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WHEN AN ENTIRE FUNCTION AND ITS LINEAR DIFFERENTIAL POLYNOMIAL SHARE TWO VALUES

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ABSTRACT. In this note, the relationship between a non-constant entire function f and its linear differential polynomial $L(f)$ has been obtained when they share two finite values, ignoring multiplicities, by applying value distribution theory,. This confirms Frank's conjecture as a special case. Entire solutions of certain types of non-linear differential equations are also discussed.

1. Introduction

Let f and g denote some non-constant meromorphic functions. We say f and g share a value b IM (CM) iff $f(z) - b = 0 \Longleftrightarrow g(z) - b = 0$, ignoring multiplicities (counting multiplicities). It has been shown [13] that if an entire function f shares two finite values CM with its derivative, then $f \equiv f'$. This result has been generalized to sharing values IM by Mues and Steinmetz (see [9]), and independently by G. Gundersen in the case when both shared values are nonzero (see [8]). Since then, many results have been obtained for this and related topics. It has been shown in [6] and [7] that if a meromorphic function f shares two distinct finite values a, b CM with $f^{(k)}$ $(k \geq 1)$, then $f \equiv f^{(k)}$. When f shares three finite values IM with its linear differential polynomial $L(f)$ was studied in [5] and [10]. For a non-constant entire function f, some relationships between f and $L(f)$ have been obtained when f and $L(f)$ share two distinct finite values CM (see [1]) or share one value IM and another value CM (see [11]). For a comprehensive collection of these results, we refer the reader to the Chinese monograph "Uniqueness theorems of meromorphic functions" by Yi-Yang [14] newly published by Science Press, China.

It was conjectured by G. Frank in [4] that if an entire function f shares two finite values IM with its k-th derivative $(k \geq 1)$, then $f \equiv f^{(k)}$. In this note, we resolve a more general problem which deals with an entire function f which shares two values IM with a linear differential polynomial of f. In particular, we confirm Frank's conjecture.

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2. The lemmas and main results

The first lemma has been used frequently in dealing with value sharing problems that can be easily derived from the lemma of the logarithmic derivative (see [12], p. 14).

LEMMA 2.1. *Let f be a transcendental meromorphic function, $P_k(f)$ denote a polynomial in f of degree k, and a_i, $i = 1, 2, \ldots, n$ denote finite distinct constants in $\mathbb{C}$. Let*

$$g = \frac{P_k(f) f^{(i)}}{(f - a_1) \cdots (f - a_n)}.$$

If $k < n$, then $m(r, g) = o(T(r, f))$, $r \to \infty$, except for a set of r of finite linear measure.

LEMMA 2.2 (see [15], p. 13). *If $f_1(z)$ and $f_2(z)$ are meromorphic in $|z| < R$ ($R \leq \infty$), then*

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right),$$

where $0 < r < R$.

LEMMA 2.3 (Clunie [2], Doeringer [3]). *Let f be a non-constant meromorphic function and $Q[f]$, $Q^*[f]$ be differential polynomials in f with $Q[f] \not\equiv 0$. Let $n \in \mathbb{N}$ and*

$$f^n Q^*[f] = Q[f].$$

If the degree of $Q[f]$ is not greater than n, then $m(r, Q^*[f]) = S(r, f)$.

THEOREM 2.1. *Let f be a non-constant entire function and a, b be two distinct complex numbers. Let $g = a_0 f + a_1 f' + \cdots + a_k f^{(k)}$, ($k \geq 1$) and*

$$\varphi = \frac{f'(f - g)}{(f - a)(f - b)}, \tag{1}$$

where $a_0, a_1, \ldots, a_k$, ($a_k \not\equiv 0$) are small entire functions of f. If f and g share a, b IM, then we have

$$\varphi \sum_{i=0}^{k} a_i \varphi_i \equiv 0.$$

Here $\varphi_0 \equiv 1$ and $\varphi_{i+1} = \varphi_i' + \varphi \varphi_i$, $i = 0, 1, \ldots, k - 1$.

Proof. Since f and g share a, b IM, it is easily seen that the function φ defined in (1) must be entire. By using Lemma 2.1, we see that $m(r, \varphi) = S(r, f) := S(r)$, where here and in the sequel, $S(r, f)$ denotes some quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, except for r in a set of finite linear measure. And thus we have

$$T(r, \varphi) = S(r). \tag{2}$$

When $\varphi \equiv 0$, there is nothing to prove. So, we assume that $\varphi \not\equiv 0$.

From (1), we deduce that

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) &= N\left(r, \frac{f'}{(f-a)(f-b)}\right)\\ &= N\left(r, \frac{\varphi}{f-g}\right) \le T(r, f-g) + S(r)\\ &= m(r, f-g) + S(r) \le T(r, f) + S(r).\end{aligned}$$

On the other hand, by Nevanlinna's Second Fundamental Theorem, we have

$$T(r, f) \le \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r).$$

Hence

$$T(r, f) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r). \tag{3}$$

By writing equation (1) as

$$\frac{1}{f} = \frac{f'}{\varphi(f-a)(f-b)}\left(1 - \frac{g}{f}\right) \text{ and } \frac{1}{f-c} = \frac{f'(f-g)}{\varphi(f-c)(f-a)(f-b)},$$

where $c \in \mathbb{C}\backslash\{a, b\}$, and using Lemma 2.1 again, we get

$$m\left(r, \frac{1}{f}\right) = S(r) \text{ and } m\left(r, \frac{1}{f-c}\right) = S(r),\ c \in \mathbb{C}\backslash\{a, b\}. \tag{4}$$

Now rewrite equation (1) as

$$f' = \frac{\varphi(f-a)(f-b)}{f-g}. \tag{5}$$

By induction, and using the above equation repeatedly, we can derive the expression

$$f^{(i)} = \frac{\sum_{j=0}^{2i} \alpha_{(i,j)} f^j + Q_i}{(f-g)^{2i-1}},\ i = 1, 2, \ldots, k, \tag{6}$$

where

$$\begin{aligned} Q_i &= Q_i(f, g, g', \ldots, g^{(i-1)}) \\ &= \sum_{\substack{l_1+l_2+j_1+\cdots+j_i \le 2i \\ l_1+l_2<2i}} \beta_{(i,l_1l_2j_1\cdots j_i)}(f-a)^{l_1}(f-b)^{l_2}(g)^{j_1}(g')^{j_2}\cdots(g^{(i-1)})^{j_i}. \end{aligned}$$

Here $\alpha_{(i,j)}$ and $\beta_{(i,l_1l_2j_1\cdots j_i)}$ are small entire functions of f and $\varphi_i := \alpha_{(i,2i)}$ satisfies recurrence formula

$$\varphi_1 = \varphi, \; \varphi_{i+1} = \varphi_i' + \varphi\varphi_i, \; i = 1, 2, \ldots, k-1. \tag{7}$$

Since $\varphi \not\equiv 0$, one can easily prove that $\varphi_i \not\equiv 0$, for $i = 1, 2, \ldots, k$. From (6), g can be expressed as

$$g = \frac{\sum_{j=1}^{2k} \gamma_j f^j + Q}{(f-g)^{2k-1}}, \tag{8}$$

where γ_j, $j = 1, \ldots, k$ are small entire functions of f, and

$$\gamma_{2k} = a_0 + \sum_{i=1}^{k} a_i\varphi_i, \tag{9}$$

and

$$Q = \sum_{\substack{l_1+l_2+j_1+\cdots+j_k \le 2k \\ l_1+l_2<2k}} \lambda_{l_1l_2j_1\cdots j_k}(f-a)^{l_1}(f-b)^{l_2}(g)^{j_1}(g')^{j_2}\cdots(g^{(k-1)})^{j_k}. \tag{10}$$

Here $\lambda_{l_1l_2j_1\cdots j_k}$ are small entire functions of f. From the expression for Q and by using (4) and the lemma of the logarithmic derivative, we can get

$$m\left(r, \frac{Q}{f^{2k-1}g}\right) = S(r). \tag{11}$$

Rewrite formula (8) as

$$\sum_{j=0}^{2k} \gamma_j f^j = g(f-g)^{2k-1} - Q. \tag{12}$$

If $\gamma_{2k} \equiv 0$, then the result is already proved. In the following, we consider the case when $\gamma_{2k} \not\equiv 0$.

In this case, it is well known that $T(r, \sum_{j=0}^{2k} \gamma_j f^j) = 2kT(r, f) + S(r)$. Hence it follows from (12) and (11) that

$$\begin{aligned} 2kT(r, f) &\le m(r, f^{2k-1}g) + m\left(r, \left(1 - \frac{g}{f}\right)^{2k-1} - \frac{Q}{f^{2k-1}g}\right) + S(r) \\ &\le m(r, f^{2k-1}g) + S(r) \\ &\le (2k-1)T(r, f) + T(r, g) + S(r). \end{aligned}$$

Hence, we have $T(r,f) \le T(r,g) + S(r)$, which implies that

$$T(r,f) = T(r,g) + S(r), \tag{13}$$

because g is a linear differential polynomial of the entire function f. Now, by applying Nevanlinna's Second Fundamental Theorem to the entire function g and noting that g shares the values a, b with f, we have

$$\begin{aligned} &2T(r,g) + m\left(r, \frac{1}{g-c}\right) \\ &\quad \le \overline{N}\left(r, \frac{1}{g-c}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) + \overline{N}\left(r, \frac{1}{g-b}\right) + m\left(r, \frac{1}{g-c}\right) + S(r) \\ &\quad \le T(r,g) + \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r) \\ &\quad \le T(r,g) + T(r,f) + S(r) \\ &\quad \le 2T(r,g) + S(r),\ c \in \mathbb{C}\backslash\{a,b\}, \end{aligned}$$

which implies that

$$m\left(r, \frac{1}{g-c}\right) = S(r),\ c \in \mathbb{C}\backslash\{a,b\}. \tag{14}$$

By applying Lemma 2.2 to the functions $f_1 = f - c$ and $f_2 = \frac{1}{g-c}$, it follows from (4) and (14) that

$$\begin{aligned} m\left(r, \frac{f-c}{g-c}\right) &= T\left(r, \frac{f-c}{g-c}\right) - N\left(r, \frac{f-c}{g-c}\right) \\ &= T\left(r, \frac{g-c}{f-c}\right) - N\left(r, \frac{f-c}{g-c}\right) + S(r) \\ &= N\left(r, \frac{g-c}{f-c}\right) - N\left(r, \frac{f-c}{g-c}\right) + S(r) \\ &= N\left(r, \frac{1}{f-c}\right) - N\left(r, \frac{1}{g-c}\right) + S(r) \\ &= T(r,f) - T(r,g) + S(r) = S(r). \end{aligned} \tag{15}$$

Now we define

$$\psi = \frac{g'(f-g)}{(g-a)(g-b)}. \tag{16}$$

Since f and g share the values a and b, we see that ψ is a nonzero entire function. By using Lemma 2.1 on (16) and (15), we get

$$m(r,\psi) \le m\left(r, \frac{(g-c)g'}{(g-a)(g-b)}\right) + m\left(r, \frac{f-c}{g-c} - 1\right) = S(r),\ c \in \mathbb{C}\backslash\{a,b\}.$$

Hence we have

$$T(r, \psi) = S(r). \tag{17}$$

Denote by $S_{(m,n)}(a)$ the set of those points $z \in \mathbb{C}$ such that z is an a-point of f with multiplicity m and an a-point of g with multiplicity n. The set $S_{(m,n)}(b)$ can be similarly defined. Let $N_{(m,n)}(r, \frac{1}{f-a})$ and $\overline{N}_{(m,n)}(r, \frac{1}{f-a})$ denote the counting function and reduced counting function of f with respect to the set $S_{(m,n)}(a)$, respectively. Let $N_{(m,n)}(r, \frac{1}{g-a})$ and $\overline{N}_{(m,n)}(r, \frac{1}{g-a})$ be similarly defined. For any $z_0 \in S_{(m,n)}(a) \cup S_{(m,n)}(b)$, from (1) and (16) we easily have

$$\varphi(z_0)(2f(z_0) - a - b) = m(f'(z_0) - g'(z_0))$$

and

$$\psi(z_0)(2f(z_0) - a - b) = n(f'(z_0) - g'(z_0)),$$

and thus $n\varphi(z_0) = m\psi(z_0)$.

If $n\varphi \equiv m\psi$, then we have

$$n\left(\frac{f'}{f-a} - \frac{f'}{f-b}\right) \equiv m\left(\frac{g'}{g-a} - \frac{g'}{g-b}\right),$$

which implies that

$$\left(\frac{f-a}{f-b}\right)^n \equiv c_1\left(\frac{g-a}{g-b}\right)^m,$$

where c_1 is a nonzero constant. If $n \neq m$, then, from the above identity, we get $nT(r, f) = mT(r, g) + S(r)$, which contradicts (13). If $n = m$, then

$$\frac{f-a}{f-b} \equiv c_2\left(\frac{g-a}{g-b}\right), \tag{18}$$

where c_2 is a nonzero constant. If $c_2 \neq 1$, then it follows from the above equation that

$$\frac{1-c_2}{c_2}\frac{f-c_3}{f-b} \equiv \frac{b-a}{g-b},$$

where $c_3 = \frac{a-bc_2}{1-c_2}$. Obviously, c_3 is different from a and b. Since f and g are entire, the above equation shows that c_3 is a Picard exceptional value of f. Thus by the Second Fundamental Theorem, we have

$$2T(r, f) \leq \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r),$$

which contradicts (3). Hence $c_2 = 1$, and thus we get $f \equiv g$ from (18). This contradicts the assumption. Hence $n\varphi \not\equiv m\psi$, for any positive integers m and n. Thus we obtain

$$\overline{N}_{(m,n)}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f-b}\right) = S(r). \tag{19}$$

It follows from (19) and (3) that

$$\begin{aligned}
T(r, f) &= \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r) \\
&= \sum_{m,n}\left(\overline{N}_{(m,n)}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f-b}\right)\right) + S(r) \\
&= \sum_{m+n\geq 5}\left(\overline{N}_{(m,n)}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f-b}\right)\right) + S(r) \\
&\leq \sum_{m+n\geq 5}\frac{1}{5}\left(N_{(m,n)}\left(r, \frac{1}{f-a}\right) + N_{(m,n)}\left(r, \frac{1}{g-a}\right)\right. \\
&\quad \left. + N_{(m,n)}\left(r, \frac{1}{f-b}\right) + N_{(m,n)}\left(r, \frac{1}{g-b}\right) + \right) + S(r) \\
&\leq \frac{4}{5}T(r, f) + S(r),
\end{aligned}$$

which implies that $T(r, f) = S(r)$, a contradiction. The proof of Theorem 2.1 is thus completed.

When $a_0, a_1, \ldots, a_k$, $(a_k \not\equiv 0)$ are constants, we have the following:

THEOREM 2.2. *Let f be a non-constant entire function and a, b be two distinct complex numbers. Let $g = a_0 f + a_1 f' + \cdots + a_k f^{(k)}$, $(k \geq 1)$ and*

$$\varphi = \frac{f'(f-g)}{(f-a)(f-b)},$$

where $a_0, a_1, \ldots, a_k$, $(a_k \not\equiv 0)$ are constants. If f and g share a, b IM, then φ must be a constant satisfying

$$a_0\varphi + a_1\varphi^2 + \cdots + a_k\varphi^{k+1} \equiv 0.$$

Proof. From the proof of Theorem 2.1, we only need to consider the case where $\gamma_{2k} \equiv 0$.

From (9) and the recurrence formula (7) for φ_i, we can easily derive the expression

$$\gamma_{2k} \equiv P[\varphi] + a_k\varphi^k,$$

where $P[\varphi]$ is a differential polynomial in φ with a degree less than or equal to $k-1$. Since $\gamma_{2k} \equiv 0$ holds in the present case, by applying Lemma 2.3 we see that φ must be a constant. Hence, by the recurrence formula (7), we have $\varphi_i = \varphi^i$, and thus the formula $\gamma_{2k} \equiv 0$ implies

$$a_0 + a_1\varphi + a_2\varphi^2 + \cdots + a_k\varphi^k \equiv 0,$$

which completes the proof of Theorem 2.2.

As a simple consequence of Theorem 2.2, we can resolve Frank's conjecture as follows.

THEOREM 2.3. *Let f be a non-constant entire function and a, b be two distinct complex number. If f and $f^{(k)}$ $(k \geq 1)$ share a, b IM, then $f \equiv f^{(k)}$.*

From Theorem 2.1, we see that the problem of the entire function f and its differential polynomial g sharing two values a, b is related to the problem of the non-linear differential equation $f'(f-g)-\varphi(f-a)(f-b)=0$ having a non-constant entire solution, where φ is a nonzero entire function. In general, it is difficult to judge whether the differential equation has a non-constant solution even for $g = f''$. However, for the very special case $g = f'$, we can solve the equation completely. We first prove a lemma.

LEMMA 2.4. *Let f be a non-constant meromorphic function and α, β, γ be small meromorphic functions of f with $\alpha \not\equiv 0$ or $\gamma \not\equiv 0$. Furthermore, let*

$$g = \alpha f^2 + \beta f + \gamma. \tag{20}$$

If $\overline{N}(r, f) = S(r, f)$, $\overline{N}(r, 1/f) = S(r, f)$ and $\overline{N}_{1)}(r, 1/g) = S(r, f)$, then $\beta^2 - 4\alpha\gamma \equiv 0$, where $\overline{N}_{1)}(r, 1/g)$ is the reduced counting function of the simple zeros of g.

Proof. If $\alpha \equiv 0, \beta \equiv 0$, then there is nothing to prove. If $\alpha \equiv 0, \beta \not\equiv 0$, then $g = \beta(f + \frac{\gamma}{\beta})$. Therefore $\overline{N}_{1)}(r, \frac{1}{f+\frac{\gamma}{\beta}}) = S(r, f)$. Hence, by the Second Fundamental Theorem, we have

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f+\frac{\gamma}{\beta}}\right) + S(r, f) \\ &\leq \frac{1}{2}N\left(r, \frac{1}{f+\frac{\gamma}{\beta}}\right) + S(r, f) \\ &\leq \frac{1}{2}T(r, f) + S(r, f), \end{aligned}$$

which leads to $T(r, f) = S(r, f)$, a contradiction.

Assuming that $\alpha \not\equiv 0$, equation (20) can be rewritten as

$$g = \alpha(F^2 - A), \tag{21}$$

where $F = f + B$, $A = \frac{\beta^2 - 4\alpha\gamma}{4\alpha^2}$ and $B = \frac{\beta}{2\alpha}$. If $A \not\equiv 0$, then by the Second Fundamental Theorem, we have

$$\begin{aligned} T(r, F^2) &< \overline{N}\left(r, \frac{1}{F^2}\right) + \overline{N}\left(r, \frac{1}{F^2 - A}\right) + \overline{N}(r, F^2) + S(r, F) \\ &= \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F^2 - A}\right) + S(r, F). \end{aligned}$$

Hence, $\overline{N}(r, \frac{1}{F^2-A}) \neq S(r, F)$, for $A \not\equiv 0$. By the assumption of Lemma 2.4, we see that $\overline{N}_{1)}(r, \frac{1}{F^2-A}) = S(r, f)$, thus

$$\overline{N}_{(2}\left(r, \frac{1}{F^2-A}\right) = \overline{N}\left(r, \frac{1}{F^2-A}\right) + S(r, F) \neq S(r, F), \tag{22}$$

where $\overline{N}_{(2}(r, f)$ is the reduced counting function of the multiple poles of f. Set

$$\lambda := \frac{f'}{f} = \frac{F'-B'}{F-B}. \tag{23}$$

Then we have $T(r, \lambda) = S(r, f)$. Let z_0 be a multiple zero of $F^2 - A$. We have

$$F^2(z_0) - A(z_0) = 0 \tag{24}$$

$$2F(z_0)F'(z_0) - A'(z_0) = 0. \tag{25}$$

In the case that $B' - \lambda B \equiv 0$, the above two equations lead to $2\lambda(z_0)A(z_0) - A'(z_0) = 0$. Therefore by (22) we have $2\lambda A - A' \equiv 0$, and thus $2AB' \equiv BA'$. That is, $A \equiv (cB)^2$, where c is a nonzero constant. From this and equation (21), we have

$$g = \alpha(F - cB)(F + cB).$$

If $B \not\equiv 0$, then, by the assumption, we have $\overline{N}_{1)}(r, \frac{1}{F-cB}) = S(r, f)$ and $\overline{N}_{1)}(r, \frac{1}{F+cB}) = S(r, f)$. Suppose that $c \neq 1$ (otherwise, $c \neq -1$). Then

$$\begin{aligned} T(r, F) &\leq \overline{N}\left(r, \frac{1}{F-B}\right) + \overline{N}\left(r, \frac{1}{F-cB}\right) + S(r, F) \\ &\leq \frac{1}{2}N\left(r, \frac{1}{F-cB}\right) + S(r, F) \\ &\leq \frac{1}{2}T(r, F) + S(r, F). \end{aligned}$$

This is impossible. Hence we have $B \equiv 0$. That is, $A \equiv 0$.

In the case that $B' - \lambda B \not\equiv 0$, equations (24), (25) and (23) lead to

$$\left(\frac{A'(z_0) - 2\lambda(z_0)A(z_0)}{2B'(z_0) - 2\lambda(z_0)B(z_0)}\right)^2 - A(z_0) = 0.$$

Hence, we must have $A \equiv D^2$, where $D = \frac{A'-2\lambda A}{2B'-2\lambda B}$. Thus equation (21) becomes

$$g = \alpha(F - D)(F + D).$$

With similar arguments, we can get $A \equiv 0$, which completes the proof of Lemma 2.4.

THEOREM 2.4. *Let $\varphi \not\equiv 0$ be an entire function and a, b be two distinct complex numbers. If f is a non-constant meromorphic function satisfying the differential equation*

$$f'(f - f') - \varphi(f - a)(f - b) = 0, \tag{26}$$

then only one of the following cases holds:

(i) $ab \neq 0$, $\varphi \equiv -\frac{ab}{(a-b)^2}$ *and*

$$f = a + ce^{\frac{bz}{b-a}} \quad \text{or} \quad f = b + ce^{\frac{az}{a-b}},$$

where c is a nonzero constant.

(ii) $ab = 0$, $\varphi \equiv \frac{1}{4}$ *and*

$$f = (a + b)(ce^{\frac{z}{4}} - 1)^2,$$

where c is a nonzero constant.

Proof. Suppose that f is a non-constant meromorphic function satisfying equation (26). Since φ is entire, we see that f is entire, too. From equation (26) and by Lemma 2.1, we have $m(r, \varphi) = S(r, f)$. That is $T(r, \varphi) = S(r, f)$. Also from (26) we easily see that

$$f = a \Longrightarrow f' = a, \quad f = b \Longrightarrow f' = b.$$

Therefore, by the Second Fundamental Theorem, we have

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{f - a}\right) + \overline{N}\left(r, \frac{1}{f - b}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{f' - a}\right) + \overline{N}\left(r, \frac{1}{f' - b}\right) + S(r, f) \\ &\leq 2T(r, f') + S(r, f) \\ &\leq 2T(r, f) + S(r, f). \end{aligned}$$

Hence $S(r, f') = S(r, f) := S(r)$.

Now we prove Part (i) of Theorem 2.4.

Since $ab \neq 0$, from equation (26), we see that any zero of f' must be a zero of φ. Therefore $\overline{N}(r, \frac{1}{f'}) = S(r)$. Rewriting equation (26) as

$$\left(f - \left(\frac{a+b}{2} + \frac{f'}{2\varphi}\right)\right)^2 = \left(\frac{1}{4\varphi^2} - \frac{1}{\varphi}\right)(f')^2 + \frac{a+b}{2\varphi} f' + \left(\frac{a-b}{2}\right)^2, \tag{27}$$

and using Lemma 2.4, we have

$$\left(\frac{a+b}{2\varphi}\right)^2 - 4\left(\frac{1}{4\varphi^2} - \frac{1}{\varphi}\right)\left(\frac{a-b}{2}\right)^2 \equiv 0.$$

That is $\varphi \equiv -\frac{ab}{(a-b)^2}$. Replacing φ by $-\frac{ab}{(a-b)^2}$ in equation (27), we get

$$\left(f - a + \frac{a-b}{b} f'\right)\left(f - b + \frac{b-a}{a} f'\right) \equiv 0,$$

which implies that

$$f - a + \frac{a-b}{b} f' \equiv 0 \quad \text{or} \quad f - b + \frac{b-a}{a} f' \equiv 0.$$

Hence

$$f \equiv a + ce^{\frac{bz}{b-a}} \quad \text{or} \quad f \equiv b + ce^{\frac{az}{a-b}},$$

where c is a nonzero constant.

Next we prove Part (ii) of Theorem 2.4.

Without loss of generality, we assume that $a = 0$, and $b = 1$. Thus, equation (26) becomes

$$f'(f - f') - \varphi f(f - 1) = 0, \tag{28}$$

which implies that any zero of f' must be a zero of f with multiplicity 2 if it is not a zero of φ. Let $h := \frac{f}{(f')^2}$. Then

$$\overline{N}\left(r, \frac{1}{h}\right) = S(r), \ \overline{N}(r, h) = S(r). \tag{29}$$

Equation (28) can be rewritten as

$$\left(f' - \frac{1}{2} f\right)^2 = \frac{1}{4} f[(1 - 4\varphi) f + 4\varphi].$$

If $\varphi \not\equiv \frac{1}{4}$, then, from above equation, we see that

$$f(z_0) = \frac{4\varphi(z_0)}{4\varphi(z_0) - 1} \Longrightarrow f'(z_0) = \frac{2\varphi(z_0)}{4\varphi(z_0) - 1},$$

and thus $h(z_0) = \frac{4\varphi(z_0)-1}{\varphi(z_0)}$, where $4\varphi(z_0) - 1 \neq 0$ and $\varphi(z_0) \neq 0$. Noting that $f = 1$ implies that $f' = 1$ and thus $h = 1$, by the Second Fundamental Theorem, we have

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{f - 1}\right) + \overline{N}\left(r, \frac{1}{f - \frac{4\varphi}{4\varphi - 1}}\right) + S(r) \\ &\leq \overline{N}\left(r, \frac{1}{h - 1}\right) + \overline{N}\left(r, \frac{1}{h - \frac{4\varphi - 1}{\varphi}}\right) + S(r) \\ &\leq 2T(r, h) + S(r). \end{aligned}$$

Therefore φ is also a small function of h. From the definition of h and equation (28), we have

$$\left(hf' - \frac{1}{2\varphi}\right)^2 = h + \frac{1}{4\varphi^2} - \frac{1}{\varphi}.$$

Therefore $h + \frac{1}{4\varphi^2} - \frac{1}{\varphi}$ has no simple zero. Hence, by Lemma 2.4, we get $\frac{1}{4\varphi^2} - \frac{1}{\varphi} \equiv 0$. That is $\varphi \equiv \frac{1}{4}$. Thus equation (28) can be written as

$$(2f' - f)^2 = f. \tag{30}$$

Let $g := 2f' - f$. We have $f = g^2$, and thus $f' = 2gg'$. From (30), we have $4g' - g = \pm 1$. Therefore $g = ce^{\frac{z}{4}} \pm 1$, and finally $f = (ce^{\frac{z}{4}} - 1)^2$, where c is a nonzero constant, which completes the proof of Theorem 2.4.

COROLLARY 2.1. *Let f be a non-constant entire function and a, b be two distinct nonzero complex numbers. If a, b are not the Picard exceptional values of f, and furthermore, $f = a \Longrightarrow f' = a$, $f = b \Longrightarrow f' = b$, then $f \equiv f'$.*

Proof. Let

$$\varphi = \frac{f'(f - f')}{(f - a)(f - b)}.$$

By assumption, we see that φ is an entire function. If $\varphi \not\equiv 0$, then f is a solution of the differential equation

$$f'(f - f') - \varphi(f - a)(f - b) = 0.$$

By Theorem 2.4, we see that either a or b is a Picard exceptional value of f. This contradicts the assumption. Hence $\varphi \equiv 0$. That is, $f \equiv f'$.

CONJECTURE. *For any entire function φ and two distinct complex numbers a, b, the entire solutions of the non-linear differential equation*

$$f'(f - f^{(k)}) - \varphi(f - a)(f - b) = 0$$

are functions of exponential type.

3. Concluding remarks

1. Note that, as assumed, g is homogenous in Theorem 2.1. However, from its proof one can verify easily that the theorem is still true when g is non-homogenous, i.e., $g = a_{-1} + a_0 f + a_1 f' + \cdots + a_k f^{(k)}$, where a_i $(i = -1, 0, 1, \ldots, k)$ is a small function of f.

2. By counting the poles of f carefully, we can prove that Theorem 2.1 is still valid for any meromorphic function f satisfying $N(r, f) = S(r, f)$, and the condition "IM" is replaced by "IM*", where "IM*" is a less restrictive concept than the "IM" concept introduced earlier in [11]. We say that f and g share a value b IM* iff

$$\overline{N}\left(r, \frac{1}{f-b}\right) - N_I\left(r, \frac{1}{f-b}\right) = S(r, f),$$

and

$$\overline{N}\left(r, \frac{1}{g-b}\right) - N_I\left(r, \frac{1}{g-b}\right) = S(r, f),$$

where $N_I(r, \frac{1}{f-b})$ is the reduced counting function of the common b-points of f and g.

3. Finally, we conjecture that Theorem 2.3 still holds when a, b are two arbitrarily distinct small functions of f.

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References

[1] C. A. Bernstein, D. C. Chang and B. Q. Li, *On uniqueness of entire functions in* $\mathbb{C}^n$ *and their partial differential polynomials*, Forum Math. **8**(1996), 379–396.
[2] J. Clunie, *On integral and meromorphic functions*, J. London Math. Soc. **37**(1962), 17–27.
[3] W. Doeringer, *Exceptional values of differential polynomial*, Pacific J. Math. **98**(1982), 55–62.
[4] G. Frank, *Lecture notes on sharing values of entire and meromorphic functions*, Workshop in Complex Analysis at Tianjing, China, 1991.
[5] G. Frank and Xinhou Hua, *Differential polynomials that share three values with their generated meromorphic function*, Michigan Math. J. **49**(1999), 175–186.
[6] G. Frank and W. Ohlenroth, *Meromorphe Funktionen, die mit einer ihrer Ableitungen Werte teilen*, Complex Variables Theory Appl. **6**(1986), pp. 23–37.
[7] G. Frank and G. Weißenborn, *Meromorphe Funktionen, die mit einer ihrer Ableitungen Werte teilen*, Complex Variables Theory Appl. **7**(1986), pp. 33–43.
[8] G. G. Gundersen, *Meromorphic functions that share finite values with their derivative*, J. Math. Anal. Appl. **75**(1980), pp. 441–446; Correction, **86**(1982), 307.
[9] E. Mues and N. Steinmetz, *Meromorphe Funktionen, die mit ihrer Ableitung Werte teilen*, Manuscripta Math. **29**(1979), pp. 195–206.
[10] E. Mues and M. Reinders, *Meromorphie funktionen, die mit einem linearen defferential polynom drei werte teilen*, Results Math. **22**(1992), pp. 725–738.
[11] Ping Li and C. C. Yang, *Value sharing of an entire function and its derivative*, J. Math. Soc. Japan **51**(1999), 781–799.
[12] M. Reinders, *Eindeutigkneitssätze für meromorphe Funktionen, die, Vier Werte teilen*, genehmigte Dissertation, Universitüt Hannover, 1990.
[13] L. A. Rubel and C. C. Yang, *Values shared by an entire function and its derivative*, Lecture Notes in Math., no. 599, Springer-Verlag, New York, 1977, pp. 101–103.
[14] H. X. Yi and C. C. Yang, *On the uniqueness theory of meromorphic functions* (in Chinese), Science Press, China, 1996.
[15] L. Yang, *Value distribution theory*, Springer-Verlag, New York, 1993.

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