A SUPPLEMENT TO "MARKOV PROCESSES WITH IDENTICAL HITTING DISTRIBUTIONS"

BY

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The purpose of this note is to correct a definition in [1] and to elaborate on a point which was given insufficient attention in the proof of Theorem 7.2 of [1]. The reader is referred to [1] for definitions, notation, and background material.

We say that two Hunt processes X and X^* on an enlarged state space $\bar{E} = E \cup \Delta$ have *identical hitting distributions* if for all x in E, and all Borel subsets B of $\bar{E}$

$$P_x(X(T_K) \in B, T_K < \infty) = P_x^*(X(T_K) \in B, T_K < \infty)$$

whenever K is a compact subset of E or the complement in $\bar{E}$ of such a set. In [1] we merely required this equality to hold for compact K. But then it does not follow that X and X^* have the same traps; and, what is more important, it does not follow that if K is compact and $P_x(T_{\bar{E}-K} < \infty) = 1$, then also $P_x^*(T_{\bar{E}-K} < \infty) = 1$. Both of these facts are needed for the proofs, so the above change in the definition is essential.

As to Theorem 7.2, it should state that X and X^* have identical hitting distributions if and only if there is a continuous additive functional ϕ of X with $\phi(t) = \phi(\sigma')$ for $t \geqq \sigma'$ such that (i) for each x with P_x probability 1, ϕ is strictly increasing in $[0, \sigma']$, and (ii) if τ is the functional inverse to ϕ, then the processes $X^*(t)$ and $X(\tau(t))$ are identical in law. In [1] the functional ϕ is constructed by transfinite induction, but the proof of (i) based on Proposition 5.5 is not valid unless we first show that $\phi(t)$ is finite for $t < \sigma'$. It is not at all obvious that the finiteness of ϕ is preserved during the passage to limit ordinals, so this point needs some attention. One can prove the finiteness directly, but it is also possible to modify slightly the construction of ϕ so that this issue does not arise. We will follow the second course, at the expense of a little extra effort. Also to save a few words we will assume that Δ is the only trap, that is, $\sigma = \sigma'$.

To start with, let $\{N_i\}$ be a family of open sets with compact closures forming a base for the topology of E, and let

$$v_i(x) = \int_0^\infty e^{-t} P(t, x, \bar{N}_i^c)\, dt.$$

The sets $U_{ij} = N_i \cap \{v_i > 1/j\}$, as i and j range over the positive integers, form a nearly open cover of E. If W_{ij} denotes $\bar{N}_i \cap \{v_i \geqq 1/j\}$, then W_{ij}^c is

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nearly open, and for each x, $P_x(T_{ij} < \infty) = 1$, where T_{ij} denotes the time of first hitting W^c_{ij}. From this it follows without much difficulty that for each x, $P^*_x(T_{ij} < \infty) = 1$. Let

$$\Phi_{ij}(x) = E_x e^{-T_{ij}} \quad \text{and} \quad W_{ijk} = \{\Phi_{ij} < 1 - 1/k\}.$$

The function Φ_{ij} is 1-excessive, and the union over k of the W_{ijk} is the state space for X terminated when it leaves W_{ij} (that is, W_{ij} less the points regular for its complement). In particular, every point of U_{ij} is in some W_{ijk}. Now list the W_{ijk} in some sequential order, and let $L(x) = (i, j)$ if W_{ijk} is the first set in this list containing the point x. Define an increasing sequence of stopping times by

$$\begin{aligned} R_1(w) &= T_{L(X(0,w))}(w), \qquad & X(0, w) \in E, \\ &= 0, & X(0, w) = \Delta, \end{aligned}$$

and

$$R_{n+1}(w) = R_n(w) + R_1(\theta_{R_n} w)$$

for $n \geqq 1$. Let R be the limit of the R_n.

Now let ϕ_{ij} denote the continuous additive functional constructed in Section 5 of [1], but relative to the terminal time T_{ij}, and note that

$$P_x(\phi_{ij}(T_{ij}) < \infty) = 1 \qquad \text{for all } x.$$

In a moment we will prove that $P_x(R = \sigma) = 1$. Thus we may obtain our additive functional ϕ by piecing together the ϕ_{ij} as we did in [1],

$$\begin{aligned} \phi(t, w) &= \phi_{L(X(0,w))}(t, w), & 0 \leqq t \leqq R_1, \\ &= \phi(R_n, w) + \phi_{L(X(0,\theta_{R_n} w))}(t - R_n, \theta_{R_n} w), \quad & R_n \leqq t \leqq R_{n+1}, \end{aligned}$$

and of course $\phi(t) = \phi(\sigma -)$ for $t \geqq \sigma$. The finiteness properties of the ϕ_{ij} imply that $P_x(\phi(R_n) < \infty) = 1$ for all n, and so the fact that $P_x(R = \sigma) = 1$ will yield the desired $P_x(\phi(t) < \infty$, for all $t < \sigma) = 1$. The verification of the other properties of ϕ proceeds as in Section 7 of [1].

THEOREM. *For all x,* $P_x(R = \sigma) = 1$.

Proof. Assume the contrary. Then for some x and (i, j), to be held fixed, $P_x(X(R) \in U_{ij}) > 0$. Now $X(t)$ has left-hand limits at finite values of t, and since $R_n < R$, if $R < \sigma$, we have as t increases to R,

$$\lim X(t) = \lim X(R_n) = X(R)$$

on $\{R < \sigma\}$. Also $v_i(X(t))$ has left-hand limits because v_i is 1-excessive, and

$$\lim e^{-R_n} v_i(X(R_n)) \geqq e^{-R} v_i(X(R)).$$

Of course the limit assertions hold almost everywhere P_x. What we have just said, together with the facts that N_i is open and U_{ij} is nearly open, imply that for some $\beta > 0$ the event

$$X(t) \in U_{ij} \quad \text{for all } t \text{ such that } | t - R | < \beta,$$

which we will call Λ, has strictly positive P_x measure. Since Φ_{ij} is 1-excessive, the composition $\Phi_{ij}(X(t))$ has left-hand limits. We will now show that

$$P_x(\lim_{t \uparrow R} \Phi_{ij}(X(t)) = 1; \ \Lambda) = 0.$$

Indeed it is clear that if $\delta > 0$, then we can find $\alpha < 1$ such that for every y in E, $\Phi_{ij}(y) > \alpha$ implies $P_y(T_{ij} \geqq \beta) < \delta$. Then for every n

$$\begin{aligned} P_x(\Phi_{ij}(X(R_n)) > \alpha; R - R_n < \beta, \Lambda) & \\ \leqq \ & P_x(\Phi_{ij}(X(R_n)) > \alpha; T_{ij}(\theta_{R_n} w) \geqq \beta) \\ < \ & \delta, \end{aligned}$$

and from this the assertion follows immediately. Therefore we can find integers k, L, and q such that

$$P_x(X(t) \in W_{ijk} \text{ for all } t \in [R_L, R], R < q) > 0.$$

It is obvious that for any i, j, k there are strictly positive numbers ξ and η such that $P_y(T_{ij} > \xi) > \eta$ for all y in W_{ijk}. Now there are only a finite number of W's appearing before W_{ijk} in our list, and so we may actually choose ξ and η so that $P_y(T_{L(y)} > \xi) > \eta$ for all y in W_{ijk}. But then for $n \geqq L$ we have

$$\begin{aligned} \eta P_x(X(t) \in W_{ijk} \text{ for all } t \in [R_L, R], R < q) & \\ \leqq \ & \eta P_x(R_n < q, X(R_n) \in W_{ijk}) \\ \leqq \ & E_x(P_{X(R_n)}(R_1 > \xi); R_n < q, X(R_n) \in W_{ijk}) \\ \leqq \ & P_x(R_{n+1} - R_n > \xi, R_n < q), \end{aligned}$$

which is impossible, since the last expression approaches 0 as $n \to \infty$. This contradiction completes the proof of the theorem.

Reference

1. R. M. Blumenthal, R. K. Getoor, and H. P. McKean, Jr., *Markov processes with identical hitting distributions*, Illinois J. Math., vol. 6 (1962), pp. 402–420.

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