

AN INFINITE PRODUCT OF ISOLS

BY

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1. Introduction

Let us denote the set of all *numbers* (i.e., nonnegative integers) by ε , and the class of all *sets* (i.e., subcollections of ε) by V . If $f(x)$ is a function from a subset of ε into ε , we write δf and ρf for its domain and its range respectively. The set α is *recursively equivalent* to the set β (written $\alpha \simeq \beta$) if there is a partial recursive one-to-one function $p(x)$ such that $\alpha \subset \delta p$ and $p(\alpha) = \beta$. This \simeq relation between sets is reflexive, symmetric, and transitive. The class of all sets σ such that $\sigma \simeq \alpha$ is denoted by $\text{Req } \alpha$. Using the \simeq relation one can extend the system $[\varepsilon, +, \cdot]$ consisting of the set ε and the binary operations of ordinary addition and multiplication to the system $[\Lambda, +, \cdot]$ of all isols. The method by which this extension can be obtained is sketched in Section 1 of [4]; for a detailed exposition, see [2].

There is a subcollection of Λ , called the collection of all regressive isols, which plays a special role in the present paper. We shall therefore recall its definition and some of its properties. A function t_n from ε into ε is *regressive* if it is one-to-one and there is a partial recursive function $p(x)$ such that $\rho t \subset \delta p$, $p(t_0) = t_0$, and $p(t_{n+1}) = t_n$, for every n . A set is *regressive* if it is finite or the range of some regressive function. Every set which is recursively equivalent to a regressive set is itself regressive; also, every regressive set is recursively enumerable or immune. An *isol* is *regressive* if it contains at least one regressive set, or equivalently, if it contains only regressive sets. The collection of all regressive isols is denoted by Λ_R . Both Λ and Λ_R have cardinality c . Let t_n and t_n^* be one-to-one functions from ε into ε . Then t_n is *recursively equivalent* to t_n^* (written $t_n \simeq t_n^*$) if there is a partial recursive one-to-one function $p(x)$ such that $\rho t \subset \delta p$ and $p(t_n) = t_n^*$ for every n . This \simeq relation between one-to-one functions from ε into ε is also reflexive, symmetric, and transitive. The basic property of regressive functions is as follows. Let t_n and t_n^* be one-to-one functions from ε into ε with respective ranges τ and τ^* . If t_n and t_n^* are regressive functions,

$$t_n \simeq t_n^* \iff \tau \simeq \tau^*.$$

This enables us to associate with every infinite regressive isol T a *denumerable* family of functions, namely the family of all regressive functions ranging over sets in T . It can be shown that if T is a regressive isol, so is 2^T .

In [4] the sum of an infinite series of finite isols (i.e., ordinary numbers) was defined, provided the summation is performed with respect to an infinite

Received July 2, 1962.

¹ The research on this paper was supported by a grant from the NSF.

regressive isol. Let a_n be a function from ε into ε , and let $T \in \Lambda_R$. If T is finite, say $T = k$,

$$\sum_T a_n = \sum_{n < k} a_n \quad (0 \text{ for } k = 0),$$

and if T is infinite,

$$\sum_T a_n = \text{Req} \sum_{n=0}^{\infty} j(t_n, \nu(a_n)),$$

where $\nu_0 = o$, $\nu_m = (0, \dots, m-1)$ for $m \geq 1$, and t_n is any regressive function ranging over any set in T . It is proved in [4] that for $T \in \Lambda_R - \varepsilon$, $x \in \varepsilon$, $x \geq 2$,

$$\underbrace{1 + x + x^2 + \dots}_T = \frac{x^T - 1}{x - 1}.$$

We shall see below (Theorem 1) that this can be generalized to

$$\underbrace{1 + X + X^2 + \dots}_T = \frac{X^T - 1}{X - 1},$$

for $T \in \Lambda_R - \varepsilon$, $X \in \Lambda$, $X \geq 2$.

2. Summary

Let $f_0(X), f_1(X), \dots$ be an infinite sequence of functions from Λ into itself. For every number $m \geq 1$ we write

$$\sum_m f_i(X) = \sum_{i=0}^{m-1} f_i(X), \quad \prod_m f_i(X) = \prod_{i=0}^{m-1} f_i(X).$$

The subscript m indicates therefore the number of functions which are to be added or multiplied. We clearly have

(1) for every number $m \geq 1$ and every number $x \geq 2$,

$$\prod_m (1 + x^{2^i}) = \sum_{2^m} x^i = (x^{2^m} - 1)/(x - 1).$$

It is the purpose of this paper to generalize (1) to

(2) for every regressive isol $T \geq 1$ and every isol $X \geq 2$,

$$\prod_T (1 + X^{2^i}) = \sum_{2^T} X^i = (X^{2^T} - 1)/(X - 1).$$

3. Notations

The well-known primitive recursive functions j, k, l defined by

$$j(x, y) = (x + y)(x + y + 1)/2 + x, \quad j(k(z), l(z)) = z,$$

can also be denoted by j_2, k_{21}, k_{22} respectively. We put

$$j_1(x) = x,$$

$$j_{n+1}(x_1, \dots, x_{n+1}) = j(j_n(x_1, \dots, x_n), x_{n+1}) \quad (n \geq 2),$$

$$j_n(k_{n1}(z), \dots, k_{nn}(z)) = z \quad (n = 1 \text{ or } n \geq 3).$$

The last relation holds therefore for all $n \geq 1$. It is often convenient to write

$$(a_1, \dots, a_n)^* = j_n(a_1, \dots, a_n) \quad (n \geq 1).$$

We recall that for every $n \geq 1$ the function j_n maps ε^n one-to-one onto ε . For any set α , the set α^n with $n \geq 0$ is defined by

$$\begin{aligned} \alpha^0 &= (0), \\ \alpha^n &= \{ (x_1, \dots, x_n)^* \mid x_1 \in \alpha, \dots, x_n \in \alpha \} \quad (n \geq 1). \end{aligned}$$

Moreover,

$$\begin{aligned} j(\alpha, \beta) &= \{j(x, y) \mid x \in \alpha \text{ and } y \in \beta\}, \\ j(a, \beta) &= \{j(a, y) \mid y \in \beta\}, \\ j(\alpha, b) &= \{j(x, b) \mid x \in \alpha\}. \end{aligned}$$

If several parentheses occur in some formula which involves the j -function, we shall sometimes use square brackets. For instance,

$$j[m, (a_1, \dots, a_n)^*] = j(m, (a_1, \dots, a_n)^*).$$

Also, association is to the right so that

$$pk(\alpha) = p(k(\alpha)), \quad fgh(x) = f(g(h(x))).$$

The words "regressive function" will only be used in the sense of "one-to-one regressive function from ε into ε ." The function t_n is sometimes written as $t(n)$ without any warning concerning a change in notation. We write " $\alpha \simeq \beta$ by p ", if $p(x)$ is a partial recursive one-to-one function such that $\alpha \subset \delta p$ and $p(\alpha) = \beta$.

4. Definitions

Let T be a nonzero regressive isol. If T is finite, say $T = k$,

$$\sum_T X^i = \sum_{i=0}^{k-1} X^i.$$

If T is infinite,

$$\sum_T X^i = \text{Req} \sum_{n=0}^{\infty} j(t_n, \xi^n),$$

where $\xi \in X$ and t_n is any regressive function ranging over any set in T .

To prove that $\sum_T X^i$ is well-defined we need only consider the case that T is infinite. Let s_n and t_n be two regressive functions, and let ξ and η be any two sets. It suffices to prove the two statements

$$(3) \quad s_n \simeq t_n \Rightarrow \sum_{n=0}^{\infty} j(s_n, \xi^n) \simeq \sum_{n=0}^{\infty} j(t_n, \xi^n),$$

$$(4) \quad \xi \simeq \eta \Rightarrow \sum_{n=0}^{\infty} j(t_n, \xi^n) \simeq \sum_{n=0}^{\infty} j(t_n, \eta^n).$$

Proof of (3). According to the hypothesis there exists a partial recursive one-to-one function $q(x)$ such that

$$\rho s \subset \delta q \quad \text{and} \quad (\forall n)[q(s_n) = t_n].$$

Let

$$f(z) = j[qk(z), l(z)] \quad \text{for } k(z) \in \delta q;$$

then $f(z)$ is a partial recursive one-to-one function such that

$$fj(s_n, y) = j(t_n, y) \quad \text{for all } n \text{ and } y.$$

This not only implies the conclusion of (3), but also the stronger result

$$\text{for every } \xi \in V, \quad \sum_{n=0}^{\infty} j(s_n, \xi^n) \simeq \sum_{n=0}^{\infty} j(t_n, \xi^n) \quad \text{by } f.$$

The gain in strength lies in the fact that f does not depend on ξ .

Proof of (4). Let $\xi \simeq \eta$ by q . Put

$$g_n(z) = j_n[qk_{n1}(z), \dots, qk_{nn}(z)];$$

then $g_n(z)$ is a partial recursive function of n and z such that

$$(\forall n)[\xi^n \simeq \eta^n \quad \text{by } g_n].$$

Let $p(x)$ be a regressing function of t_n . We recall that $p^*(x)$ is a partial recursive function with δp as domain such that for every n , $p^*(t_n) = n$. Writing $g(n, z)$ for $g_n(z)$ we define

$$f(z) = j[k(z), g(p^*k(z), l(z))].$$

Then $f(z)$ is a partial recursive function and

$$fj(t_n, y) = j[t_n, g_n(y)] \quad \text{for every } n.$$

It follows that f maps $\sum_{n=0}^{\infty} j(t_n, \xi^n)$ onto $\sum_{n=0}^{\infty} j(t_n, \eta^n)$. Let $z_1 = j(x_1, y_1)$ and $z_2 = j(x_2, y_2)$ be two distinct elements of δf . Clearly,

$$x_1 \neq x_2 \Rightarrow k(z_1) \neq k(z_2) \Rightarrow f(z_1) \neq f(z_2).$$

If, on the other hand, $x_1 = x_2$, then $y_1 \neq y_2$. Put $m = x_1$, then $g(p^*(m), y)$ is a one-to-one function of y , hence

$$g(p^*(m), y_1) \neq g(p^*(m), y_2) \quad \text{and} \quad f(z_1) \neq f(z_2).$$

Thus $f(z)$ is a one-to-one function. This completes the proof of (4).

In order to define the infinite product we use ultimately vanishing sequences of numbers, i.e., infinite sequences of numbers which have only finitely many nonzero elements. For each such sequence $\{x_n\}$, we put

$$\{x_n\}^* = \Theta\{x_n\} = \prod_{n \leq k} p_{x_n}^{p_n} - 1,$$

where $p_0 = 2$, $p_n = n^{\text{th}}$ odd prime number (for $n \geq 1$), and k is any number such that $x_n = 0$ for $n > k$. The mapping Θ maps the collection of all ultimately vanishing sequences effectively and one-to-one onto ε . In particular it maps the sequence $\{0, 0, 0, \dots\}$ which we also denote by $\{0\}$ onto the number 0. With every sequence $\{\sigma_n\}$ of sets we associate the set

$$\prod_{n=0}^{\infty} \sigma_n = \{\{x_n\}^* \mid x_0 \in \sigma_0, \quad x_1 \in \sigma_1, \quad \dots\}.$$

For any one-to-one function t_n from ε into ε and any set ξ we define

$$\prod (t, \xi) = \prod_{n=0}^{\infty} [(0) + j(t_n, \xi^{2^n})].$$

Let T be any nonzero regressive isol. If T is finite, say $T = k$,

$$\prod_T (1 + X^{2^i}) = \prod_{i=0}^{k-1} (1 + X^{2^i}).$$

If T is infinite,

$$\prod_T (1 + X^{2^i}) = \text{Req} \prod (t, \xi),$$

where $\xi \in X$ and t_n is any regressive function ranging over any set $\tau \in T$ such that $0 \notin \tau$. The purpose of the condition $0 \notin \tau$ is to ensure that $0 \notin j(t_n, \xi^{2^n})$ for every n .

To prove that this infinite product is well-defined we may restrict our attention to the case that T is infinite. Let s_n and t_n be two regressive functions, and let ξ and η be any two sets. It suffices to prove the two statements

$$(5) \quad s_n \simeq t_n \quad \text{and} \quad 0 \notin \rho s + \rho t \Rightarrow \prod (t, \xi) \simeq \prod (s, \xi),$$

$$(6) \quad \xi \simeq \eta \quad \text{and} \quad 0 \notin \rho t \Rightarrow \prod (t, \xi) \simeq \prod (t, \eta).$$

Proof of (5). It follows from the hypothesis that there exists a partial recursive one-to-one function $q(x)$ such that

$$0 \notin \delta q + \rho q, \quad \rho s \subset \delta q, \quad \text{and} \quad (\forall n)[q(s_n) = t_n].$$

Let the partial recursive functions $a(x)$ and $b(x)$ be defined by

$$\delta a = \delta q + (0), \quad a(0) = 0, \quad x \in \delta q \Rightarrow a(x) = q(x),$$

$$\delta b = j(\delta a, \varepsilon), \quad bj(x, y) = j[a(x), y].$$

The function $b(x)$ is one-to-one, $b(x) = 0$ if and only if $x = 0$, and for every ultimately vanishing sequence $\{x_n\}$,

$$(7) \quad \{x_n\}^* \in \prod (s, \xi) \Leftrightarrow \{b(x_n)\}^* \in \prod (t, \xi).$$

We claim that there exists a partial recursive function $f(x)$ such that

$$(8) \quad \delta f = \{\{x_n\}^* \mid (\forall x_n)[x_n \in \delta b]\}, \quad \{x_n\}^* \in \delta f \Rightarrow f(\{x_n\}^*) = \{b(x_n)\}^*.$$

For there exist recursive functions $c(n)$ and $d(n, x)$ such that

$$c(\{0\}^*) = c(0) = 0,$$

$$(9) \quad c(\{x_0, \dots, x_k, 0, 0, \dots\}^*) = k + 1 \quad \text{for } x_k \neq 0,$$

$$d(k, \{x_n\}^*) = x_k \quad \text{for every } k \geq 0.$$

Thus the function

$$f(x) = \prod_{n=0}^{c(x)^{-1}} p_n^{bd(n, x)} - 1 \quad \text{for } x \in \{\{x_n\}^* \mid (\forall x_n)[x_n \in \delta b]\},$$

satisfies (8). The function $f(x)$ is one-to-one in view of (7), (8), and the

fact that $b(x) = 0$ if and only if $x = 0$. This does not only imply the conclusion of (5), but the stronger statement

$$\text{for every } \xi, \quad \prod (t, \xi) \simeq \prod (s, \xi) \quad \text{by } f.$$

Proof of (6). Let $\xi \simeq \eta$ by p . There exists a partial recursive function $p_k(x)$ of k and x such that for every k

$$\begin{aligned} \delta p_k &= j[\varepsilon - (0), (\delta p)^k] + (0), & p_k(0) &= 0, \\ p_k j[x, (y_1, \dots, y_k)^*] &= j[x, (p(y_1), \dots, p(y_k))^*] & \text{for } x \neq 0. \end{aligned}$$

This implies that for every k , the function $p_k(x)$ is a partial recursive one-to-one function of x which for every $a \neq 0$ maps $j(a, \xi^k)$ onto $j(a, \eta^k)$. Put $p(k, x) = p_k(x)$. Let q be a regressing function of t_n . We use the partial recursive function q^* with δq as domain and the property that $q^*(x) = t^{-1}(x)$ for $x \in \rho t$. Consider the partial recursive function $w(z)$ defined by

$$\begin{aligned} w(0) &= 0, \\ w(z) &= j[k(z), p(2^{q^*k(z)}, l(z))] \quad \text{for } k(z) \neq 0 \quad \text{and} \quad k(z) \in \delta q^*. \end{aligned}$$

It follows that

$$(10) \quad \text{for every number } n, \quad w(z) \text{ maps } (0) + j(t_n, \xi^{2^n}) \text{ one-to-one onto } (0) + j(t_n, \eta^{2^n}).$$

Let $z_1 = j(x_1, y_1)$ and $z_2 = j(x_2, y_2)$ be two distinct elements of δw . Note that $z = 0$ if and only if $w(z) = 0$. Thus, if exactly one of z_1 and z_2 equals 0, we have $w(z_1) \neq w(z_2)$. Consider the case that z_1 and z_2 are both different from 0. Clearly,

$$x_1 \neq x_2 \Rightarrow k(z_1) \neq k(z_2) \Rightarrow w(z_1) \neq w(z_2).$$

Now assume $x_1 = x_2$; then $y_1 \neq y_2$. Putting $m = 2^{q^*(x_1)}$ we know that $p(m, y)$ is a one-to-one function of y . Hence

$$y_1 \neq y_2 \Rightarrow p(m, y_1) \neq p(m, y_2) \Rightarrow w(z_1) \neq w(z_2).$$

Thus $\prod (t, \xi) \simeq \prod (t, \eta)$ by $w(z)$.

5. Theorems

In order to establish (2) we have to evaluate an infinite series and to expand an infinite product in an infinite series.

THEOREM 1. *For every regressive isol $T \geq 1$ and every isol $X \geq 2$,*

$$(11) \quad \sum_T X^i = (X^T - 1)/(X - 1).$$

Proof. If T is finite, say $T = k$, we can deduce (11) from the relation

$$(X - 1)S_k = X^k - 1, \quad \text{where } S_k = \sum_{n=0}^{k-1} X^n,$$

which can be proved as in elementary algebra. Henceforth we assume that T is infinite. Let $\tau \in T$, and let t_n be a regressive function ranging over τ . For the notion of a finite function and the definitions of $r_n(x)$ and α^β , see [2, pp. 181, 182]. Let $0 \in \xi \in X$. Put $\xi_0 = \xi - (0)$,

$$\alpha = \sum_{n=0}^{\infty} j(t_n, \xi^n), \quad \mathfrak{F} = \{r_n \mid n \in \xi^\tau - (0)\}.$$

Thus $\xi^\tau - (0)$ is the set of all indices with which the functions in \mathfrak{F} occur in the infinite sequence $r_0(x), r_1(x), \dots$ of finite functions. We have

$$\begin{aligned} \xi_0 \in X - 1, \quad \alpha \in \sum_{\tau} X^i, \\ \xi^\tau - (0) \in X^T - 1, \quad j(\xi_0, \alpha) \in (X - 1) \cdot \sum_{\tau} X^i. \end{aligned}$$

Hence it suffices to prove

$$(12) \quad j(\xi_0, \alpha) \simeq \xi^\tau - (0).$$

With every $f \in \mathfrak{F}$ we associate the number

$$d(f) = \max \{n \mid f(t_n) \neq 0\}.$$

Every function $f \in \mathfrak{F}$ with $d(f) = d$ is completely characterized by the finite sequence y_0, \dots, y_d , where $y_i = f(t_i)$ for $0 \leq i \leq d$. With every number u of the form

$$u = j[x, j(t_n, (y_0, \dots, y_{n-1})^*)]$$

we associate the function f_u such that

$$\begin{aligned} f_u(t_i) &= y_i \quad \text{for } 0 \leq i \leq n-1, \\ f_u(t_n) &= x, \\ f_u(w) &= 0 \quad \text{for } w \notin (t_0, \dots, t_n). \end{aligned}$$

Thus we see that

$$u \in j(\xi_0, \alpha) \implies f \in \mathfrak{F}.$$

We claim that the mapping $u \rightarrow f_u$ maps the set $j(\xi_0, \alpha)$ onto the family \mathfrak{F} . For let $f \in \mathfrak{F}$. If $d(f) = 0$, we put $x_0 = f(t_0)$. Then

$$f_u = f \quad \text{for } u = j[x_0, j(t_0, 0)] \in j(\xi_0, \alpha).$$

If $d(f) = n > 0$, we put $x_i = f(t_i)$ for $0 \leq i \leq n$. Then $x_n \neq 0$, and

$$f_u = f \quad \text{for } u = j[x_n, j(t_n, (x_0, \dots, x_{n-1})^*)] \in j(\xi_0, \alpha).$$

For every number $u \in j(\xi_0, \alpha)$ we denote the unique number m such that $f_u = r_m$ by $g(u)$. It follows that

$$(13) \quad g(u) \text{ maps } j(\xi_0, \alpha) \text{ onto } \xi^\tau - (0).$$

We wish to show that $g(u)$ has an extension $g_0(u)$ which is *partial recursive* and *one-to-one*. Let $p(x)$ be a regressing function of t_n , let $\varepsilon_0 = \varepsilon - (0)$

and $\alpha_0 = j(\delta p^*, \varepsilon)$. We take the recursively enumerable set $j(\varepsilon, \alpha_0)$ as the domain of $g_0(u)$. Assume

$$u = j[x, j(y, z)] \in j(\varepsilon_0, \alpha_0).$$

Let $n = p^*(y)$. Then z determines a unique ordered n -tuple (z_0, \dots, z_{n-1}) such that $z = (z_0, \dots, z_{n-1})^*$. Hence

$$u = j[x, j(y, (z_0, \dots, z_{n-1})^*)].$$

We define $g_0(u)$ as the unique number m such that

$$r_m(y) = x, \quad r_m p(y) = z_{n-1}, \quad r_m p^2(y) = z_{n-2}, \quad \dots, \quad r_m p^n(y) = z_0, \\ r_m(w) = 0 \quad \text{for } w \notin (y, p(y), \dots, p^n(y)).$$

It is readily seen that $g_0(u)$ is a partial recursive function which is an extension of $g(u)$. We proceed to prove that the function $g_0(u)$ is one-to-one. Assume

$$u_1 = j[x_1, j(y_1, (v_0, \dots, v_{a-1})^*)] \in j(\varepsilon_0, \alpha_0), \\ u_2 = j[x_2, j(y_2, (w_0, \dots, w_{b-1})^*)] \in j(\varepsilon_0, \alpha_0).$$

Let $m(1) = g_0(u_1)$, $m(2) = g_0(u_2)$, $f_1 = r_{m(1)}$, $f_2 = r_{m(2)}$. Suppose $f_1 = f_2$. We shall show that $u_1 = u_2$. We have

$$f_1(y_1) = x_1, \quad a = p^*(y_1), \\ f_1 p(y_1) = v_{a-1}, \quad f_1 p^2(y_1) = v_{a-2}, \quad \dots, \quad f_1 p^a(y_1) = v_0, \\ f_1(w) = 0 \quad \text{for } w \notin (y_1, p(y_1), \dots, p^a(y_1)), \\ f_2(y_2) = x_2, \quad b = p^*(y_2), \\ f_2 p(y_2) = w_{b-1}, \quad f_2 p^2(y_2) = w_{b-2}, \quad \dots, \quad f_2 p^b(y_2) = w_0, \\ f_2(w) = 0 \quad \text{for } w \notin (y_2, p(y_2), \dots, p^b(y_2)).$$

Since $f_1 = f_2$, we have $f_2(y_1) = f_1(y_1)$; however, $f_1(y_1) = x_1$, where $x_1 \neq 0$. Thus $f_2(y_1) \neq 0$, and this implies

$$y_1 \in (y_2, p(y_2), \dots, p^b(y_2)), \\ (14) \quad y_1 = p^i(y_2) \quad \text{for some } i \text{ with } 0 \leq i \leq b.$$

Applying p^{b-i} to both sides of (14) yields $p^{b-i}(y_1) = p^b(y_2)$; however, $b = p^*(y_2)$; hence $p^{b+1}(y_2) = p^b(y_2)$. We conclude

$$(15) \quad p^{b-i+1}(y_1) = p^{b-i}(y_1) \quad \text{for some } i \text{ with } 0 \leq i \leq b.$$

Now suppose b were less than a . Then $b - i < a$, and (15) would be false. Thus $b \geq a$. Similarly we can prove $a \geq b$. Hence $a = b$, and (15) implies

$$p^{a-i+1}(y_1) = p^{a-i}(y_1) \quad \text{for some } i \text{ with } 0 \leq i \leq a.$$

The last equality is false for $0 < i < a$; also, it implies $a = 0$, in case $i = a$.

Thus $i = 0$ and $y_1 = y_2$ in view of (14). From $a = b$, $y_1 = y_2$, $f_1 = f_2$, it readily follows that $u_1 = u_2$. Since the function $g(u)$ mentioned in (13) has a partial recursive one-to-one function as extension, namely $g_0(u)$, the proof of (12) is complete.

THEOREM 2. *For every regressive isol $T \geq 1$ and every isol X ,*

$$(16) \quad \prod_T (1 + X^{2^i}) = \sum_{2^T} X^i.$$

Proof. If T is finite, say $T = k$, the relation (16) reduces to

$$(1 + X)(1 + X^2)(1 + X^4) \cdots (1 + X^{2^{k-1}}) \\ = 1 + X + X^2 + X^3 + \cdots + X^{2^k-1},$$

which can be verified by expressing the left side as a polynomial in X . If $X = 0$, both sides of (16) equal 1. We therefore restrict our attention to the case: T infinite, $X \geq 1$. Let $\eta = (1, 2, \dots)$. Put

$$\begin{aligned} \gamma_0 &= (0) + j(0, \eta), & \delta_0 &= [j(0, 0)] = (0), \\ \gamma_1 &= (0) + j(1, \eta^2), & \delta_1 &= j(1, \eta), \\ \gamma_2 &= (0) + j(2, \eta^4), & \delta_2 &= j(2, \eta^2), \\ \gamma_3 &= (0) + j(3, \eta^8), & \delta_3 &= j(3, \eta^3), \\ &\vdots & &\vdots \\ \gamma_n &= (0) + j(n, \eta^{2^n}), & \delta_n &= j(n, \eta^n), \\ &\vdots & &\vdots \\ \gamma &= \prod_{n=0}^{\infty} \gamma_n, & \delta &= \sum_{n=0}^{\infty} \delta_n. \end{aligned}$$

We proceed to prove

- (17) there exists a partial recursive one-to-one function with γ as domain and δ as range.

It is clear that γ and δ are infinite recursively enumerable sets. Let $x = \{x_n\}^* \in \gamma$. If $x = 0$, we define $f(x) = 0$. Assume $x \neq 0$, say

$$x = \{x_0, \dots, x_k, 0, 0, \dots\}^*, \quad \text{where } x_k \neq 0.$$

We then define

$$e_i = sg(x_i) \quad \text{for } 0 \leq i \leq k, \quad m = e_0 \cdot 2^0 + \cdots + e_k \cdot 2^k.$$

Note that for every i with $0 \leq i \leq k$, either $x_i = 0$ or $x_i \in j(i, \eta^{2^i})$ and that the value of e_i tells us whether the first or second alternative is realized; these two alternatives exclude each other, since $0 \notin \eta^n$ for every $n \geq 1$. We have

$$\begin{aligned} \text{if } e_0 &= 1, \quad x_0 = j(0, y_{11}) && \text{for some } y_{11} \in \eta, \\ \text{if } e_1 &= 1, \quad x_1 = j[1, (y_{21}, y_{22})^*] && \text{for some } y_{21}, y_{22} \in \eta, \\ \text{if } e_2 &= 1, \quad x_2 = j[2, (y_{41}, \dots, y_{44})^*] && \text{for some } y_{41}, \dots, y_{44} \in \eta, \\ &\vdots && \\ \text{if } e_k &= 1, \quad x_k = j[k, (y_{2^k,1}, \dots, y_{2^k,2^k})^*] && \text{for some } y_{2^k,1}, \dots, y_{2^k,2^k} \in \eta, \end{aligned}$$

where the hypothesis of the last conditional is true. For each i with $0 \leq i \leq k$ we have: $e_i = 1$ implies that $l(x_i)$ represents an ordered 2^i -tuple of elements of η . We define

$$f(x) = j[m, (z_1, \dots, z_m)^*],$$

where z_1, \dots, z_m is the ordered m -tuple of elements of η obtained by concatenating the sequences represented by those $l(x_i)$'s among

$$(I) \quad l(x_0), \dots, l(x_k),$$

for which $e_i = 1$ (in the order in which they occur in (I)). Thus $f(x) \in \delta_m \subset \delta$. Hence $f(x)$ is a partial recursive function from γ into δ . Let x_1 and x_2 be two distinct elements of γ . If exactly one of x_1 and x_2 equals 0, exactly one of $f(x_1)$ and $f(x_2)$ equals 0; hence $f(x_1) \neq f(x_2)$. Now assume both x_1 and x_2 are different from 0, say

$$\begin{aligned} x_1 &= \{x_{10}, \dots, x_{1p}, 0, 0, \dots\}^*, & \text{where } x_{1p} \neq 0, \\ x_2 &= \{x_{20}, \dots, x_{2q}, 0, 0, \dots\}^*, & \text{where } x_{2q} \neq 0. \end{aligned}$$

Then

$$kf(x_1) = m_1 = \sum_{i=0}^p sg(x_{1i}) \cdot 2^i, \quad kf(x_2) = m_2 = \sum_{i=0}^q sg(x_{2i}) \cdot 2^i.$$

Hence $m_1 \neq m_2$ implies $f(x_1) \neq f(x_2)$, and we may restrict our attention to the case $m_1 = m_2$, i.e., the case

$$p = q \quad \text{and} \quad (\forall i)[i \leq p \Rightarrow sg(x_{1i}) = sg(x_{2i})].$$

Writing m for m_1 (and m_2) we have

$$f(x_1) = j[m, (z_{11}, \dots, z_{1m})^*], \quad f(x_2) = j[m, (z_{21}, \dots, z_{2m})^*].$$

Since $x_1 \neq x_2$, there exists a number v such that $0 \leq v \leq p$, $sg(x_{1v}) = sg(x_{2v}) = 1$, while $l(x_{1v})$ and $l(x_{2v})$ represent distinct ordered 2^v -tuples of elements in η . In view of the fact that the elements of these ordered 2^v -tuples occupy the same positions in z_{11}, \dots, z_{1m} as in z_{21}, \dots, z_{2m} we see that

$$(z_{11}, \dots, z_{1m})^* \neq (z_{21}, \dots, z_{2m})^*;$$

hence $f(x_1) \neq f(x_2)$. Thus $f(x)$ maps γ one-to-one into δ . Finally, let $y \in \delta$. If $y = 0$, then $y = f(x)$ for $x = 0$. Now assume $y \neq 0$, say

$$y = j[m, (y_1, \dots, y_m)^*],$$

$$\text{where } m = e_0 \cdot 2^0 + \dots + e_s \cdot 2^s, \text{ with } e_s = 1.$$

We can effectively deconcatenate the ordered m -tuple y_1, \dots, y_m into (reading from the left to the right)

an ordered 1-tuple, in case $e_0 = 1$,

an ordered 2-tuple, in case $e_1 = 1$,

an ordered 4-tuple, in case $e_2 = 1$,
 \vdots
 an ordered 2^{s-1} -tuple, in case $e_{s-1} = 1$,
 an ordered 2^s -tuple (*we know that $e_s = 1$*).

For $0 \leq i \leq s$, let in case $e_i = 1$,

$$y_{2^i,1}, \dots, y_{2^i,2^i}$$

be the ordered 2^i -tuple arising in the deconcatenation of y_1, \dots, y_m . Define $x = \{x_n\}^*$, where $x_n = 0$ for $n > s$, and for $0 \leq i \leq s$,

$$\begin{aligned} x_i &= 0, & \text{in case } e_i &= 0, \\ x_i &= j[i, (y_{2^i,1}, \dots, y_{2^i,2^i})^*], & \text{in case } e_i &= 1. \end{aligned}$$

We then see that $y = f(x)$, where $x \in \gamma$. Thus f is a partial recursive one-to-one function with γ as domain and δ as range. This completes the proof of (17).

With every strictly increasing function $a(n)$ from ε into ε we associate the function

$$a'(n) = e_{n0} \cdot 2^{a(0)} + \dots + e_{nn} \cdot 2^{a(n)},$$

where e_{n0}, \dots, e_{nn} is the sequence of zeros and ones such that

$$n = e_{n0} \cdot 2^0 + \dots + e_{nn} \cdot 2^n.$$

This implies that if $i(n)$ denotes the identity function, so does $i'(n)$. We use the well-known enumeration ρ_0, ρ_1, \dots without repetitions of the class Q of all finite sets defined by

$$\rho_0 = o, \quad \rho_{x+1} = (y_1, \dots, y_k),$$

where y_1, \dots, y_k are the distinct numbers such that

$$x + 1 = 2^{y_1} + \dots + 2^{y_k}.$$

It follows that

$$\begin{aligned} \rho_{x+1} &= \{i \mid 0 \leq i \leq x+1 \text{ and } e_{x+1,i} = 1\}, \\ \rho_{a'(n)} &= \{a(i) \mid 0 \leq i \leq n \text{ and } e_{ni} = 1\} \\ &= a\{i \mid 0 \leq i \leq n \text{ and } e_{ni} = 1\} = a(\rho_n). \end{aligned}$$

If n assumes successively the values $0, 1, \dots$, then ρ_n runs over the class Q of all finite sets, $\rho_{a'(n)} = a(\rho_n)$ over the class of all finite subsets of $\alpha = \rho a$, and $a'(n)$ over the set

$$\{x \mid \rho_x \subset \alpha\} \in 2^{\text{Req}(\alpha)}.$$

It follows from the definition of $a'(n)$ that $a'(n)$ is strictly increasing because $a(n)$ is. Using the three facts

$$a'(n+1) = e_{n+1,0} \cdot 2^{a(0)} + \dots + e_{n+1,n+1} \cdot 2^{a(n+1)},$$

$$a'(n) = e_{n0} \cdot 2^{a(0)} + \cdots + e_{nn} \cdot 2^{a(n)},$$

$$\max \{i \mid e_{ni} = 1\} \leq \max \{i \mid e_{n+1,i} = 1\},$$

we see that if $a(n)$ is regressive, so is $a'(n)$. Let us now return to the proof of (16) for $X \geq 1$, T infinite and regressive. Every regressive isol contains some retraceable set which does not contain 0. Let $\xi \in X$, $0 \notin \tau$, $\tau \in T$, τ retraceable, t_n the (regressive) function which enumerates τ according to size. Put

$$\begin{aligned} \gamma_0^* &= (0) + j(t_0, \xi), & \delta_0^* &= [j(t'_0, 0)] = [j(0, 0)], \\ \gamma_1^* &= (0) + j(t_1, \xi^2), & \delta_1^* &= j(t'_1, \xi), \\ \gamma_2^* &= (0) + j(t_2, \xi^4), & \delta_2^* &= j(t'_2, \xi^2), \\ &\vdots & &\vdots \\ \gamma_n^* &= (0) + j(t_n, \xi^{2^n}), & \delta_n^* &= j(t'_n, \xi^n), \\ &\vdots & &\vdots \\ \gamma^* &= \prod_{n=0}^{\infty} \gamma_n^*, & \delta^* &= \sum_{n=0}^{\infty} \delta_n^*. \end{aligned}$$

The set γ^* belongs to the left side of (16). Also, t'_n is a strictly increasing regressive function which ranges over the set $\{x \mid \rho_x \subset \tau\} \in 2^T$. Hence δ^* belongs to the right side of (16). It therefore suffices to prove

$$(18) \quad \gamma^* \simeq \delta^*.$$

We shall construct a mapping f^* from γ^* into δ^* using the particular partial recursive one-to-one function f which we used in the proof of (17). Let $x^* = \{x_n^*\}^* \in \gamma^*$. If $x^* = 0$, let $f^*(x^*) = 0$. If $x^* \neq 0$, say

$$x^* = \{x_0^*, \dots, x_b^*, 0, 0, \dots\}^*, \quad \text{where } x_b^* \neq 0,$$

the $b+2$ numbers

$$e_{mi} = sg(x_i^*) \quad \text{for } 0 \leq i \leq b, \quad m = e_{m0} \cdot 2^0 + \cdots + e_{mb} \cdot 2^b,$$

can be computed from x^* . Let for every $i \geq 0$,

$$\begin{aligned} x_i &= 0 & \text{if } x_i^* &= 0, \\ x_i &= j[i, l(x_i^*)] & \text{if } x_i^* &\neq 0; \end{aligned}$$

then we have for every $i \geq 0$,

$$\text{either } x_i = x_i^* = 0,$$

$$\text{or } x_i^* = j(t_i, y) \quad \text{for some } y \in \xi^{2^i} \text{ and } x_i = j(i, y).$$

Let $p(x)$ be a regressing function of t_n , then

$$\begin{aligned} x_i &= 0 & \text{if } x_i^* &= 0, \\ x_i &= j[p^*k(x_i^*), l(x_i^*)] & \text{if } x_i^* &\neq 0. \end{aligned}$$

Define $x = \{x_n\}^*$; then x can be computed from x^* . Now compute $f(x)$, say

$$f(x) = j[m, (z_1, \dots, z_m)^*].$$

Then $f^*(x^*)$ is defined by

$$f^*(x^*) = j[t'_m, (z_1, \dots, z_m)^*].$$

Note that

$$(z_1, \dots, z_m)^* = lf(x), \quad t'_m = e_{m0} \cdot 2^{t(0)} + \dots + e_{mb} \cdot 2^{t(b)},$$

where e_{m0}, \dots, e_{mb} can be computed from x^* , and $t(0), \dots, t(b)$ from $t(b) = k(x_b^*)$; here x_b^* is the exponent of the highest prime which divides $x^* + 1$. Thus $f^*(x^*)$ can be computed from x^* . We leave it to the reader to verify that f^* maps γ^* one-to-one onto δ^* . Using the fact that t_n and t'_n are strictly increasing regressive functions, one can also prove that both f^* and its inverse have partial recursive extensions. It then follows by [4, Proposition 1] that $\gamma^* \simeq \delta^*$, i.e., that (18) is correct. This completes the proof of (16).

COROLLARY. For every regressive isol $T \geq 1$ and every isol $X \geq 2$,

$$\prod_T (1 + X^{2^i}) = (X^{2^T} - 1) / (X - 1).$$

Remark. Let Λ_Z denote the collection of all *cosimple* isols, i.e., of all isols which contain a set with a recursively enumerable complement. We know that

$$X, Y \in \Lambda_Z \Rightarrow X^Y \in \Lambda_Z,$$

$$X \leq Y \text{ and } Y \in \Lambda_Z \Rightarrow X \in \Lambda_Z,$$

by [2, Theorems 56, 140]. Clearly, for $X \geq 2$, $T \geq 1$,

$$(X^T - 1) / (X - 1) \leq X^T - 1.$$

It follows therefore from Theorem 1 that $\sum_T X^i$ is cosimple for every regressive cosimple isol $T \geq 1$ and every cosimple isol $X \geq 2$. For $X = 0$ this infinite series equals 1, and for $X = 1$ it equals T (cf. [4, Theorem 2]). The condition $X \geq 2$ may therefore be omitted. If T is regressive and cosimple, so is 2^T . We conclude by Theorem 2 that $\prod_T (1 + X^{2^i})$ is cosimple for every regressive cosimple isol $T \geq 1$ and every cosimple isol X .

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