}{\epsilon})^L$ is the index of the last average we use, we need to make certain that the length of the associated block, ℓ_{N_L} satisfies $\frac{1}{n_{N_0}} \ell_{N_L} \leq \alpha$. This means, since $n_k \geq \lambda^k$, that we will be able to achieve our goal if $\frac{1}{\lambda^{N_0}} (N_0(\frac{1}{\epsilon})^L)^d < \alpha$. Since the bottom is growing geometrically, and the top is growing only like a polynomial, we see we can achieve our goal. \square

Remark 2.15. If in the above example we were to take $\ell_k = \rho^k$ where $1 < \rho < \lambda$, then to make the above argument work we would need

$$\frac{1}{\lambda^{N_0}} \rho^{N_0/\epsilon^L} < \alpha.$$

This cannot be achieved in general if ϵ is small enough, or L is large enough, since we can have $\rho^{1/\epsilon^L} > \lambda$. Hence it is now natural to ask the question: do we have strong sweeping out (or even divergence) in the case of Cesaro averages of moving averages where both the starting points of the blocks, and the lengths are lacunary?

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